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SOME GLOBAL PROPERTIES OF $M(f_1, f_2, f_3)_{2n+1}$–MANIFOLDS

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ABSTRACT : In this paper, we examine the global properties of generalized Sasakian space forms and obtained some interesting results.

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1. INTRODUCTION

The notions of weakly symmetric and weakly Ricci symmetric manifolds were introduced by L. Tamassy and T. Q. Binh in ([4], [5]). A non flat $(2n+1)$-dimensional differentiable manifold $(M^{2n+1}, g), n > 2$, is called pseudo symmetric ([4], [5]) it there exists a 1-form α on M^{2n+1}such that

$$\begin{aligned}(\nabla_X R)(Y, Z, V) &= 2\alpha(X)R(Y, Z)V + \alpha(V)R(X, Z)V + \alpha(Z)R(Y, X)V \\ &+\alpha(V)R(Y, Z)X + g(R(Y, Z)V, X)A, \end{aligned} \tag{1}$$

where $X, Y, Z, V \in \chi(M^{2n+1})$ are vector fields and α is a 1-form on M^{2n+1}, $A \in \chi(M^{2n+1})$ is the vector field corresponding through g to the 1-form which is defined as $g(X, A) = \alpha(X)$.
A non flat $(2n + 1)$-dimensional differentiable manifold $(M^{2n+1}, g), n > 2$, is called weakly symmetric ([4], [5]), it there exists a 1-forms α, β, ρ and γ on M^{2n+1} such that the condition

$$\begin{aligned}(\nabla_X R)(Y, Z, V) &= \alpha(X)R(Y, Z)V + \beta(Y)R(X, Z)V + \gamma(Z)R(Y, X)V \\ &+\sigma(V)R(Y, Z)X + g(R(Y, Z)V, X)P, \end{aligned} \tag{2}$$

holds for all vector fields $X, Y, Z, V \in \chi(M^{2n+1})$. A weakly symmetric manifold (M^{2n+1}, g) is pseudo symmetric if $\beta = \gamma = \sigma = {}^1\!/_2\alpha$ and $P = A$,locally symmetric if$\alpha = \beta = \gamma = \sigma = 0$. and a weakly symmetric manifold is said to be proper if at least one of the 1-form α, β, γ and σ is not zero or $P \neq 0$.

A non flat $(2n+1)$-dimensional differentiable manifold $(M^{2n+1}, g), n > 2$ is called weakly Ricci symmetric ([4], [5]), it there exists a 1-form ρ, μ and υ such that the condition

$$(\nabla_X S)(Z,V) \quad = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \upsilon(Z)S(X,Y), \qquad (3)$$

holds for all vector fields $X, Y, Z, V \in \chi(M^{2n+1})$, if $\rho = \mu = \upsilon$ then (M^{2n+1}) is called pseudo Ricci symmetric ([12]). If M is weakly symmetric, from (2), we have ([5]).

$$(\nabla_X S)(Z,V) = \alpha(X)S(Z,V) + \beta(R(X,Z)V) + \gamma(Z)S(X,V) + \sigma(V)S(Z,X)$$
$$+g(R(X,V,Z), \qquad (4)$$

In [5], Tamassy and et all studied weakly symmetric and weakly Ricci symmetric Einstein and Sasakian manifold. In ([14], [2], [9]) authors studied weakly symmetric and weakly Ricci symmetric K-contact, Lorentzian Para-Sasakian and Lorentzian β-Kenmotsu manifolds respectively. The notion of special weakly Ricci symmetric manifold was introduced and studied by H. Sinh and Q. Khan ([3]). An n-dimensional Riemannian manifold is called a special weakly Ricci symmetric manifold if

$$(\nabla_X S)(Y,Z) \quad = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(X,Y), \qquad (5)$$

where α is a 1-form and is defined by

$$\alpha(X) = g(X, \rho), \qquad (6)$$

where ρ is the associated vector field.

2. PRILIMANARIES

In [7], the author has defined a generalized Sasakian space forms as a contact metric manifolds $(M, \varphi, \zeta, \eta, g)$ whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

where f_1, f_2 , f_3 are some differentiable functions on M and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g\,(X, \varphi\, Z)\,\varphi\, Y - g\,(Y, \varphi\, Z) \varphi\, X + 2g\,(X, \varphi\, Y) \varphi\, Z,$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\zeta - g(Y,Z)\eta(X)\zeta$$

for any vector fields X, Y, Z on M. We denote it by $M(f_1, f_2, f_3)_{2n+1}$. In [7], the authors cited the several examples of such manifolds if $f_{1.} = \frac{c+1}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then generalized Sasakian space forms with Sasakian structure becomes Sasakian space forms A $(2n+1)$-dimensional Riemannian manifold (M, g) is called an almost contact manifold if the following results hold ([7], [12]):

$$\varphi^2(X) = -X + \eta(X)\zeta, \varphi\zeta = 0 \tag{7}$$

$$g(X, \zeta) = \eta(X), \eta(\zeta) = 1, \eta(\varphi X) = 0, \tag{8}$$

$$g(\varphi\, X, \varphi\, Y) = g(X,Y) - \eta(X)\eta(Y), \tag{9}$$

$$g(\varphi\, X, Y) = -g(X, \varphi\, Y), g(\varphi X, X) = 0, \tag{10}$$

$$(\nabla_X \eta)\,(Y) = g(\nabla_X \zeta, Y). \tag{11}$$

An almost contact metric manifold is called contact metric manifold if $d\eta(X,Y) = \Phi(X,Y) = g(X, \varphi Y)$, where Φ is called the fundamental two-form of the manifold. If ζ is a killing vector field the manifold is called a K-contact manifold. It is well known that a contact metric manifold is K-contact if and only if $\nabla_X \zeta = -\varphi X$, for any vector field X on (M, g). An almost contact metric manifold is Sasakian if and only if $(\nabla_X \varphi)\,(Y) = g(X,Y)\zeta - \eta(Y)X$, for any vector fields X, Y. In 1967, D. E. Blair introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds [4]. An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \zeta = -\beta \varphi X, \tag{12}$$

for all $X \in TM$ and a function β such that $\zeta\,\beta = 0$. As the consequence of (12), we get

$$(\nabla_X \eta)\,(Y) = g(\nabla_X \zeta, Y) = -\beta g(\varphi X, Y), \tag{13}$$

$$(\nabla_X \eta)\,(\zeta) = -\beta g(\varphi X, \zeta) = 0, \tag{14}$$

Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. It is known that [11] for a three-dimensional quasi-Sasakian manifold the Riemannian curvature tensor satisfies

$$R(X,Y)\zeta = \beta^2 \left\{\eta(Y)X - \eta(X)Y\right\} + d\beta(Y)\varphi X - d\beta(X)\varphi Y, \tag{15}$$

For a$(2n+1)$-dimensional generalized Sasakian spaceforms we have

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\zeta - g(Y,Z)\eta(X)\zeta\}, \end{aligned} \tag{16}$$

$$R(X,Y)\zeta = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{17}$$

$$R(\zeta,X)Y = (f_1 - f_3)\{g(X,Y)\zeta - \eta(Y)X\} \tag{18}$$

$$g(R(\zeta,X)Y,\zeta) = (f_1 - f_3)g(\varphi X,\varphi Y) \tag{19}$$

$$R(\zeta,X)\zeta = (f_1 - f_3)\varphi^2 X \tag{20}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \tag{21}$$

$$S(X,\zeta) = 2n(f_1 - f_3)\eta(X), \tag{22}$$

$$Q\zeta = 2n(f_1 - f_3)\zeta, \tag{23}$$

$$S(\varphi X,\varphi Y) = S(X,Y) + 2n(f_3 - f_1)\eta(X)\eta(Y) \tag{24}$$

here S is the Ricci tensor and r is the scalar curvature tensor of the space-form. It is known that an $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space forms is conformally flat if and only if $f_2 = 0$[13].

3. MAIN RESULTS

Theorem.1 *In a weakly symmetric generalized Sasakian space forms* $M(f_1, f_2, f_3)_{2n+1}$ *the sum of* 1*-forms* α, γ *and* σ *is zero everywhere.*

Proof. Let $M(f_1, f_2, f_3)_{2n+1}$ is a weakly symmetric generalized Sasakian space forms. Taking covariant differentiation of the Ricci tensor S with respect to X, we get

$$(\nabla_X S)(Z,V) = \nabla_X S(Z,V) + S(\nabla_X Z,V) + S(Z,\nabla_X.V), \tag{25}$$

Taking $V = \zeta$ in (25) and using (22), we have

$$(\nabla_X S)(Z,\zeta) = 2n\beta(f_1 - f_3)g(\varphi X,Z) - 2n(f_1 - f_3)\eta(\nabla_X Z) + \beta S(Z,\varphi X), \tag{26}$$

On the other hand taking $V = \zeta$ in (4) and using (22), we obtained

$$(\nabla_X S)(Z,\,\zeta) = 2n(f_1 - f_3)\alpha(X)\eta(Z) + \beta(R(X,Z)\zeta) + \gamma(Z)S(X,\zeta)$$

$$+\sigma(\zeta)S(Z,X) + g(R(X,\zeta,Z), \tag{27}$$

In view of (26) and (27), we have

$$2n\beta(f_1-f_3)g(\varphi X,Z)-2n(f_1-f_3)\eta(\nabla_X Z)+\beta S(Z,\varphi X) = 2n(f_1-f_3)\alpha(X)\eta(Z)$$

$$+\beta(R(X,Z)\zeta) + \gamma(Z)S(X,\zeta) + \sigma(\zeta)S(Z,X) + g(R(X,\zeta,Z), \tag{28}$$

Now taking $X = Z = \zeta$ in (28) and (17), (18) and (22), we yields

$$2n(f_1 - f_3)\left[\alpha(\zeta) + \gamma(\zeta) + \sigma(\zeta)\right] = 0, \tag{29}$$

which implies that $2n(f_1 - f_3) \neq 0$, so we have

$$\alpha(\zeta) + \gamma(\zeta) + \sigma(\zeta) = 0. \tag{30}$$

Now we will show that $\alpha + \gamma + \sigma = 0$ hold for all vector fields on M^{2n+1}. Taking $Z = \zeta$ in (4), similar to previous calculations it follows that

$$\begin{aligned} 0 \ &= 2n(f_1 - f_3)\alpha(X)\eta(V) + (f_1 - f_3)\{\eta(V)\beta(X) - g(X,V)\beta(\xi)\} \\ &+\gamma(\zeta)S(X,V) + 2n(f_1 - f_3)\eta(X)\sigma(V) + (f_1 - f_3)\{\eta(V)P(X) - \eta(X)P(V)\} \end{aligned} \tag{31}$$

$$\begin{aligned} 0 \ &= 2n(f_1 - f_3)\alpha(X) + (f_1 - f_3)\{\beta(X) - \eta(X)\beta(\zeta)\} \\ &+\gamma(\zeta)S(X,V) + 2n(f_1 - f_3)\eta(X)\sigma(\zeta) + (f_1 - f_3)\{P(X) - \eta(X)P(\zeta)\} \end{aligned} \tag{32}$$

Replacing $V = \zeta$ in (31) and using (6), (8) and (22), we have

Now taking $X = \zeta$ in (31) we obtained

$$\begin{aligned} 0 \ &= 2n(f_1 - f_3)\alpha(\zeta)\eta(V) + (f_1 - f_3)\{\eta(V)\beta(\zeta) - \eta(V)\beta(\zeta)\} \\ &+\gamma(\zeta)2n(f_1 - f_3)\eta(V) + 2n(f_1 - f_3)\sigma(V) + (f_1 - f_3)\{\eta(V)P(\zeta) - P(V)\} \end{aligned} \tag{33}$$

Interchanging V with X in (33) and summing with (32), in view of (30), we get

$$0 = 2n(f-f)\left[\alpha(X) + \sigma(X) + \eta(X)\gamma(\zeta)\right]+(f_1-f_3)\left(\beta(X) - \eta(X)\beta(\zeta)\right), \tag{34}$$

Now putting $X = \zeta$ in (28), we have

$$0 = 2n(f_1-f_3)\alpha(\zeta)\eta(Z)-\beta(Z)+\eta(Z)\beta(\zeta)+2n(f_1-f_3)\gamma(Z)+2n(f_1-f_3)\eta(Z)\sigma(\zeta), \tag{35}$$

Replacing Z with X in (35) and taking summations with (34), we find

$$0 = 2n(f_1 - f_3)\left[\alpha(X) + \sigma(X) + \gamma(X)\right] + 2n(f - f)\left[\gamma(\zeta) + \sigma(\zeta) + \alpha(\zeta)\right], \quad (36)$$

In view of (30) and (36), we get

$$\alpha(X) + \gamma(X) + \sigma(X) = 0, \forall X.$$

This proves the theorem 1.

Theorem 2. *In a weakly Ricci symmetric generalized Sasakian space forms $M(f_1, f_2, f_3)_{2n+1}$ the sum of 1-forms ρ, μ and υ is zero everywhere.*
Proof. We suppose that $M(f_1, f_2, f_3)_{2n+1}$ is a weakly Ricci symmetric generalized Sasakian space forms. Then putting $Z = \zeta$ in (3) and using (22), we have

$$(\nabla_X S)(\zeta, Y) = 2n(f_1 - f_3)\{\eta(Y)\rho(X) + \eta(X)\mu(Y)\} + \upsilon(\zeta)S(X, Y) \quad (37)$$

In view of (26) and (37), we get

$$2n\beta(f_1 - f_3)g(\varphi X, Y) + \beta S(Z, \varphi X)$$

$$= 2n(f_1 - f_3)\{\eta(Y)\rho(X) + \eta(X)\mu(Y)\} + \upsilon(\zeta)S(X, Y), \quad (38)$$

Taking $X = Y = \zeta$ in (38) and by use of (7) and (22), we yields

$$0 = 2n(f_1 - f_3)\left[\rho(\zeta) + \mu(\zeta) + \upsilon(\zeta)\right], \quad (39)$$

This implies that $(2n(f_1 - f_3) \neq 0)$

$$\rho(\zeta) + \mu(\zeta) + \upsilon(\zeta) = 0. \quad (40)$$

Now putting $X = \zeta$ in (38), and by use of (7) and (22), we get

$$0 = 2n(f_1 - f_3)\eta(Y)\{\rho(\zeta) + \upsilon(\zeta)\} + 2n(f_1 - f_3)\mu(Y), \quad (41)$$

In view of (40), the equations(41) reduces t o$(2n(f_1 - f_3) \neq 0)$

$$\mu(Y)\} = \mu(\zeta)\eta(Y), \quad (42)$$

Again putting $Y = \zeta$ in (38), and by virtue of (40), we also have

$$\rho(X) = \rho(\zeta)\eta(X), \tag{43}$$

Since $(\nabla_X S)(\zeta, X) = 0$, from (3), we obtain

$$\eta(X)[\rho(\zeta) + \mu(\zeta)] = -\upsilon(X) \tag{44}$$

In view of (40) and (43), we get

$$\upsilon(X) = \eta(X)\upsilon(\zeta), \tag{45}$$

Therefore replacing Y with X in (42) and by summation of (42), (43) and (44), we get

$$\rho(X) + \mu(X) + \upsilon(X) = \eta(X)\left[\rho(\zeta) + \mu(\zeta) + \upsilon(\zeta)\right], \tag{46}$$

In view of (40), it follows that

$$\rho(X) + \mu(X) + \upsilon(X) = 0.$$

for all X, which implies that $\rho + \mu + \upsilon = 0$ on M^{2n+1}.

Theorem.3 *If a special weakly Ricci symmetric generalized Sasakian space forms $M(f_1, f_2, f_3)_{2n+1}$ admits a cyclic Ricci tensor then* 1*-form α must vanishes.*
Proof. Taking cyclic sum of (5), we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y .S)(Z, X) + (\nabla_Z .S)(X, Y)$$

$$= 4\left[\alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y)\right] \tag{47}$$

We suppose that $M(f_1, f_2, f_3)_{2n+1}$ admits a cyclic Ricci condition. Then (47) reduces to

$$0 = 4\left[\alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y)\right], \tag{48}$$

Putting $Z = \zeta$ in (48) and using (22), we get

$$2n(f_1 - f_3)\left[\eta(Y)\alpha(X) + \eta(X)\alpha(Y)\right] + \alpha(\zeta)S(X, Y) = 0, \tag{49}$$

Again taking $Y = \zeta$ in (49) and using (22), we obtain

$$\alpha(X) = -2\eta(X)\alpha(\zeta), \tag{50}$$

Replacing $X = \zeta$ in (21) and by virtue of (15), we get

$$\alpha(X) = 0, \tag{51}$$

for all X. This proves the theorem 3.

Theorem 4. *A special weakly Ricci symmetric generalized Sasakian space forms $M(f_1, f_2, f_3)_{2n+1}$ can not be an Einstein manifold provided* 1*-form* $\alpha \neq 0$.
Proof . We know that for Einstein manifold, $(\nabla_X S)(Y, Z) = 0$ and $S(Y, Z) = kg(Y, Z)$. Then from (5) gives

$$0 = 2\alpha(X)g(Y, Z) + \alpha(Y)g(X, Z) + \alpha(Z)g(Y, X), \tag{52}$$

Replacing $Z = \zeta$ in (52) and using(6), we have

$$0 = 2\alpha(X)\eta(Y) + \alpha(Y)\eta(X) + \eta(\rho)g(X, Y), \tag{53}$$

Again replacing $X = \zeta$ in (53) and using (6), we get

$$3\eta(\rho)\eta(Y) = \alpha(Y), \tag{54}$$

Taking $X = \zeta$ (54), we have

$$\eta(\rho) = 0, \tag{55}$$

This implies that $\alpha(Y) = 0$, for all Y. this proves the theorem 4.

Theorem 5. *A special weakly Ricci symmetric generalized Sasakian space forms $M(f_1, f_2, f_3)_{2n+1}$ is an Einstein manifold.*
Proof. Finally taking $Z = \zeta$ in (5), we have

$$(\nabla_X S)(Y, \zeta) = 4n(f_1 - f_3)\eta(Y)\alpha(X) + 2n(f_1 - f_3)\eta(X)\alpha(Y) + \alpha(\zeta)S(X, Y), \tag{56}$$

The left hand side can be written in the form

$$(\nabla_X S)(Y, \zeta) = X\, S(Y, \zeta) - S(\nabla_X Y, \zeta) - S(Y, \nabla_X \zeta), \tag{57}$$

In view of (22), (56) and (57), we get

$$4n(f_1 - f_3)\eta(Y)\alpha(X) + 2n(f_1 - f_3)\eta(X)\alpha(Y) + \alpha(\xi)S(X,Y)$$
$$= -2n\beta(f_1 - f_3)g(\phi X, Y) + \beta S(Y, \phi X), \tag{58}$$

Taking $Y = \zeta$ in (58) and by use of (6), (12) and (22), we get

$$\alpha(X) = 0. \tag{59}$$

Using (59) in (5), we obtain $(\nabla_X S)(Y,Z) = 0$, this proves the theorem 5.

Corollary: *A special weakly Ricci symmetric generalized Sasakian space forms $M(f_1, f_2, f_3)_{2n+1}$ is R.hormonic.*

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