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GLOBAL BEHAVIOR OF A RATIONAL DIFFERENCE EQUATION

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Abstract. In this paper, we investigate the local asymptotic stability, global stability, the periodic character, semicycle analysis and the boundedness nature of solutions of the following $(k+1)$-order rational difference equation:

$$x_{n+1} = \frac{A + Bx_n + Cx_{n-k}}{1 + x_n + x_{n-k}},\ n = 0, 1, 2, \cdots,$$

where $A,\ B,\ C$ are positive real numbers, the initial conditions $x_{-k},\ x_{-k+1},\ \cdots,\ x_{-1},\ x_0$ are arbitrary positive real numbers and k is positive integer. Some numerical examples are given to illustrate our results.

2000 *Mathematics Subject Classification* 39A10, 40A05.

1. Introduction

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology and resource management. It is very interesting to investigate the behavior of solutions of a higher-order rational difference equation and to discuss the local asymptotic stability of its equilibrium points. Rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such equations. For more results for the rational difference equations, we refer the interested reader to [6–10].

2. Preliminaries and definitions

Recently, there has been great interest in studying difference equations. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in many applied sciences [9, 10].

Su and Li [1] studied the global asymptotic stability of the nonlinear difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}.$$

Saleh and Baha [2] investigated the behavior of nonlinear rational difference equation:

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}.$$

Yan, Li and Zhao [3] investigated the boundedness, periodic character, invariant intervals and the global asymptotic stability of all nonnegative solutions of the difference equation:

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_n}.$$

Tang, Hu and Ma [4] considered the following higher-order nonlinear difference equation:

$$x_{n+1} = \frac{p + qx_{n-k}}{1 + x_n + rx_{n-k}}.$$

Dehghan and Rastegar [5] investigated the qualitative behavior of the higher-order non-linear difference equation:

$$y_{n+1} = \frac{p + qy_n + ry_{n-k}}{1 + y_{n-k}}.$$

To be motivated by the above studies, our aim in this paper is to investigate the behavior of a rational difference equation:

$$x_{n+1} = \frac{A + Bx_n + Cx_{n-k}}{1 + x_n + x_{n-k}},\ n = 0, 1, 2, \cdots, \tag{1}$$

where $A,\ B,\ C$ are positive real numbers, the initial conditions $x_{-k},\ x_{-k+1},\ \cdots,\ x_{-1},\ x_0$ are arbitrary positive real numbers and k is positive integer. A difference equation of order $(k+1)$ is an equation of the form:

$$x_{n+1} = F(x_n, x_{n-1}, \cdots, x_{n-k}),\ n = 0, 1, \cdots, \tag{2}$$

where F is a continuously differentiable function which maps some set I^{k+1} into I. The set I is usually an interval of real numbers.

A solution of the equation (2) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies the equation (2) for all $n \geq 0$.

Definition 1 *A solution $\{x_n\}_{n=-k}^{\infty}$ of the equation (2) which is constant for all $n \geq -k$ is called an equilibrium solution of the equation (2). If $x_n = \bar{x}$ for all $n \geq -k$ is an equilibrium solution of the equation (2), then $\bar{x}$ is an equilibrium point of the equation (2), or equivalently a point $\bar{x} \in I$ is an equilibrium point of the equation (2) if*

$$\bar{x} = F(\bar{x}, \bar{x}, \cdots, \bar{x}).$$

Definition 2 *A solution $\{x_n\}_{n=-k}^{\infty}$ of difference the equation (2) is bounded and persists if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, \cdots, x_{-1}, x_0$ there exists a positive integer $\bar{n}$ such that $m \leq x_n \leq M$ for all $n \geq \bar{n}$.*

Definition 3 *(Stability)*

(i) An equilibrium point $\bar{x}$ of the equation (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of the equation (2) with $|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\bar{x}$ of the equation (2) is locally asymptotically stable if it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Equation (2) with $|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma$, then $\lim\limits_{n\to\infty} x_n = \bar{x}$.

(iii) An equilibrium point $\bar{x}$ of the equation (2) is a global attractor if for every solution $\{x_n\}_{n=-k}^{\infty}$ of the equation (2), we have $\lim\limits_{n\to\infty} x_n = \bar{x}$.

(iv) An equilibrium point $\bar{x}$ of the equation (2) is globally asymptotically stable if it is locally stable, and $\bar{x}$ is also global attractor of the equation (2).

(v) An equilibrium point $\bar{x}$ of the equation (2) is unstable if $\bar{x}$ is not locally stable.

Definition 4 *A solution $\{x_n\}_{n=-k}^{\infty}$ is periodic with period p if there exists an integer $p \geq 1$ such that*

$$x_{n+p} = x_n \text{ for all } n \geq -k. \tag{3}$$

A solution is periodic with prime period p if p is the smallest positive integer for which (3) holds.

3. Linearized stability analysis

Suppose F is continuously differentiable in some open neighborhood of $\bar{x}$. Let $p_i = \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x})$ for $i = 0, 1, \cdots, k$ denote the partial derivatives of $F(u_0, u_1, \cdots, u_k)$ with respect to u_i evaluated at $\bar{x}$. The equation

$$z_{n+1} = p_0 z_n + p_1 z_{n-1} + \cdots + p_k z_{n-k}, \ n = 0, 1, \cdots \tag{4}$$

is called linearized equation of (2) about $\bar{x}$, and the equation

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{5}$$

is called characteristic equation of (4) about $\bar{x}$.

The following result is known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point of the equation (2).

Lemma 1 *[11] Assume that F is continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$ and if all roots of the equation (5) have absolute value less than one, then the equilibrium point of the equation (2) is locally asymptotically stable.*

The following result is a sufficient condition for all roots of an equation of any order to lie inside the unit disk.

Lemma 2 *[11] Assume that $p_0, p_1, \cdots, p_k$ are real numbers such that $|p_0| + |p_1| + \cdots + |p_k| < 1$. Then all roots of the equation (5) lie inside the open unit disk $|\lambda| < 1$.*

To study the local stability character of the solutions of equation (1), we let I be some interval of real numbers and let $f : I^2 \to I$ be a continuously differentiable function defined by

$$f(x, y) = \frac{A + Bx + Cy}{1 + x + y}.$$

Let $\bar{x}$ be an equilibrium point of (1), then $\bar{x} = f(\bar{x}, \bar{x})$ and this implies that

$$\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$$

be unique positive equilibrium point of the equation (1). Moreover,

$$\frac{\partial f}{\partial x}(x, y) = \frac{B - A + (B - C)y}{(1 + x + y)^2},$$

and
$$\frac{\partial f}{\partial y}(x,y)=\frac{C-A+(C-B)x}{(1+x+y)^2}.$$
Furthermore,
$$q_0=\frac{\partial f}{\partial x}(\bar{x},\bar{x})=\frac{-1-4A+C+(2B+1)\sqrt{8A+(-1+B+C)^2}-B(1+2B+2C)}{8A-4(B+C)},$$
and
$$q_1=\frac{\partial f}{\partial y}(\bar{x},\bar{x})=\frac{-1-4A+B+(2C+1)\sqrt{8A+(-1+B+C)^2}-C(1+2B+2C)}{8A-4(B+C)}.$$
The linearized equation of (1) about $\bar{x}$ is given by
$$y_{n+1}=q_0y_n+q_1y_{n-k}. \tag{6}$$
Moreover, characteristic equation of (6) is given by
$$\lambda^{k+1}-q_0\lambda^k-q_1=0. \tag{7}$$

Theorem 1 *The unique positive equilibrium point*
$$\bar{x}=\frac{1}{4}\left(-1+B+C+\sqrt{8A+(-1+B+C)^2}\right)$$
of the equation (1) is locally asymptotically stable if one of the following cases holds

(i) $B>2A$.

(ii) $B<2A$ *and* $C<2A-B$.

(iii) $B<2A$ *and* $C>2A-B$.

Proof. Let $P(\lambda)=\lambda^{k+1}-q_0\lambda^k-q_1$, $\Phi(\lambda)=\lambda^{k+1}$ and $\Psi(\lambda)=q_0\lambda^k+q_1$.
(i) Assume that $B>2A$ and $|\lambda|=1$. Then, one has
$$\begin{aligned}|\Psi(\lambda)| &\le \left|\frac{-1-4A+C+(2B+1)\sqrt{8A+(-1+B+C)^2}-B(1+2B+2C)}{8A-4(B+C)}\right|\\ &+\left|\frac{-1-4A+B+(2C+1)\sqrt{8A+(-1+B+C)^2}-C(1+2B+2C)}{8A-4(B+C)}\right|\\ &=\frac{(B-C)\left(-\sqrt{8A+(B+C-1)^2}+B+C+1\right)}{4A-2(B+C)}<1.\end{aligned}$$

Hence, $|\Psi(\lambda)| < 1 = |\Phi(\lambda)|$ for $B > 2A$ and $|\lambda| = 1$.

Then, by Rouche's Theorem $\Phi(\lambda)$ and $\Phi(\lambda) - \Psi(\lambda)$ have same number of zeroes in an open unit disk $|\lambda| < 1$. Hence, all the roots of (7) satisfies $|\lambda| < 1$, and it follows from lemma 1 that the unique positive equilibrium point $\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$ of the equation (1) is locally asymptotically stable for $B > 2A$.

Similarly, one can prove (ii) and (iii).

4. Periodicity

In this section we discuss the periodic nature of the solutions of the equation (1).

Theorem 2 *Let k is even, then the equation (1) has no prime period two solutions.*

Proof. Assume that k is even and

$$\cdots,\ p,\ q,\ p,\ q,\ \cdots (p \neq q)$$

be distinctive prime period-two solutions of the equation (1). Then, from (1) we have

$$p = \frac{A + (B + C)q}{1 + 2q},\ q = \frac{A + (B + C)p}{1 + 2p}. \tag{8}$$

From (8), we obtain

$$p(1 + 2q) = A + (B + C)q,\ q(1 + 2p) = A + (B + C)p. \tag{9}$$

On subtraction (9) implies that

$$(p - q)(1 + B + C) = 0.$$

Since $1 + B + C \neq 0$, it follows that $p = q$, which is a contradiction.

Theorem 3 *Let k is odd, then the equation (1) has no prime period two solutions.*

Proof. Assume that k is odd and

$$\cdots,\ p,\ q,\ p,\ q,\ \cdots (p \neq q)$$

be distinctive prime period-two solutions of the equation (1). Then, from (1) we have

$$p = \frac{A + Bq + Cp}{1 + p + q},\ q = \frac{A + Bp + Cq}{1 + p + q}. \tag{10}$$

From (10), we obtain

$$p(1+p+q) = A + Bq + Cp,\ q(1+p+q) = A + Bp + Cq. \tag{11}$$

Assume that p and q are two positive distinct real roots of the quadratic equation

$$t^2 - (p+q) + pq = 0. \tag{12}$$

From the equation (11), we obtain

$$p + q = \frac{1}{2}\left(-1 + B + C + \sqrt{8A + (-1+B+C)^2}\right),$$

and

$$pq = \frac{1}{16}\left(-1 + B + C + \sqrt{8A + (-1+B+C)^2}\right).$$

Then, $(p+q)^2 - 4pq = 0$ and it follows that the equation (12) has equal real roots $p = q$, which is a contradiction.

5. Boundedness

In this section we prove that every solution of the equation (1) is bounded and persists.

Theorem 4 *Every solution of the equation (1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be an arbitrary solution of the equation (1). Then, it follows from the equation (1) that

$$\begin{aligned} x_{n+1} &= \frac{A + Bx_n + Cx_{n-k}}{1 + x_n + x_{n-k}} \\ &= \frac{A}{1 + x_n + x_{n-k}} + \frac{Bx_n}{1 + x_n + x_{n-k}} + \frac{Cx_{n-k}}{1 + x_n + x_{n-k}} \\ &\le A + B + C = M. \end{aligned}$$

Hence,

$$x_n \le M \quad \text{for all} \quad n \ge 1.$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \ge m \quad \text{for all} \quad n \ge 1.$$

Due to transformation $x_n = \frac{1}{y_n}$ equation (1) becomes

$$\begin{aligned} y_{n+1} &= \frac{1+y_n+y_{n-k}}{A+By_n+Cy_{n-k}} \\ &= \frac{1}{A+By_n+Cy_{n-k}} + \frac{y_n}{A+By_n+Cy_{n-k}} + \frac{y_{n-k}}{A+By_n+Cy_{n-k}} \\ &\leq \frac{1}{A}+\frac{1}{B}+\frac{1}{C} \\ &= \frac{AB+BC+CA}{ABC}. \end{aligned}$$

Thus

$$x_n \geq \frac{ABC}{AB+BC+CA} = m \text{ for all } n \geq 1.$$

Hence,

$$0 < m \leq x_n \leq M \text{ for all } n \geq 1.$$

6. Semi-cycle analysis

In this section, we study the analysis of semi-cycles of solutions of the equation (1) relative to the unique positive equilibrium point

$$\bar{x} = \frac{1}{4}\left(-1+B+C+\sqrt{8A+(-1+B+C)^2}\right).$$

Definition 5 *A positive semi-cycle of the solution $\{x_n\}_{n=-k}^{\infty}$ of the equation (1) consists of a " string" of terms $\{x_s,\ x_{s+1},\ \cdots,\ x_t\}$ all greater than or equal to $\bar{x}$, with $s \geq -k$ and $t \leq \infty$ such that*

$$\textit{either } \ s=-k \ \textit{ or } \ s>-k \ \textit{ and } \ x_{s-1}<\bar{x},$$

and

$$\textit{either } \ t=\infty \ \textit{ or } \ t<\infty \ \textit{ and } \ x_{t-1}<\bar{x}.$$

A negative semi-cycle of the solution $\{x_n\}_{n=-k}^{\infty}$ of the equation (1) consists of a " string" of terms $\{x_s,\ x_{s+1},\ \cdots,\ x_t\}$ all less than $\bar{x}$, with $s \geq -k$ and $t \leq \infty$ such that

$$\textit{either } \ s=-k \ \textit{ or } \ s>-k \ \textit{ and } \ x_{s-1}\geq\bar{x},$$

and

$$\textit{either } \ t=\infty \ \textit{ or } \ t<\infty \ \textit{ and } \ x_{t-1}\geq\bar{x}.$$

Lemma 3 *[5] Consider the following difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \;\; n = 0, 1, 2, \cdots, \tag{13}$$

where k is some positive integer and $f : (0,\infty)^2 \to (0,\infty)$ is a continuous function such that $f(x,y)$ is increasing in x for each y, and $f(x,y)$ is decreasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of the equation (13). Then, except possibly for the first semi-cycle, every oscillatory solution of the equation (13) has semi-cycle of length at least $k+1$, or of length of most $k-1$.

Theorem 5 *Let $C < A < B$, then the function $f : (0,\infty)^2 \to (0,\infty)$ be a continuously differentiable function defined by*

$$f(x,y) = \frac{A + Bx + Cy}{1 + x + y}$$

is increasing in x for each $y \in (0,\infty)$, and $f(x,y)$ is decreasing in y for every $x \in (0,\infty)$.

Proof. Let $C < A < B$, then it follows that

$$\frac{\partial f}{\partial x}(x,y) = \frac{B - A + (B - C)y}{(1 + x + y)^2} > 0,$$

for each $y \in (0,\infty)$, and

$$\frac{\partial f}{\partial y}(x,y) = \frac{C - A + (C - B)x}{(1 + x + y)^2} < 0,$$

for each $x \in (0,\infty)$.

Theorem 6 *Let $C < A < B$, then for $k \geq 2$ every oscillatory solution of the equation (1) has semi-cycle of length at least $k+1$, or of length of most $k-1$.*

Proof. Let $C < A < B$, then from theorem 5 the function $f(x,y) = \frac{A+Bx+Cy}{1+x+y}$ is increasing in x for each $y \in (0,\infty)$, and $f(x,y)$ is decreasing in y for every $x \in (0,\infty)$. Let $\{x_n\}_{n=-k}^{\infty}$ be an oscillatory solution of the equation (1). Then, for the unique positive equilibrium point $\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$ of the equation (1) such that

$$x_{m-1} < \bar{x} \leq x_m.$$

Assume that

$$x_{m+1}, \cdots, x_{m+k-1} \geq \bar{x}.$$

Then, from the equation (1), we obtain

$$x_{m+k} = \frac{A + Bx_{m+k-1} + Cx_{m-1}}{1 + x_{m+k-1} + x_{m-1}} > f(\bar{x}, \bar{x}) = \bar{x},$$

which shows that x_{m+k} also belongs to positive semi-cycle. The proof in the case $x_m < \bar{x} \leq x_{m-k}$, is similar and omitted.

7. Global stability

In this section we discuss the global stability of the unique positive equilibrium point $\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$ of the equation (1).

Lemma 4 *[11] Let $f : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \ n = 0, 1, 2, \cdots. \tag{14}$$

Suppose that f satisfies the following conditions

(i) $f(x, y)$ is non-decreasing in x for each fixed $y \in [a, b]$, and $f(x, y)$ is non-increasing in y for each fixed $x \in [a, b]$.

(ii) If (m, M) is the solution of the system

$$m = f(m, M), \ M = f(M, m),$$

then $m = M$.

Then there exists exactly one equilibrium point $\bar{x}$ of the equation (14), and every solution of the equation (14) converges to $\bar{x}$.

Theorem 7 *Assume that $C < A < B$ and $B > 2A$, then the unique positive equilibrium point $\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$ of the equation (1) is globally asymptotically stable.*

Proof. Assume that $B > 2A$, then from (i) of theorem 1 the unique positive equilibrium point $\bar{x} = \frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right)$ of the equation (1) is locally asymptotically stable. Moreover, assuming $C < A < B$, then from the theorem 5 the function $f(x,y) = \frac{A+Bx+Cy}{1+x+y}$ is non-decreasing in x for each fixed $y \in (0,\infty)$, and $f(x,y)$ is non-increasing in y for every fixed $x \in (0,\infty)$. If (m, M) is the solution of the system

$$m = f(m, M), \; M = f(M, m),$$

then one has

$$M = \frac{A + BM + Cm}{1 + M + m},$$

and

$$m = \frac{A + Bm + CM}{1 + M + m}.$$

This implies that

$$Mm = \frac{m(A + BM + Cm)}{1 + M + m}, \tag{15}$$

and

$$mM = \frac{M(A + Bm + CM)}{1 + M + m}. \tag{16}$$

Comparing (15) and (16), we obtain

$$\frac{m(A + BM + Cm)}{1 + M + m} = \frac{M(A + Bm + CM)}{1 + M + m}.$$

This implies that

$$(M - m)\left(A + C(M + m)\right) = 0.$$

Since $A + C(M + m) \neq 0$, it follows that $M = m$ and the proof is completed.

8. Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent qualitative behavior of solutions of the nonlinear difference equation (1).

Example 1 *Let $A = 5,\; B = 5000,\; C = 0.3,\; k = 5$, then the equation (1) can be written as*

$$x_{n+1} = \frac{5 + 5000x_n + 0.3x_{n-5}}{1 + x_n + x_{n-5}}, \tag{17}$$

with initial conditions $x_0 = 0.1,\; x_{-1} = 0.2,\; x_{-2} = 0.3,\; x_{-3} = 0.4,\; x_{-4} = 0.5,\; x_{-5} = 0.6$. The unique positive equilibrium point of the equation (17) is given by $\frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right) = 2499.65$. Moreover, the plot of the equation (17) is shown in Fig. 1.

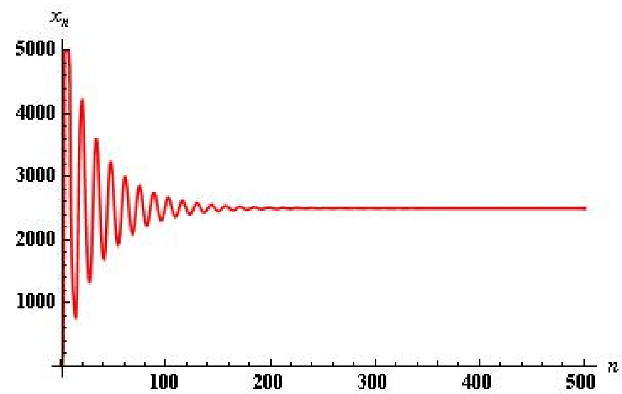

Figure 1: Plot of the equation (17)

Example 2 *Let* $A = 2,\ B = 1500000,\ C = 700,\ k = 10$*, then the equation (1) can be written as*

$$x_{n+1} = \frac{2 + 1500000x_n + 700x_{n-10}}{1 + x_n + x_{n-10}}, \tag{18}$$

with initial conditions $x_0 = 0.1,\ x_{-1} = 0.2,\ x_{-2} = 0.3,\ \cdots,\ x_{-10} = 1.1$*. The unique positive equilibrium point of the equation (18) is given by*

$$\frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right) = 750350.$$

Moreover, the plot of the equation (18) is shown in Fig. 2.

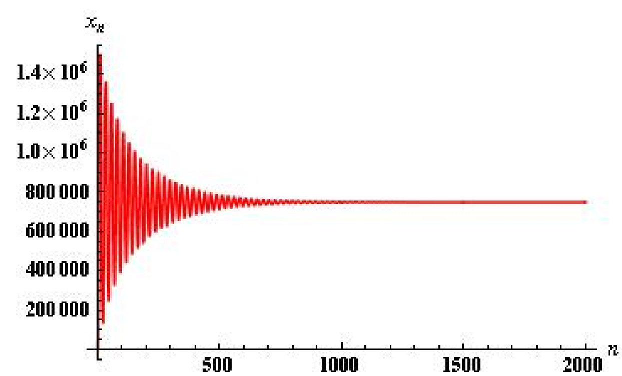

Figure 2: Plot of the equation (18)

Example 3 *Let* $A = 8,\ B = 60000,\ C = 17,\ k = 15$, *then the equation (1) can be written as*

$$x_{n+1} = \frac{8 + 60000x_n + 17x_{n-15}}{1 + x_n + x_{n-15}}, \tag{19}$$

with initial conditions $x_0 = 0.1,\ x_{-1} = 0.2,\ x_{-2} = 0.3,\ \cdots,\ x_{-15} = 1.6$. *The unique positive equilibrium point of the equation (19) is given by*

$$\frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right) = 30008.$$

Moreover, the plot of the equation (19) is shown in Fig. 3.

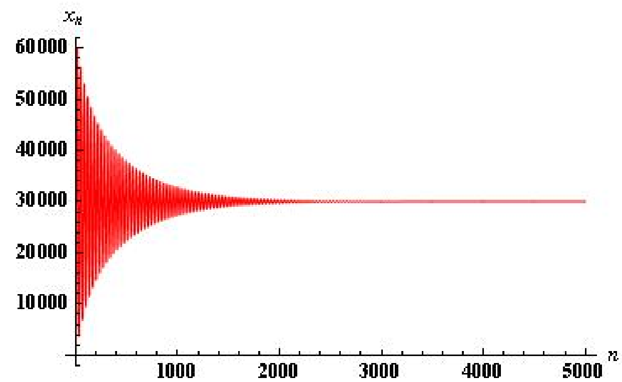

Figure 3: Plot of the equation (19)

Example 4 *Let* $A = 11,\ B = 90,\ C = 800000,\ k = 30$, *then the equation (1) can be written as*

$$x_{n+1} = \frac{11 + 90x_n + 800000x_{n-30}}{1 + x_n + x_{n-30}}, \tag{20}$$

with initial conditions $x_0 = 0.1,\ x_{-1} = 0.2,\ x_{-2},\ \cdots,\ x_{-30} = 3.1$. *The unique positive equilibrium point of the equation (20) is given by*

$$\frac{1}{4}\left(-1 + B + C + \sqrt{8A + (-1 + B + C)^2}\right) = 400045.$$

Moreover, the plot of the equation (20) is shown in Fig. 4.

Example 5 *Let* $A = 55,\ B = 150,\ C = 80000,\ k = 35$, *then the equation (1) can be written as*

$$x_{n+1} = \frac{55 + 150x_n + 80000x_{n-35}}{1 + x_n + x_{n-35}}, \tag{21}$$

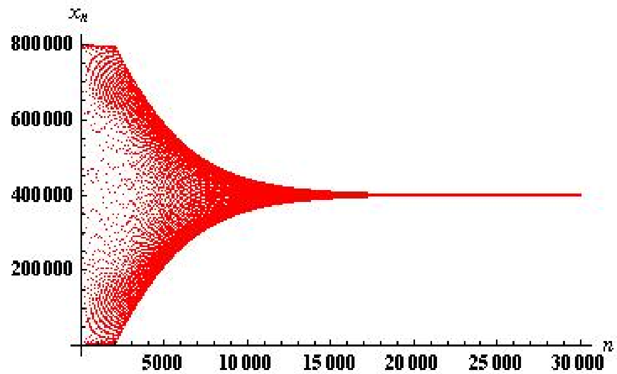

Figure 4: Plot of the equation (20)

with initial conditions $x_0 = 0.1,\ x_{-1} = 0.2,\ \cdots,\ x_{-35} = 3.6$. *The unique positive equilibrium point of the equation (21) is given by*

$$\frac{1}{4}\left(-1+B+C+\sqrt{8A+(-1+B+C)^2}\right) = 40074.5$$

Moreover, the plot of the equation (21) is shown in Fig. 5.

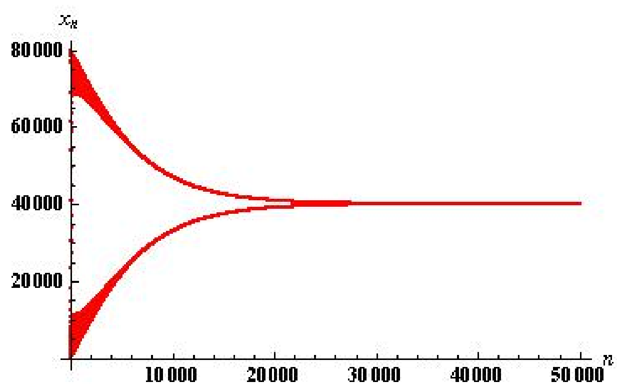

Figure 5: Plot of the equation (21)

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