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PENALTY METHOD ALGORITHM FOR FUZZY LINEAR FRACTIONAL OPTIMIZATION

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Abstract. In the classical problems of mathematical programming the coefficients are assumed to be exactly known. This assumption is seldom satisfied by great majority of real-life problems. Starting from the idea of K.D. Jamison and W.A. Lodwick which solve a fuzzy linear programming problem (2001), a linear fractional programming problem with fuzzy coefficients is described and a solving algorithm is obtained. The method is based on a penalty approach which could transform the objective and constraints into an unconstraint function defined in the space of fuzzy numbers.

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1. Introduction

Fuzzy approaches to solve deterministic problems could be developed and also fuzzy models, implying fuzzy goals and fuzzy coefficients, could be defined and solved [4]. In [1] Buckley and Feuring considered the fully fuzzified linear programming problem (FFLP) by establishing all the coefficients and variables of a linear program as being fuzzy quantities. They transform the fully fuzzified programming problem in a multi-objective deterministic problem which is solved using a genetic algorithm leading to feasible solutions for the initial problem.

In this paper the case of a linear fractional programming problem with crisp variables and fuzzy coefficients is considered.

Starting from the idea of K.D. Jamison and W.A. Lodwick [3] which solve a fuzzy linear programming problem (2001), a linear fractional programming problem with fuzzy coefficients is described here and a solving algorithm is obtained. The method is based on a penalty approach which could transform the objective and constraints into an unconstraint function defined in the space of fuzzy numbers.

The paper is divided into 5 sections. The aggregation of triangular fuzzy numbers is presented in Section 3. In Section 4 we propose a method of solving problem (1)-(2) when all initial fuzzy quantities are described with triangular fuzzy numbers. The unconstraint linear fractional programming problem is obtained by penalizing the objective for possible constraint violation. In Section 5, to illustrate our method, we consider a numerical example.

2. Fractional programming problem with fuzzy coefficients

Let us consider the linear fractional programming problem

$$\max\left(\overline{Z}=\frac{\sum\limits_{j=1}^{n}\overline{C_j}X_j+\overline{C_0}}{\sum\limits_{j=1}^{n}\overline{D_j}X_j+\overline{D_0}}\right) \tag{1}$$

subject to

$$\begin{cases}\overline{M_i}=\sum\limits_{j=1}^{n}\overline{A_{ij}}X_j-\overline{B_i}\leq\overline{0}, & i=1,...,m\\ X_j\geq 0, \quad j=1,...,n\end{cases} \tag{2}$$

where

(i) $\left(\overline{C_j}\right)_{j=1,...,n}, \overline{C_0}$ and $\left(\overline{D_j}\right)_{j=1,...,n}, \overline{D_0}$ represent the coefficients of the linear fractional objective function,

(ii) $\left(\overline{A_{ij}}\right)_{i=1,...,m}^{j=1,...,n}$ and $\left(\overline{B_i}\right)_{i=1,...,m}$ represent the coefficients and the right hand side of the linear constraints respectively,

(iii) $(X_j)_{j=1,...,n}$ represents the decision variables.

Here it is customary to assume that the denominator in (1) is strictly positive for any X_j in the feasible region. Moreover, in this paper we will assume that the nominator in (1) is strictly positive. The notation $\overline{Y}$ means that Y represents a fuzzy quantity.

The aggregating operators for fuzzy quantities are defined using the Zadeh's extension principle (see Zimmermann [7]). We apply the extension principle of Zadeh to add fuzzy numbers and an approximate version of the same principle to multiply and divide fuzzy numbers. As we will see in the next section, if $\overline{C_j}$, $\overline{C_0}$, $\overline{D_j}$, $\overline{D_0}$, $\overline{B_i}$, $\overline{A_{ij}}$ are triangular fuzzy numbers for each $i=1,...,m$ and $j=1,...,n$, then $\overline{Z}$ and $\overline{M_i}$ could be approximated by triangular fuzzy numbers for each $i=1,...,m$.

2. The aggregation of triangular fuzzy numbers

The purpose of this section is to recall some concepts which will be needed in the sequel. Also, we define the aggregating way of triangular fuzzy numbers.

Definition 1. [7]*A triangular fuzzy number* $\overline{Y}$ *is a triplet* $(y^1, y^2, y^3) \in R^3$*. The membership function of* $\overline{Y}$ *is defined in connection with the real numbers* y^1, y^2, y^3 *as follows:*

$$\overline{Y}(x) = \begin{cases} 0, & x \in (-\infty, y^1) \\ \dfrac{x - y^1}{y^2 - y^1}, & x \in [y^1, y^2] \\ \dfrac{x - y^3}{y^2 - y^3}, & x \in (y^2, y^3] \\ 0, & x \in (y^3, \infty) \end{cases}$$

$\overline{Y}(x)$ represents a number in $[0, 1]$, which is the membership function of $\overline{Y}$ evaluated in x. It can be easily verified that graph $y = \overline{Y}(x)$ of $\overline{Y}$ is a triangle with base on $[y^1, y^3]$ and vertex at $x = y^2$ for $y^1 < y^2 < y^3$.

The extension principle was formulated by Zadeh in order to extend the known models implying fuzzy elements to the case of fuzzy entities. Applying this principle the following definitions of the addition and subtraction of triangular fuzzy numbers result:

Definition 2. *Being given two triangular fuzzy numbers* $\overline{A} = (a^1, a^2, a^3)$, $\overline{B} = (b^1, b^2, b^3)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, *we have:*

(i) $\overline{A} + \overline{B} = (a^1 + b^1, a^2 + b^2, a^3 + b^3)$,

(ii) $\overline{A} - \overline{B} = (a^1 - b^3, a^2 - b^2, a^3 - b^1)$.

Applying the principle of extension to multiply triangular fuzzy numbers it is not obtained a triangular fuzzy number. According [2,6] we could use $\alpha-$cuts method to describe the membership function of the result.

The $\alpha-$cuts of fuzzy numbers $\overline{A} = (a^1, a^2, a^3)$ and $\overline{B} = (b^1, b^2, b^3)$ are the intervals

$$\left[\left(a^2 - a^1\right)\alpha + a^1, \left(a^2 - a^3\right)\alpha + a^3\right],$$

$$\left[\left(b^2 - b^1\right)\alpha + b^1, \left(b^2 - b^3\right)\alpha + b^3\right]$$

respectively. Considering this intervals with non-negative endpoints and making a multiplication of them an interval $[l, r]$ is obtained, where

$$l = \left(a^2 - a^1\right)\left(b^2 - b^1\right)\alpha^2 + \left[\left(a^2 - a^1\right)b^1 + \left(b^2 - b^1\right)a^1\right]\alpha + a^1 b^1,$$

$$r = \left(a^2 - a^3\right)\left(b^2 - b^3\right)\alpha^2 + \left[\left(a^2 - a^3\right)b^3 + \left(b^2 - b^3\right)a^3\right]\alpha + a^3 b^3.$$

Consequently, the membership function for fuzzy number $\overline{A} \cdot \overline{B}$ is

$$\overline{A \cdot B}\,(x) = \begin{cases} 0, & x \in (-\infty, a^1 b^1] \\ -p^2 + \sqrt{m^2 + n^2 x}, & x \in [a^1 b^1, a^2 b^2] \\ q^2 - \sqrt{t^2 + s^2 x}, & x \in [a^2 b^2, a^3 b^3] \\ 0, & x \in [a^3 b^3, +\infty) \end{cases}$$

where $p, m, n, q, t, s \in R$ and could be computed starting from the parameters of triangular fuzzy numbers $\overline{A}$ and $\overline{B}$. Moreover, $\overline{A \cdot B}\,(x)$ increases on $[a^1 b^1, a^2 b^2]$ and decreases on $[a^2 b^2, a^3 b^3]$. In Figure 2 it is shown a possible approximate version of function $\overline{AB}\,(x)$ by a function which describe a triangular fuzzy numbers.

Considering again the $\alpha-$cuts of fuzzy numbers $\overline{A}$ and $\overline{B}$ with non-negative endpoints and making a division of them, an interval

$$\left[\frac{(a^2 - a^1)\,\alpha + a^1}{(b^2 - b^3)\,\alpha + b^3}, \frac{(a^2 - a^3)\,\alpha + a^3}{(b^2 - b^1)\,\alpha + b^1}\right]$$

is obtained. Consequently, the membership function for fuzzy number $\dfrac{\overline{A}}{\overline{B}}$ is

$$\overline{\frac{\overline{A}}{\overline{B}}}\,(x) = \begin{cases} 0, & x \in \left(-\infty, \dfrac{a^1}{b^3}\right] \\ \dfrac{x - l^2}{m^2 x + n^2}, & x \in \left[\dfrac{a^1}{b^3}, \dfrac{a^2}{b^2}\right] \\ \dfrac{-x + q^2}{t^2 x + s^2}, & x \in \left[\dfrac{a^2}{b^2}, \dfrac{a^3}{b^1}\right] \\ 0, & x \in \left[\dfrac{a^3}{b^1}, +\infty\right) \end{cases}$$

where $p, m, n, q, t, s \in R$ and could be computed starting from the parameters of triangular fuzzy numbers $\overline{A}$ and $\overline{B}$. Moreover, $\overline{\dfrac{\overline{A}}{\overline{B}}}\,(x)$ increases on $\left[\dfrac{a^1}{b^3}, \dfrac{a^2}{b^2}\right]$ and decreases on $\left[\dfrac{a^2}{b^2}, \dfrac{a^3}{b^1}\right]$. An approximate version of function $\overline{\dfrac{\overline{A}}{\overline{B}}}\,(x)$ by a function which describe a triangular fuzzy numbers is also possible in this case.

Taking into consideration these approximate versions for Zadeh's principle, we work in further with Definition 3 in order to obtain a triangular fuzzy number as a result for a multiplication or a division of two triangular fuzzy numbers.

Definition 3.*Being given two triangular fuzzy numbers* $\overline{A} = (a^1, a^2, a^3)$, $\overline{B} = (b^1, b^2, b^3)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, *we have:*

(i) $\overline{A} \cdot \overline{B} = (a^1 b^1, a^2 b^2, a^3 b^3)$,

(ii) $\dfrac{\overline{A}}{\overline{B}} = \left(\dfrac{a^1}{b^3}, \dfrac{a^2}{b^2}, \dfrac{a^3}{b^1} \right)$.

3. The solving method

After aggregating the fuzzy quantities according to Definition 2 and Definition 3 we transform the problem of maximizing a fuzzy number under some constraints into a deterministic unconstraint programming by penalizing the objective for possible constraint violation.

$$\max \left(\overline{Z} = \frac{\sum\limits_{j=1}^{n} \overline{C_j} X_j + \overline{C_0}}{\sum\limits_{j=1}^{n} \overline{D_j} X_j + \overline{D_0}} - \overline{h} \max \left\{ 0, \sum_{j=1}^{n} \overline{A_{ij}} X_j - \overline{B_i} \right\} \right) \tag{3}$$

Considering that $\overline{h}$ is an m-dimensional vector with triangular fuzzy numbers as components, the function to be optimized in (3) could be approximated by a triangular fuzzy number having parameters (z^1, z^2, z^3) where

(i) $z^1 = \dfrac{\sum\limits_{j=1}^{n} c_j^1 x_j + c_0^1}{\sum\limits_{j=1}^{n} d_i^3 x_j + d_0^3} - \sum\limits_{i=1}^{m} h_i^3 \max \left\{ 0, \sum\limits_{j=1}^{n} a_{ij}^3 x_j - b_i^1 \right\}$,

(ii) $z^2 = \dfrac{\sum\limits_{j=1}^{n} c_j^2 x_j + c_0^2}{\sum\limits_{j=1}^{n} d_j^2 x_j + d_0^2} - \sum\limits_{i=1}^{m} h_i^2 \max \left\{ 0, \sum\limits_{j=1}^{n} a_{ij}^2 x_j - b_i^2 \right\}$,

$$\text{(iii)}\ \ z^3 = \frac{\sum\limits_{j=1}^{n} c_j^3 x_j + c_0^3}{\sum\limits_{j=1}^{n} d_j^1 x_j + d_0^1} - \sum_{i=1}^{m} h_i^1 \max\left\{0, \sum_{j=1}^{n} a_{ij}^1 x_j - b_i^3\right\}.$$

Two kinds of approaches could be taken into consideration here. We can optimize the midpoint of (z^1, z^2, z^3) (Jamison and Lodwick did it for a linear problem) using results of possibility theory or, we can solve the multidimensional unconstraint problem $(\max z^1, \max z^2, \max z^3)$(Buckley and Feuring suggested it for a fuzzy number).

In the first case Problem (3) is equivalent with maximization of function

$$\frac{(c^1 + 2c^2 + c^3)\, x - h^1\,(0, A^1 x - b^3) - 2h^2\,(0, A^2 x - b^2) - h^3\,(0, A^3 x - b^1)}{4}$$

representing

$$\frac{1}{2}\int_0^1 \left(\alpha z^2 + (1-\alpha)\, z^1 + \alpha z^2 + (1-\alpha)\, z^3\right) d\alpha = \frac{z^1 + 2z^2 + z^3}{4}.$$

Consequently, Problem (1)-(2) is reduced to an unconstraint deterministic programming problem which could be solved using classical methods ([5]).

4. Computational results

In order to illustrate our solving method let us consider the following deterministic linear fractional program

$$\max\left(z = \frac{2x_1 + x_2 + 1}{x_1 + x_2 + 5}\right) \tag{4}$$

subject to

$$\left\{ \begin{array}{c} x_1 + \frac{8}{3}x_2 \leq 4 \\ x_1 + x_2 \leq 2 \\ 2x_1 \leq 3 \\ x_1, x_2 \geq 0 \end{array} \right. . \tag{5}$$

The optimal solution of this problem is $x^1 = 1.5$, $x^2 = 0.5$ and the optimal value of z is 0.643.

We attach now to this problem a new problem considering its real number coefficient c as being symmetric triangular fuzzy number $\overline{c}$ of spread 1, having the following form

$$\overline{c} = \left(c^1, c^2, c^3\right), \; c^1 = c - 0.5, \; c^2 = c, \; c^3 = c + 0.5.$$

Thus the problem with triangular fuzzy numbers as coefficients to be solved is

$$\max\left(z = \frac{\overline{c_1}x_1 + \overline{c_2}x_2 + \overline{c_0}}{\overline{d_1}x_1 + \overline{d_2}x_2 + \overline{d_0}}\right) \tag{6}$$

subject to

$$\begin{cases} \overline{a_{11}}x_1 + \overline{a_{12}}x_2 - \overline{b_1} \leq \overline{0} \\ \overline{a_{21}}x_1 + \overline{a_{22}}x_2 - \overline{b_2} \leq \overline{0}, \\ \overline{a_{31}}x_1 + \overline{a_{32}}x_2 - \overline{b_3} \leq \overline{0}, \\ x_1, x_2 \geq 0 \end{cases} \tag{7}$$

where the coefficients' values are

$$\overline{a} = \begin{bmatrix} (0.5, 1, 1.5) & (2.166, 2.666, 3.166) \\ (0.5, 1, 1.5) & (0.5, 1, 1.5) \\ (1.5, 2, 2.5) & (-0.5, 0, 0.5) \end{bmatrix}, \overline{b} = \begin{bmatrix} (3.5, 4, 4.5) \\ (1.5, 2, 2.5) \\ (2.5, 3, 3.5) \end{bmatrix},$$

$$\overline{c} = \begin{bmatrix} (1.5, 2, 2.5) & (0.5, 1, 1.5) & (0.5, 1, 1.5) \end{bmatrix},$$

$$\overline{d} = \begin{bmatrix} (0.5, 1, 1.5) & (0.5, 1, 1.5) & (4.5, 5, 5.5) \end{bmatrix}.$$

According to the method described in Section 4, we choose $h = [3, 2, 3]$. In order to obtain the solution of this problem we have to analyse the triangular fuzzy number (z^1, z^2, z^3) where

(i) $z^1 = \dfrac{1.5x^1 + 0.5x^2 + 0.5}{1.5x^1 + 1.5x^2 + 5.5} - 3.5\max\{0, 1.5x^1 + 3.166x^2 - 3.5\} -$
$- 2.5\max\{0, 1.5x^1 + 1.5x^2 - 1.5\} - 3.5\max\{0, 2.5x^1 + 0.5x^2 - 2.5\},$

(ii) $z^2 = \dfrac{2x^1 + 1x^2 + 1}{x^1 + x^2 + 5} - 3\max\{0, x^1 + 2.666x^2 - 4\} -$
$- 2\max\{0, x^1 + x^2 - 2\} - 3\max\{0, 2x^1 - 3\},$

(iii) $z^3 = \dfrac{2.5x^1 + 1.5x^2 + 1.5}{0.5x^1 + 0.5x^2 + 4.5} - 2.5\max\{0, 0.5x^1 + 2.166x^2 - 4.5\} -$
$- 1.5\max\{0, 0.5x^1 + 0.5x^2 - 2.5\} - 2.5\max\{0, 1.5x^1 - 0.5x^2 - 3.5\}.$

We obtain the midpoint $(0.28, 0.5, 0.8)$ for $x^1 = 1$ and $x^2 = 0$ when $\frac{z^1 + 2z^2 + z^3}{4}$ is maximized.

The optimal values $z^1\,(1,0) = 1.5$, $z^2\,(1.5, 0.5) = 3.5$ and $z^3\,(2.8, 1.43) = 9.16$ are obtained when the components (z^1, z^2, z^3) are independently maximized.

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