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AN ADDITIVE DIVISOR PROBLEM

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Abstract. In this work we give an asymptotic formula for the number of solutions of equation $x_1x_2 + x_3x_4 = N,\ x_1x_2 \equiv l_1 \,(\mathrm{mod}\ r_1),\ x_3x_4 \equiv l_2\,, r_1 = r_1(N),\ r_2 = r_2(N)$.

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1. First Section

In the modern Number theory, the problems, concerning representation of natural numbers as a sum of two summands of a special type, are of interest.

In 1927 A. Ingham, applying standard technique, obtained the asymptotic formulae for the number of solutions $\nu(N)$ to the following diophantine equation

$$x_1x_2 + x_3x_4 = N,$$

namely when $N \to \infty$

$$\nu(N) = \frac{6}{\pi^2}\sigma(N)\ln^2 N\left(1 + O\left(\frac{\ln\ln N}{\ln N}\right)\right),$$

$$\sigma(N) = \sum_{d/N} d.$$

In 1930 T. Estermann using the circle method obtained asymptotic estimation with degree reduction to the reminder term $R(N)$

$$\nu(N) = N\sum_{r=0}^{2}\ln^r N\sum_{j=0}^{2-r} a_{rj}\sigma_{-1}^{j}(N)\,,$$

$$\sigma_{-1}^{j}(N) = \sum_{d/N} d^{-1}\ln^j d\,,$$

$$a_{rj} = \frac{(-2)^j}{r!j!}\sum_{\substack{l,h,q>0\\ l+h+q=2-r-j}}\frac{2\gamma^l 2^h}{l!h!q!}\left(\frac{1}{\Gamma(s)}\right)^{(q)}\left(\frac{1}{\zeta(s)}\right)^{(h)}_{|s=2},$$

where γ is the Euler constant and

$$R(N) \ll N^{\frac{7}{8}} \ln^{\frac{23}{4}} N\sigma_{-\frac{3}{4}}(N)\,, \qquad \sigma_{-\frac{3}{4}}(N) = \sum_{d/N} d^{-\frac{3}{4}}.$$

The best known estimation of $R(N)$ is the Y. Motohashi estimate (1994). He evaluated sums of sums of Kloosterman and derived

$$R(N) \ll N^{\frac{7}{10}+\varepsilon}.$$

In the present paper we consider the diophantine equation

$$\begin{aligned} &x_1x_2 + x_3x_4 = N,\\ &x_1x_2 \equiv l_1 \,(\mathrm{mod}\; r_1),\; x_3x_4 \equiv l_2 \,(\mathrm{mod}\; r_2),\\ &(l_1,\, r_1) = 1,\; (l_1,\, r_1) = 1,\; (r_1,\, r_2) = 1. \end{aligned} \tag{1}$$

The following result is obtained:

$$\nu(N,\, l_1,\, r_1,\, l_2,\, r_2) = G(N,\, l_1,\, r_1,\, l_2,\, r_2) + O\left(N^{\frac{3}{4}+\varepsilon}(r_1r_2)^{\frac{1}{2}}\right),$$

where $\nu(N,\, l_1,\, r_1,\, l_2,\, r_2)$ is number of solutions of diophantine equation (1) and

$$\frac{N\ln^2 N \ln r_1 \ln r_2}{r_1r_2} \le G(N,\, l_1,\, r_1,\, l_2,\, r_2) \ll \frac{N\ln^2 N \ln r_1 \ln r_2}{r_1r_2},$$
$$N^{\frac{1}{8}-\varepsilon} \ll r_1r_2 \ll N^{\frac{1}{6}-\varepsilon}.$$

The considered problems is a generalization of Ingham-Estermann problem.

2.The second section

To solve the stated problem we perform the following:

1. The number of solutions of equation (1) is given by the following sum

$$\Gamma = \sum_{\substack{n\equiv l_1 (mod\, r_1),\\ m\equiv l_2 (mod\, r_2)\\ n+m=N}} \tau(n)\tau(m)\,.$$

The above sum we represent as:

$$\Gamma = \sum_{\substack{n\equiv l_1 (mod r_1),\\ m\equiv l_2 (mod r_2),\\ n+m=N}} \tau(n)\tau(m)\omega\left(\frac{n}{N}\right)\omega\left(\frac{m}{N}\right)\omega_1\left(\frac{n+m}{N}\right)\Delta(n+m-N)+O(\delta N^{1+\varepsilon})\,,$$

where

$$\omega : [0, \infty) \to [0, 1],$$
$$\omega(u) = 1 \text{ for } u \in [\delta, 1],\, \delta \in (0, 1),$$
$$0 \le \omega(u) \le 1 \text{ for } u \in [0, \delta] \text{ and } u \in [1, \infty);$$
$$\omega_1 : [0, 2] \to [0, 1],$$
$$\omega_1(u) = 1 \text{ for } u \in [1-\delta, 1+\delta],$$
$$0 \le \omega_1(u) \le 1 \text{ for } u \in [0, 1-\delta] \text{ and } u \in [1+\delta, 2].$$

According to Heath-Brown (1996) we have that

$$\Delta(n) = C_Q . Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a (mod\, q)} e_q(an) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right),$$

where $Q > 1$ is a positive number, $C_Q = 1 + O_M(Q^{-M})$, $M \in \mathbb{N}$ is arbitrary, $e_q(a) = e^{\frac{2\pi i a}{q}}$, $h(x, y)$ is infinitely differentiable function defined on the set $(0, \infty) \times \mathbb{R}$ and the sign $*$ means that $(a, q) = 1$. We set $Q = N^{\frac{1}{2}}$ and $\delta = Q^{-\frac{1}{2}}$.

2. The Mellin and Fourier transforms to the function $\omega(u_1)\omega(u_2)\omega_1(u_1 + u_2)h\left(\frac{q}{Q}, u_1 + u_2 - 1\right)$ and $\omega_1(v+1)h\left(\frac{q}{Q}, v\right)$ yield

$$\begin{aligned}\Gamma = \frac{1}{N} \sum_{1 \le q \le 3Q} \sideset{}{^*}\sum_{1 \le a \le q} e_q(-aN) \times \\ \int_{\mathbb{R}} P_{\frac{q}{Q}}(t) e^{-2\pi i t} \left(\frac{1}{2\pi i} \int_{(c)} N^s \left(\sum_{n \equiv l_1 (mod\, r_1)} \frac{\tau(n) e_q(an)}{n^s} \right) J(s, t)\, ds \right) \times \\ \left(\frac{1}{2\pi i} \int_{(c)} N^s \left(\sum_{m \equiv l_2 (mod\, r_2)} \frac{\tau(m) e_q(am)}{n^s} \right) J(s, t)\, ds \right) dt + O\left(N^{\frac{3}{4}+\varepsilon}\right), \quad (2)\end{aligned}$$

$$P_{\frac{q}{Q}} = \int_{\mathbb{R}} \omega_1(v+1) h\left(\frac{q}{Q}, v\right) e^{2\pi i t v}\, dv, \quad J(s, t) = \int_0^{\infty} u^{s-1} \omega(u) e^{-2\pi i t u}\, du.$$

After some processing we obtain

$$\sum_{n\equiv l_i (mod\, r_i)} \frac{\tau(n)e_q(an)}{n^s} = \frac{N}{[q,\, r_i]^2} \sum_{\substack{1\le\alpha_i,\, \beta_i\le[q,\, r],\\ \alpha_i\beta_i\equiv l_i (mod\, r_i)}} e_q(a\alpha_i\beta_i)\zeta\left(s,\, \frac{\alpha_i}{[q,\, r_i]}\right)\zeta\left(s,\, \frac{\beta_i}{[q,\, r_i]}\right),$$

$$i = 1, 2,$$

where $\zeta(s,\, \theta) = \sum\limits_{n=0}^{\infty} \frac{1}{(n+\theta)^s}$ is the Hurwitz zeta function.

3. We apply to every inner integral of (2) the Cauchy residual theorem:

$$\int\limits_{(c)} \left(\frac{N}{[q,\, r_i]^2}\right)^s \zeta\left(s,\, \frac{\alpha_i}{[q,\, r_i]}\right)\zeta\left(s,\, \frac{\beta_i}{[q,\, r_i]}\right) J(s,\, t)\, ds = R(r_i,\, \alpha_i,\, \beta_i) + I(r_i,\, \alpha_i,\, \beta_i)$$

where

$$R(r_i,\, \alpha_i,\, \beta_I) = Res_{s=1}\left(\left(\frac{N}{[q,\, r_i]^2}\right)^s \zeta\left(s,\, \frac{\alpha_i}{[q,\, r_i]}\right)\zeta\left(s,\, \frac{\beta_i}{[q,\, r_i]}\right) J(s,\, t)\right),$$

$$I(r_i,\, \alpha_i,\, \beta_i) = \frac{1}{2\pi i}\int\limits_{(\sigma)} \left(\frac{N}{[q,\, r_i]^2}\right)^s \zeta\left(s,\, \frac{\alpha_i}{[q,\, r_i]}\right)\zeta\left(s,\, \frac{\beta_i}{[q,\, r_i]}\right) J(s,\, t)\, ds,$$

$$\sigma < 0,\ i = 1, 2.$$

The next representation of Γ is valid:

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4\,, \tag{3}$$

where

$$\Gamma_1 = \frac{1}{N} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{1\le a\le q} e_q(-aN) \sum_{\substack{1\le\alpha_1,\, \beta_1\le[q,\, r_1]\\ \alpha_1\beta_1\equiv l_1 (mod\, r_1)}} e_q(a\alpha_1\beta_1) \sum_{\substack{1\le\alpha_2,\, \beta_2\le[q,\, r_2]\\ \alpha_2\beta_2\equiv l_2 (mod\, r_2)}} e_q(a\alpha_2\beta_2)\times$$

$$\int\limits_{\mathbb{R}} P_{\frac{q}{Q}} e^{2\pi it}\Big(R(r_1,\, \alpha_1,\, \beta_1)R(r_2,\, \alpha_2,\, \beta_2)\Big)\, dt\,,$$

$$\Gamma_2 = \frac{1}{N} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{1\le a\le q} e_q(-aN) \sum_{\substack{1\le\alpha_1,\, \beta_1\le[q,\, r_1]\\ \alpha_1\beta_1\equiv l_1 (mod\, r_1)}} e_q(a\alpha_1\beta_1) \sum_{\substack{1\le\alpha_2,\, \beta_2\le[q,\, r_2]\\ \alpha_2\beta_2\equiv l_2 (mod\, r_2)}} e_q(a\alpha_2\beta_2)\times$$

$$\int\limits_{\mathbb{R}} P_{\frac{q}{Q}} e^{2\pi it}\Big(R(r_1,\, \alpha_1,\, \beta_1)I(r_2,\, \alpha_2,\, \beta_2)\Big)\, dt\,,$$

$$\Gamma_3 = \frac{1}{N} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{1\le a\le q} e_q(-aN) \sum_{\substack{1\le\alpha_1,\,\beta_1\le[q,\,r_1]\\ \alpha_1\beta_1\equiv l_1(mod\, r_1)}} e_q(a\alpha_1\beta_1) \sum_{\substack{1\le\alpha_2,\,\beta_2\le[q,\,r_2]\\ \alpha_2\beta_2\equiv l_2(mod\, r_2)}} e_q(a\alpha_2\beta_2)\times$$
$$\int\limits_{\mathbb{R}} P_{\frac{q}{Q}} e^{2\pi it}\Big(R(r_2,\,\alpha_2,\,\beta_2)I(r_1,\,\alpha_1,\,\beta_1)\Big)\,dt\,,$$

$$\Gamma_4 = \frac{1}{N} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{1\le a\le q} e_q(-aN) \sum_{\substack{1\le\alpha_1,\,\beta_1\le[q,\,r_1]\\ \alpha_1\beta_1\equiv l_1(mod\, r_1)}} e_q(a\alpha_1\beta_1) \sum_{\substack{1\le\alpha_2,\,\beta_2\le[q,\,r_2]\\ \alpha_2\beta_2\equiv l_2(mod\, r_2)}} e_q(a\alpha_2\beta_2)\times$$
$$\int\limits_{\mathbb{R}} P_{\frac{q}{Q}} e^{2\pi it}\Big(I(r_1,\,\alpha_1,\,\beta_1)I(r_2,\,\alpha_2,\,\beta_2)\Big)\,dt\,.$$

The summand Γ_1 give the main term of the sum (3) and Γ_4 give the reminder term.

4. We express the main term as

$$G(N,\,l_1,\,r_1,\,l_2,\,r_2) = \frac{N}{r_1^2r_2^2}\left(r_1\sum_{d/r_1}\frac{\mu(d)}{d}\ln\frac{r_1}{d}+\left(\frac{3}{2}\ln 2+\gamma-2\right)\varphi(r_1)\right)\times$$
$$\left(r_2\sum_{d/r_2}\frac{\mu(d)}{d}\ln\frac{r_2}{d}+\left(\frac{3}{2}\ln 2+\gamma-2\right)\varphi(r_2)\right)\times$$
$$\sum_{d_1/r_1}\frac{1}{d_1^2}\frac{\varphi(d_1)}{\varphi\left(\frac{d_1}{(N-l_1,\,d_1)}\right)}\mu\left(\frac{d_1}{(N-l_1,\,d_1)}\right)\sum_{d_2/r_2}\frac{1}{d_2^2}\frac{\varphi(d_1)}{\varphi\left(\frac{d_2}{(N-l_2,\,d_2)}\right)}\mu\left(\frac{d_2}{(N-l_2,\,d_2)}\right)\times$$
$$\sum_{d\le\frac{Q}{d_1d_2}}\frac{1}{d^4}\frac{\varphi(d)}{\varphi\left(\frac{d}{(N,\,d)}\right)}\mu\left(\frac{d_1}{(N,\,d)}\right)\left(A\left(\ln\frac{N}{[q,\,r_1]^2}\ln\frac{N}{[q,\,r_2]^2}\right)+\right.$$
$$\left.+\,2\ln\frac{N}{[q,\,r_1][q,\,r_2]}+B\ln\frac{N}{[q,\,r_1][q,\,r_2]}+C\right),$$

where A, B, C are absolute constants. For the $G(N,\,l_1,\,r_1,\,l_2,\,r_2)$ we have that

$$\frac{N\ln^2 N\ln r_1\ln r_2}{r_1r_2}\le G(N,\,l_1,\,r_1,\,l_2,\,r_2)\ll\frac{N\ln^2 N\ln r_1\ln r_2}{r_1r_2}$$

5. We estimate the sum Γ_4 as follows:

   For $Re\, s = \sigma < 0$ the next formulae is valid:

$$\zeta\left(s,\, \frac{b}{c}\right) = -i\Gamma(1-s)(2\pi)^{s-1} \sum_{\mu=\pm 1} \mu \sum_{m=1}^{\infty} \frac{e_4(\mu s) e_c(\mu m b)}{m^{1-s}}\,, \quad b, c \in \mathbb{N}.$$

   Substituting the above expression for ζ in Γ_4 we have:

$$\Gamma_4 = \frac{1}{(4\pi^2)^2} \frac{1}{N} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{1\le a\le q} e_q(-aN) \sum_{\mu_1,\, \nu_1,\, \mu_2,\, \nu_2 = \pm 1} \mu_1\nu_1\mu_2\nu_2 \sum_{m_1,\, k_1,\, m_2,\, k_2=1}^{\infty} \times$$
$$\sum_{\substack{1\le \alpha_1,\, \beta_1 \le [q,\, r_1],\\ \alpha_1\beta_1 \equiv l_1 (mod\, r_1)}} e\left(\frac{a\alpha_1\beta_1}{q} + \frac{\mu_1 m_1 \alpha_1 + \nu_1 k_1 \beta_1}{[q,\, r_1]}\right) \times$$
$$\sum_{\substack{1\le \alpha_2,\, \beta_2 \le [q,\, r_2],\\ \alpha_2\beta_2 \equiv l_2 (mod\, r_2)}} e\left(\frac{a\alpha_2\beta_2}{q} + \frac{\mu_2 m_2 \alpha_2 + \nu_2 k_2 \beta_2}{[q,\, r_2]}\right) \times$$
$$\int\limits_{\mathbb{R}} P_{\frac{q}{Q}}(t) e^{2\pi i t} \frac{1}{2\pi i} \int\limits_{(\sigma)} \left(\frac{4\pi^2 N}{[q,\, r_1]^2}\right)^s \Gamma(1-s)^2 J(s,\, t) \frac{e_4((\mu_1+\nu_1)s)}{(m_1 k_1)^{1-s}}\, ds \times$$
$$\int\limits_{(\sigma)} \left(\frac{4\pi^2 N}{[q,\, r_2]^2}\right)^s \Gamma(1-s)^2 J(s,\, t) \frac{e_4((\mu_2+\nu_2)s)}{(m_2 k_2)^{1-s}}\, ds\, dt.$$

   The sum

$$S_i = \sum_{\substack{1\le \alpha_i,\, \beta_i \le [q,\, r_i],\\ \alpha_i\beta_i \equiv l_i (mod\, r_i)}} e\left(\frac{a\alpha_i\beta_i}{q} + \frac{\mu_i m_i \alpha_1 + \nu_i k_i \beta_1}{[q,\, r_i]}\right)$$

   is represented as $S_i = \Phi_1^i \Phi_2^i \Phi_3^i$, $\;i = 1, 2$, where

$$\Phi_1^i = \sum_{1\le x_1,\, y_1 \le \tilde{q}_i} e\left(\frac{a x_1 y_1 (\tilde{r}_i \Delta_i)^2}{\tilde{q}_i q_i'} + \frac{\mu_i m_i x_1 + \nu_i k_i y_1}{\tilde{q}_i}\right)$$

$$\tilde{q_i} = \prod_{\substack{p^j/q, \\ p \text{ not divide } r_i}} p^j, \quad \tilde{r_i} = \prod_{\substack{p^j/r_i, \\ p \text{ not divide } q}} p^j, \quad q_i' = \frac{q}{\tilde{q_i}}, \quad r_i' = \frac{r_i}{\tilde{r_i}}, \quad \Delta_i = [q_i', r_i'],$$

$$\Phi_2^i = \sum_{\substack{1 \le x_2, \, y_2 \le \tilde{r_i} \\ x_2 y_2 (\tilde{q_i}\Delta_i)^2 \equiv l_i (mod \, \tilde{r_i})}} e\left(\frac{\mu - i m_i x_2 + \nu_i k_i y_2}{\tilde{r_i}} \right),$$

$$\Phi_3^i = \sum_{\substack{1 \le x_3, \, y_3 \le \Delta_i, \\ x_3 y_3 (\tilde{r_i}\tilde{q_i})^2 \equiv l_i (mod \, r_i')}} e\left(\frac{a x_3 y_3 (\tilde{q_i}\tilde{r_i})^2}{q} + \frac{\mu_i m_i x_3 + \nu_i k_i y_3}{\Delta_i} \right).$$

For Φ_1^i, Φ_2^i and Φ_3^i we obtain the representations:

$$\Phi_1^i = \tilde{q_i} e\left(- \frac{\mu_i \nu_i m_i k_i \bar{a}_q \overline{\left(\frac{(\tilde{r_i}\Delta_i)^2}{q_i'} \right)}_{\tilde{q_i}}}{\tilde{q_i}} \right),$$

where $\bar{h}_s$ is the natural number such that $h\bar{h}_s \equiv 1 (mod \, s)$;

$$\Phi_2^i = K\left(\tilde{r_i}, \, m_i, \, k_i l_i \overline{(\tilde{r_i}\tilde{q_i})^2}_{\tilde{r_i}} \right),$$

where $K(c, \, a, \, b) = \sum_{1 \le n \le c}^{*} e_n \left(\frac{an + b\bar{n}}{c} \right)$ is the Kloosterman sum;

$$\Phi_3^i = e_q(a\Delta_i) K\left(\Delta_i, \, m_i, \, k_i l_i \overline{(\tilde{r_i}\tilde{q_i})^2}_{\Delta_i} \right).$$

For the integral

$$\int_{\mathbb{R}} P_{\frac{q}{Q}}(t) e^{2\pi i t} \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{4\pi^2 N}{[q, \, r_1]^2} \right)^s \Gamma(1-s)^2 J(s, \, t) \frac{e_4((\mu_1 + \nu_1)s)}{(m_1 k_1)^{1-s}} \, ds \times$$
$$\frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{4\pi^2 N}{[q, \, r_2]^2} \right)^s \Gamma(1-s)^2 J(s, \, t) \frac{e_4((\mu_2 + \nu_2)s)}{(m_2 k_2)^{1-s}} \, ds \, dt$$

we derive the estimation $O\left(\frac{Q^{3-4\sigma+\varepsilon}}{q^{3-4\sigma+\varepsilon}} \right)$. Hence

$$\Gamma_4 \ll N^{\frac{3}{4}+\varepsilon} (r_1 r_2)^{\frac{1}{2}}$$

and $N^{\frac{1}{8}-\varepsilon} \ll r_1 r_2 \ll N^{\frac{1}{6}-\varepsilon}$.

6. The sums Γ_2 and Γ_3 can be estimated similarly to Γ_4.

   Finally, according to this algoritm we obtain the desired result.

References

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