SOME REMARKS ON THE STABILITY PROBLEMS FOR DE RHAM CURRENTS

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Abstract. This paper contains two paragraphs. The first one presents the De Rham p-currents stability problem as an extension for the idea of stability problem for differentiable C^∞ forms. The second paragraph contains the definition and some properties for the notion of infinitesimal stability problem for currents.

1. The global stability problem for currents

For the beginning we will remind some results from stability problem for differentiable C^∞ forms. Let M be a smooth, n-dimensional, orientable manifold.

Definition 1. *A differentiable C^∞ p-form ω on M, is says stable if there exists a neighborhood V_ω of ω in C^∞ p-forms topology such as for every $\theta \in V_\omega$ it exists $f : M \longrightarrow M$ continuous diffeomorfism with*:

$$f^*(\theta) = \omega$$

Definition 2. *It says that two p-forms ω and θ are equivalents if it exists continuous diffeomorfism $f : M \longrightarrow M$ with:*

$$f^*(\theta) = \omega$$

Remark. *For a compact M it was proven the nonexistence of global stability for p-forms.*

These stability concepts for the differentiable C^∞ forms can be expanded for the currents space.

Definition 3. *We will say that $T \in D^{p\prime}(M)$ is stable if there exists V_T in $D^{p\prime}(M)$ such as for every $S \in V_T$ it exists $f : M \longrightarrow M$ continuous diffeomorfism with*:

$$f(S) = T$$

Even if the De Rham p-currents space is the dual of $(n-p)$-forms space, we will specify that the techniques for proving the stability properties of p-currents are totally different. So we can prove:

Theorem 1. *There are not stable De Rham currents on M.*

Proof. We will suppose that $T \in D^{p\prime}(M)$ is a stable De Rham p-current. Then it exists a neighborhood V_T of T as in definition 3.

Case 1.

If T is continuous we have $T = T_\omega$, and we'll consider a Riemannian metric on M. The linear measure of Dirac p-current is dense on $D^{p\prime}(M)$, so we choose:

$$S_1 = \sum_{i=1}^{k} c_i \delta_{\alpha_i}$$

From the hypothesis we have $f \in Diff(M)$ such as $f(S_1) = T_\omega \Longrightarrow S_1 = T_{f^*(\omega)}$, which is not possible. So, our supposition is false.

Case 2.

We will suppose that T is not continuous. Using the density of smooth p-currents on $D^{p\prime}(M)$, we can choose a smooth p-current $S_\omega \in V_T$ such as $h(S_\omega) = T$, where $h \in Diff(M)$. So $T = S_{(h^{-1})^*\omega}$.

We have again a contradiction, which is proven our theorem.

Corollary 1. *If M is compact, using the Hodge-De Rham factorization theorem, we'll obtain the same result for closed currents.*

Proof. We will suppose that $T \in Z^{p\prime}(M)$ is closed and stable current (we denote by $Z^{p\prime}(M)$ the set of closed currents).

Then it exists a neighborhood V_T of T in $Z^{p\prime}(M)$ as in definition 3, and let $S_\omega \in V_T$, where ω is a C^∞ p-form, closed in M.

If T is continuous, i.d $T = T_\omega$, then it exists $f \in Diff(M)$ such as $f(S_\omega) = T_\omega$. It result that $(f^{-1})^* \omega = \omega$. But this is in contradiction with the non existence of global stability for closed p-forms. So, our supposition is false.

Every time that the continuity hypothesis is missing, the density property allow us to apply a similar technique, as in the proof of theorem 1, for obtaining again a contradiction.

Proposition 1. *For M a compact Riemannian manifold, not all of De Rham currents from the same cohomology class are equivalents.*

Proof. We will suppose the reverse, and let S be a De Rham current.

Let ω a unique form corresponding to the cohomology class of S. If S and T_ω are in the same cohomology class, from hypothesis it exists $h \in Diff(M)$ such as $h(S) = T_\omega$, i.e. $S = T_{h^*(\omega)}$, which is absurd.

Remark. *Let ω a differentiable C^∞ n-form. For M a compact Riemannian manifold, the De Rham n-current $T = T_\omega$ is not stable in its cohomology class.*

Indeed, supposing the reverse, let V_T a neighborhood of T as in definition 3, and $S_1 \in V_T$ as in the proof of theorem 1. Because the n-dimensional real cohomology group is isomorphic with R, it results that S_1 is in the same cohomology class as T_ω, except a multiplication by a scalar, so there exists $f \in Diff(M)$ and $h \in M$ such as $f(S_1) = T_{h\omega}$, that means $S_1 = T_{f^*(h\omega)}$, which is not possible.

However, we can obtain some positive results about the stability of some properties for the currents. For example, if M is compact, $T \in D^{p\prime}(M)$, then the condition $\int_M T \neq 0$ is stable. Analogous, if $T \in D^{n-1\prime}(M)$ it results that the condition $dT \neq 0$ is stable.

2.The infinitesimal stability problem for currents

Appropriate with the stability idea, we have the idea of infinitesimal stability, which for a differentiable C^∞ p-form ω on M, is defined as bellow:

Definition 4. *We call ω infinitesimal stable if there exists the application*:

$$\aleph(M) \ni X \longrightarrow L_X(\omega) \in A^p(M)$$

where $\aleph(M)$ is the real vectors space of the continuous vector fields, $A^p(M)$ is the real vectors space of the differentiable p-form on M and $L_X(\omega)$ is Lie derivative for ω respecting to X.

Hsiung, in 1973, proved the non existence for infinitesimal stable differentiable C p-form.

Definition 5. *We call $T \in D^{p\prime}(M)$ infinitesimal stable if there exists the application*:

$$\aleph\left(M\right) \ni X \longrightarrow L_X\left(T\right) \in D^{p\prime}\left(M\right).$$

Proposition 2. *The smooth currents are not infinitesimal stable.*

Proof. Indeed, supposing that $T_{\ \omega} \in D^{p\prime}(M)$ is infinitesimal stable. Then for Dirac p-current δ_{α,x_0}, exists $X \in \aleph\left(M\right)$ with $L_X\left(T\right) = \delta_{\alpha,x_0} \Longleftrightarrow T_{(-1)^{n+1}}L_X\left(\omega\right) = \delta_{\alpha,x_0}$, contradiction. So the supposition is false.

The next definitions help us to say more about the infinitesimal stability problem for De Rham currents:

Definition 6. *For U an open set, we will say that the p-current T is null on U if for every $\rho \in D^{n-p}(M)$ with Supp$(\rho) \subseteq U$, we have$T\left(\rho\right) = 0$.*

Definition 7. *Supp(T) represents the complement of the largest open set from M where $T(\rho) = 0$.*

Theorem 2. *The currents with a compact support are not infinitesimal stable.*

Proof. We will suppose the reverse. Let $T_{\ \omega} \in D^{p\prime}(M)$ with Supp$(T) = K$-compact and T infinitesimal stable.

Let U an open neighborhood for K and b a C^∞ real function on M with

$$\text{Supp}(b) \subset C_K \text{ the complement of } K\ ;\ b|_K = 0; b|_{C_V} = 1.$$

Then for every $\omega \in A^p\left(M\right)$ we have:

$$Supp(b) \subset C_K \tag{1}$$

Because T is infinitesimal stable, it exists $X \in \aleph\left(M\right)$ with:

$$\text{Supp}(b\omega) \subset C_K$$

This will take us to a contradiction because of (1) and Supp$(L_XT) \subset K$.

Corollary 1. *The Dirac current is not infinitesimal stable.*

Corollary 2. *For M compact, De Rham currents are not infinitesimal stable.*

The same reasoning can be used to prove that the forms defined on a compact set are not infinitesimal stable. We can conclude that De Ram currents are extremely unstable objects. When we affirm that, we referring at Definition 3 and 5 for the stability problem.

References

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