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ON SOME EXTENSIONS OF JORDAN'S ARITHMETIC FUNCTIONS

by
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Abstract. In this paper we introduce and study the arithmetic functions ,$J_k^{(1)}$ and $J_k^{(2)}$ of Jordan's type. The basic theory of Jordan's totient function J_k is reobtained by using some properties of our second function.

Keywords: Jordan totient, greatest common divisor

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1. Introduction

An arithmetic function generalizing the well-known Euler totient function φ is the Jordan's function of order *k,* where *k* is a positive integer. This function is denoted by J_k and it is defined by $J_k(n)$ = the number of all vectors (a_1,..., a_k)$\in Z_+^k$ with the properties $a_i \leq n$, i= *1,2, . . . , k* and $gcd(a_1,\ldots,a_1,n) = 1$. It is clear that J_i=φ. The early history of the function J_k is presented in [4].

The function J_k has some interesting properties and numerous applications. In what follows we recall few of them.

1.The function J_k is multiplicative, i.e. for any positive integers *m, n* with *gcd(m,n)* = 1 the relation $J_k(mn) = J_k(m)J_k(n)$ holds ([7], [8]).

2. If *p* is a primp and a is a positive integer, then

$$J_k(p^\alpha) = p^\alpha - p^{k(\alpha-)}$$

3. If the unique prime decomposition of n is $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ then

$$J_k(n) = n^k\left(1-\frac{1}{p_1^k}\right)\dots\left(1-\frac{1}{p_m^k}\right)$$

An easy argument for this formula is the inclusion-exclusion principle (see [7], [8]).

4. (Gauss' type formula) The following formula holds

$$\sum_{d/n} J_k(d) = n^k$$

(see [7] and [8]).

5. The following formula holds

$$\sum_{d/n}\frac{\mu(d)}{d^k} = \frac{J_k(n)}{n^k}$$

where *n* is the Möbius inversion function. That is for all positive integers *n*

$$J_k(n) = \sum_{d/n}\left(\frac{n}{d}\right)^k \mu(d) = \sum_{d/n} d^k \mu\left(\frac{n}{d}\right) = (\zeta_k \times \mu)(n)$$

where $\zeta_k(n) = n^k$ and "*" is the Dirichlet convolution defined by

$$(f * g)(n) = \sum_{d/n} f(d) g\left(\frac{n}{d}\right)$$

for any functions $f,g : Z_+ \to C$ ([8, pp. 12-13]).

6. Recall that the Riemann ζ function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}, \text{ Re } z > 1$$

The following formula holds

$$\frac{\zeta(z-k)}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{J_k(n)}{n^z}, \text{ Re } z > 1.$$

7. The following asymptotic formula holds ([8, pp. 265-272])

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{s=1}^{n} J_k(s) = \frac{1}{(k+1)\zeta(k+1)}$$

In the case $k=2m-1$ we get

$$\lim_{n\to\infty} \frac{1}{n^{2m}} \sum_{s=1}^{n} J_{2m-1}(s) = \frac{(2m-1)!}{2^{m-1}|B_{2m}|\pi^{2m}}$$

8. The von Sterneck function H_k is defined by

$$H_k(n) = \sum_{\substack{[s_1,\ldots,s_k]=n \\ 1\le s_1,\ldots,s_k\le n}} (s_1)\ldots\ (s_k)$$

where $[s_1,\ldots, s_k]$ denotes the latest common multiple of integers $s_1,\ldots, s_k$. For all positive integers k the following formula is true ([8. Proposition 1.7, pp. 15]):

$$J_k = H_k$$

9. The interpretation of the integer $J_k(t)$ in the theory of finite groups is the following. Consider the Abelian group defined as the cross product $Z_n^k = Z_n \times \ldots Z_n$, where $(Z_n, +)$ is the well-known group of residues modulo n. Then for $t\backslash n$ we have (see [11])

$$J_k(t) = \#\{g \in Z_n^k : ord(g) = t\}$$

10. Some interesting applications in determining the order of some matrices finite groups are given by

$$|GL(m, Z_n)| = n^{\frac{m(m-1)}{2}} \prod_{k=1}^{m} J_k(n)$$

$$|SL(m, Z_n)| = n^{\frac{m(m-1)}{2}} \prod_{k=2}^{m} J_k(n)$$

$$|Sp(2m, Z_n)| = n^{m^2} \prod_{k=1}^{m} J_{2k}(n)$$

where $GL(m,Z_n)$, $SL(m,Z_n)$, $Sp(2m, Z_n)$ are the general linear group, the special linear group and the symplectic group, respectively, of matrices of order m with elements in the ring Z_n. The first two formulas are obtained by C. Jordan [7] and they are also contained in [1]. The third formula is given in [11]. The multiplicative group *G(n)* is defined by

$$G(n) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in Z_n \text{ and } \alpha\delta - \beta\gamma = \pm 1\}$$

For any positive integer $n \geq 3$ the order of *G(n)* is given by

$$|G(n)| = 2nJ_2(n).$$

11. Other applications of the Jordan's function J_2 are given in Diophantine Analysis (see [3]). Some special properties of J_k are obtained in the paper [5], [6] and [10].

There are few generalizations of Jordan's totient function. We mention here the recent one given in [12] and defined by

$$S_m^k(n) = \sum_{\substack{1 \leq a_1, \ldots, a_m \leq n \\ \gcd(a_1, \ldots, a_m, k) = 1}} 1$$

where *m* and *k* are fixed positive integers. It is dear that. $S_k^k(n) = J_k(n)$.

In this paper we introduce two functions of Jordan' type and we make the connection with the function J_k. The basic theory for Jordan's function J_k is reobtained by using our second function.

2. The arithmetic function $J_k^{(1)}$

For a fixed positive integer *k* define $Z_+^{k+1} = \underbrace{Z_+ \times \ldots \times Z_+}_{k+1\,times}$ and consider the sets

$$M_{k+1}(n) = \{(a_1, \ldots a_{k+1}) \in Z_+^{k+1} : 1 \leq a_1 \leq \ldots \leq a_{k+1} \leq n \text{ and } \gcd(a_1, \ldots a_{k+1}, n) = 1\}$$

$$N_k(n) = \{(a_1, \ldots a_k, n) \in Z_+^{k+1} : 1 \leq a_1 \leq \ldots \leq a_{k+1} \leq n \text{ and } \gcd(a_1, \ldots a_k, n) = 1\}$$

The cardinal numbers of these finite sets are denoted by

$$F_{k+1}(n) = \# M_{k+1}(n) \text{ and } J_k^{(1)}(n) = \# N_k(n)$$

It is clear that for $k = 1$ we obtain $J_1^{(1)} = J_1 = \varphi$, the well-known Euler totient function.

The Gauss' type formula for the function $J_k^{(1)}$ is given in

Theorem 2.1. The following formula holds

$$\sum_{d/n} J_k^{(1)}(d) = \binom{n+k-1}{k} \tag{2.1}$$

Proof. First of all let us note the following relation

$$F_{k+1}(n) = F_{k+1}(n-1) + J_k^{(1)}(n) \tag{2.2}$$

Consider the set

$$S_d(n) = \{(a_1, \dots a_{k+1}) \in Z_+^{k+1} : 1 \le a_1 \le \dots \le a_{k+1} \le n \text{ and } \gcd(a_1, \dots a_{k+1}) = d\}$$

We have the relations

$$\binom{n+k}{k+1} = \sum_{d=1}^{n} \# S_d(n) = \sum_{s=1}^{n} F_{k+1}\left(\left[\frac{n}{d}\right]\right) \tag{2.3}$$

Replacing n by $n-1$ in the above relation we get

$$\binom{n+k-1}{k+1} = \sum_{s=1}^{n-1} F_{k+1}\left(\left[\frac{n-1}{s}\right]\right) \tag{2.4}$$

From (2.3) and (2.4) and then by using (2.2) it follows

$$\binom{n+k-1}{k+1} = \binom{n+k}{k+1} - \binom{n+k-1}{k+1} = \sum_{d/n} F_{k+1}\left(F_{k+1}\left(\left[\frac{n}{d}\right]\right) - F_{k+1}\left(\left[\frac{n-1}{d}\right]\right)\right)$$

$$= \sum_{d/n} \left(F_{k+1}(d) - F_{k+1}(d-1)\right) = \sum_{d/n} J_k^{(1)}(d)$$

For $k = 1$. from (2.1) we obtain the classical Gauss' formula.

Theorem 2.2. Fur any positive, integer k $\ge$ 2 the following relation is satisfied

$$F_k(n) = \sum_{m=2} J_{k-1}^{(1)}(m) \tag{2.5}$$

Proof. From (2.1) and from the well- known Mobius inversion formula we have

$$J_k^{(1)}(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right) f_k(d) \tag{2.6}$$

where $f_k(m) = \binom{m+k-1}{k}$ i.e. $F_k^{(1)}(n) = f_k * \mu$. Using the relations (2.2) and (2.6) the formula (2.5) quickly follows.

Theorem 2.3. The following formula holds

$$J_k^{(1)}(n) = \sum_{\substack{s<n \\ \gcd(s,n)=1}} \sum_{m=1}^{\left[\frac{n}{s}\right]} J_{k-1}^{(1)}(m) \tag{2.7}$$

Proof. We have

$$J_k^{(1)}(n) = \# N_k(n) = \sum_{\substack{s<n \\ \gcd(s,n)=1}} \# S_S(n) = \sum_{\substack{s<n \\ \gcd(s,n)=1}} F_k\left(\left[\frac{n}{s}\right]\right) = \sum_{\substack{s<n \\ \gcd(s,n)=1}} \sum_{m=1}^{\left[\frac{n}{s}\right]} J_{k-1}^{(1)}(m)$$

and the formula is proved.

Corollary 2.4. If n is a prime, then

$$\sum_{s=1}^{n-1} \sum_{m=1}^{\left[\frac{n}{s}\right]} \varphi(m) = \frac{1}{2}\left(n^2 + n - 2\right) \tag{2.8}$$

Proof. Consider *k = 2* in (2.7).

3. The arithmetic function $J_k^{(2)}$ and the connection to Jordan's function J_k

Consider the set

$$P_k(n) = \{(a_1, \ldots a_k) \in Z_+^k : \gcd(a_1, \ldots a_k, n) = 1\}$$

and define $G_k(n) = \# P_k(n)$. Let us define the integer

$$J_k^{(2)}(n) = \# \{(a_1, \ldots a_k) \in P_k : \textit{at least a component } a_j \textit{ is equal } n\}$$

The Gauss' type formula for the function $J_k^{(2)}$ is given by

Theorem 3.1. The fallowing formula holds

$$\sum_{d/n} J_k^{(2)}(d) = n^k - (n-1)^k \tag{3.1}$$

Proof. Note that the following relation is valid

$$G_k(n) = G_k(n-1) + J_k^{(2)}(n) \tag{3.2}$$

Consider the set

$$N_d(n) = \{(a_1,\dots a_k) \in Z_+^k : 1 \le a_1 \le \dots \le a_k \le n \textit{ and } \gcd(a_1,\dots a_k) = d\}$$

and obtain

$$n^k = \sum_{s=1}^{n} \# N_s(n) = \sum_{s=1}^{n} G_k\left(\left[\frac{n}{s}\right]\right) \tag{3.3}$$

Replacing n by n - 1 we get

$$(n-1)^k = \sum_{s=1}^{n} G_k\left(\left[\frac{n-1}{s}\right]\right) \tag{3.4}$$

It follows

$$\sum_{d/n} J_k^{(2)}(d) = \sum_{d/n} (G_k(d) - G_k(d-1)) = n^k - (n-1)^k$$

Remarks.

1) If $k = 2$, then

$$J_k^{(2)}(n) = \begin{cases} 2\ (n) & \textit{if } n > 1 \\ 2\ (n) - 1 = 1 & \textit{if } n = 1 \end{cases}$$

and from relation (3.1) we obtain

$$\sum_{d/n} J_k^{(2)}(d) = \sum_{d/n} 2\ (d) - 1 = n^2 - (n-1)^2 = 2n - 1$$

that is the classical Gauss' formula for Eulers totient function.

2) The functions $J_k^{(1)}$, $J_k^{(2)}$ are not multiplicative. Indeed, if $f : Z_+ \to C$ is a numerical function with *f(1) = 1*, define its summation function *S* by formula

$S(n)\sum_{d/n} f(d)$ **.** It is easy to see that if *f* is multiplicative then *S* is multiplicative.

From formulas (2.1) and (3.1) the summation functions of $J_k^{(1)}$ and $J_k^{(2)}$ are not multiplicative**,** hence these functions are not multiplicative.

Theorem 3.2. The fallowing formla holds

$$G_k(n) = 1 + \sum_{m=2}^{n} J_k^{(2)}(m) \tag{3.5}$$

Proof. Applying the Möbius inversion formula, from (3.1) we obtain

$$J_k^{(2)}(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right) g_k(d) \tag{3.6}$$

where $g_k(m) = m^k - (m - 1)^k$. That is $J_k^{(2)} = g_k * \mu$. From (3.6) and (3.2) it follows relation (3.5).

The connection between $J_k^{(2)}$ and the Jordan's functions J_i is given by

Theorem 3.3. *The following relation holds*

$$J_k^{(2)}(n) = \sum_{s=1}^{k+1} (-1)^{s+1} \binom{k+1}{s} J_{k+1-s}(m) \tag{3.7}$$

Proof. Note that we can write

$$J_k^{(1)}(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right)\left(d^{k+1} - (d-1)^{k+1}\right) = \sum_{d/n} \mu\left(\frac{n}{d}\right) d^{k+1}\mu - \sum_{d/n} (d-1)^{k+1} \mu\left(\frac{n}{d}\right) =$$

$$= J_{k+1}(n) - \sum_{d/n} (d-1)^{k+1} \mu\left(\frac{n}{d}\right) = J_{k+1}(n) - \sum_{d/n} \sum_{m=0}^{k+1} (-1)^{k+1} \binom{k+1}{s} \mu\left(\frac{n}{d}\right) d^{k+1-s} =$$

$$= \sum_{S=1}^{k+1} (-1)^{s+1} \binom{k+1}{s} J_{k+1-s}(n)$$

Remarks.

1) An other argument for formula (3.7) can he obtained by using inclusion-exclusion principle as follows. Denote by *M* the set of all vectors

$(a_1,\ldots,a_{k+1}) \in Z_+^{k+1}$ such that $1 \le a_1 \le \ldots \le a_{k+1} \le n, \gcd(a_1,\ldots a_{k+1}) = 1$ and n is a component of the vector $(a_1,\ldots a_{k+1})$ at least once. Also, consider the sets *M,* consisting in all vectors $(a_1,\ldots a_{k+1}) \in Z_+^{k+1}$,
$1 \le a_1 \le \ldots \le a_{k+1} \le n, \gcd(a_1,\ldots a_{k+1}) = 1$, and n is the s^{th} component of the vector, *s = 1,2,..., k+1*.

It is clear that $M = \bigcup_{s=1}^{k+1} M_s$ and from inclusion-exclusion principle we have

$$\# M = \sum_{s=1}^{k+1} \# M_s - \sum_{1 \le i \le j \le k+1} \# M_i \bigcap M_j + \ldots$$

That is

$$J_k^{(2)}(n) = \binom{k+1}{1} J_k(n) - \binom{k+1}{1} J_k(n) + \ldots$$

i.e. the connection between $J_k^{(2)}$ and Jordan's functions given in the formula (3.7).

2) Consider $n = p_1^{\alpha 1} \ldots p_1^{\alpha 1}$ the prime factorization of *n.* Using formula (3.7) and a simple mathematical induction argument it follows

$$J_k^{(2)}(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right) d^k = n^k \left(1 - \frac{1}{p_1^k}\right) \ldots \left(1 - \frac{1}{p_m^k}\right) \qquad (3.8)$$

From formula (3.8) we deduce immediately that the Jordan function J_s is multiplicative.

Also, by using formula (3.8) and Möbius inversion formula we obtain the Gauss' formula for J_k, i.e.

$$\sum_{d/n} J_k(d) = n^k$$

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