SOME SUMABILITY METHODS

by
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Let $A = \|\rho_k(n)\|_{n\in\mathbf{N},\, k=0,\, \ldots,\, n}$ be a matrix with $\rho_k(n)\in$R. A sequence $s = \{s_n\}_{n\in\mathrm{N}}$ is said to be A-summable to ρ if $\lim_{n\to\infty}\sum_{k=0}^{n}\rho_k(n)\cdot s_k = \rho$.

A sequence $a = (a_n)_{n\geq 0}$ is called p, q-convex if

$$a_{n+2} - (p+q)a_{n+1} + pqa_n \geq 0 \quad , \quad (\forall)\ n\geq 0.$$

Let K be set of all real sequences, K_+ the set of all real sequences positive, $K_{p,\,q}$ the set of all p, q-convex sequences and T : $K_{p,\,q}\to$ K be a linear operator, defined as:

$$\mathrm{T}(a;\, n) = \sum_{k=0}^{n}\rho_k(n)\cdot a_{s+k}$$

where $\rho_k(n)\in$R ($n\in\mathbf{N}$) and $s\in\mathbf{N}$ are arbitrary.

The purpose of this work is to determine sufficient conditions for a real triangular matrix $\|\rho_k(n)\|_{n\in\mathbf{N},\, k=0,\, \ldots,\, n}$ such that $\mathrm{T}(K_{p,\,q}) \subseteq K_+$.

Theorem 1

Let $a = (a_n)_{n\geq 0}\in K_{p,\,q}$ be given arbitrary. T(a; n)$\in K_+$ if

i)

$$\begin{cases}\sum_{i=0}^{n}\rho_i(n)\cdot q^i = 0\\ \sum_{i=0}^{n}\rho_i(n)\cdot p^i = 0\end{cases} \qquad p\neq q\ ,\ \ p\neq 0\ ,\ \ q\neq 0$$

or

$$\begin{cases}\sum_{i=0}^{n}\rho_i(n)\cdot q^i = 0\\ \sum_{i=0}^{n}\rho_i(n)\cdot i\cdot p^i = 0\end{cases} \qquad p = q\neq 0$$

ii)

$$\frac{1}{q\cdot p^{k+1}}\cdot\sum_{i=0}^{k}\rho_i(n)p^i\cdot\sum_{r=0}^{k-i}\left(\frac{p}{q}\right)^r\geq 0 \quad , \qquad k=\overline{0,n-2}$$

Proof. For the computations we need the following notation: $\rho_k(n)=\rho_k$, $(\forall)\,k=\overline{0,n}$, $n\in\mathbf{N}$. Consider

$$\mathrm{T}(a;n)=\sum_{k=0}^{n-2}c_k\cdot[a_{s+k+2}-(p+q)a_{s+k+1}+p\cdot qa_{s+k}],$$

$$\mathrm{T}(a;n)=\sum_{k=0}^{n-2}c_k\cdot a_{s+k+2}-(p+q)\cdot\sum_{k=0}^{n-2}c_k\cdot a_{s+k+1}+p\cdot q\cdot\sum_{k=0}^{n-2}c_k\cdot a_{s+k}$$

$$\mathrm{T}(a;n)=\sum_{k=0}^{n-2}a_{s+k}\cdot[c_{k-2}-(p+q)c_{k-1}+p\cdot q\cdot c_k]+a_{s+n-1}\cdot[c_{n-3}-(p+q)c_{n-2}]+$$

$$+\,c_{n-2}\cdot a_{s+n}+a_{s+1}\cdot[p\cdot qc_1-(p+q)\cdot c_0]+p\cdot q\cdot c_0\cdot a_s.$$

Therefore

$$\begin{cases} pqc_0=\rho_0 \\ pqc_1-(p+q)c_0=\rho_1 \\ c_{k-2}-(p+q)c_{k-1}+pqc_k=\rho_k \quad , \quad k=\overline{2,n-2} \\ c_{n-3}-(p+q)c_{n-2}=\rho_{n-1} \\ c_{n-2}=\rho_n \end{cases} \quad (1)$$

From (1), it follows:

$$c_{r-2}-(p+q)c_{r-1}+pqc_r=\rho_r \quad , \quad r=\overline{2,n-2}$$
$$(c_{r-2}-pc_{r-1})-q(c_{r-1}-pc_r)=\rho_r$$
$$q^{r-2}(c_{r-2}-pc_{r-1})-q^{r-1}(c_{r-1}-pc_r)=\rho_r q^{r-2}$$

Let add those equalities for $r=\overline{2,k}$. We obtain:

$$c_0 - pc_1\text{-}q^{k-1}\cdot(c_{k-1} - p\cdot c_k) = \sum_{r=2}^{k}\rho_r\cdot q^{r-2}\ ,\quad k=\overline{2,n-2}$$

$$p\cdot c_k - c_{k-1} = \frac{pc_1 - c_0}{q^{k-1}} + \frac{1}{q^{k-1}}\sum_{r=2}^{k}\rho_r\cdot q^{r-2}\ ,$$

$$p^k\cdot c_k - p^{k-1}\cdot c_{k-1} = (pc_1\text{-}c_0)\cdot\left(\frac{p}{q}\right)^{k-1} + \left(\frac{p}{q}\right)^{k-1}\sum_{r=2}^{k}\rho_r\cdot q^{r-2}\ ,$$

$$p^i\cdot c_i - p^{i-1}\cdot c_{i-1} = (pc_1\text{-}c_0)\cdot\left(\frac{p}{q}\right)^{i-1} + \left(\frac{p}{q}\right)^{i-1}\sum_{r=2}^{i}\rho_r\cdot q^{r-2}\ .$$

Let add those equalities for $i = 2, \ldots, k$. We obtain:

$$p^k\cdot c_k - pc_1 = (pc_1\text{-}c_0)\sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i-1} + \sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i-1}\cdot\sum_{r=2}^{i}\rho_r\cdot q^{r-2}$$

$$c_k = \frac{c^1}{q^{k-1}} + \frac{pc_1 - c_0}{p^k}\sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}}\sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i}\cdot\sum_{r=2}^{i}\rho_r\cdot q^{r}$$

By the formula $\sum_{i=2}^{k}a_i\cdot\sum_{r=2}^{i}b_r = \sum_{i=2}^{k}b_i\cdot\sum_{r=i}^{k}a_r$ we can write:

$$c_k = \frac{c^1}{p^{k-1}} + \frac{pc_1 - c_0}{p^k}\sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}}\sum_{i=2}^{k}\rho_i\cdot q^i\sum_{r=i}^{k}\left(\frac{p}{q}\right)^{r}$$

$$c_k = \frac{c^1}{p^{k-1}} + \frac{pc_1 - c_0}{p^k}\sum_{i=2}^{k}\left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}}\sum_{i=2}^{k}\rho_i\cdot q^i\sum_{r=0}^{k-i}\left(\frac{p}{q}\right)^{r+i}\ ,\quad k=\overline{2,n-2}$$

From (1) we have

$$c_0 = \frac{\rho_0}{p \cdot q}\ ,\ c_1 = \frac{\rho_1}{p \cdot q} + \frac{p+q}{p^2 \cdot q^2} \cdot \rho_0$$

Therefore

$$c_k = \frac{\rho^1}{q \cdot p^k} + \frac{p+q}{q^2 p^{k+1}} \cdot \rho_0 + \frac{1}{p^k}\left[p\left(\frac{\rho_1}{pq} + \frac{p+q}{p^2 \cdot q^2} \cdot \rho_0\right) - \frac{\rho_0}{pq}\right] \sum_{i=2}^{k} \left(\frac{p}{q}\right)^{i-1} +$$

$$+ \frac{1}{qp^{k+1}} \sum_{i=2}^{k} \rho_i \cdot q^i \sum_{r=0}^{k-i} \left(\frac{p}{q}\right)^{r+i} ,$$

$$c_k = \frac{1}{qp^{k+1}} \sum_{i=0}^{k} \rho_i \cdot q^i \cdot \sum_{r=0}^{k-i} \left(\frac{p}{q}\right)^{r+i} , \quad k = \overline{2, n-2} \qquad (1')$$

From (1) and (1') it follows that:

$$\begin{cases} c_{n-2} = \dfrac{1}{qp^{n-1}} \displaystyle\sum_{i=0}^{n-2} \rho_i \cdot q^i \sum_{r=0}^{n-2-i} \left(\frac{p}{q}\right)^{r+i} = \rho_n \\ c_{n-3} = \dfrac{1}{qp^{n-2}} \displaystyle\sum_{i=0}^{n-3} \rho_i \cdot q^i \sum_{r=0}^{n-3-i} \left(\frac{p}{q}\right)^{r+i} = \rho_{n-1} + (p+q)\rho_n \end{cases} \qquad (1'')$$

Because

$$c_{n-2} = \frac{1}{qp^{n-1}} \sum_{i=0}^{n-2} \rho_i \cdot q^i \cdot w_{n-1-i} = \rho_n$$

where

$$w_j = \begin{cases} \dfrac{\left(\frac{p}{q}\right)^j - 1}{\frac{p}{q} - 1}, & p \neq q \\ j, & p = q \end{cases} \qquad j = \overline{1, n-1}$$

it results:

$$c_{n-2} = \frac{1}{qp^{n-1}} \sum_{i=0}^{n} \rho_i \cdot p^i \cdot w_{n-1-i} - \frac{\rho_n p_n}{qp^{n-1}} \cdot \frac{\frac{q}{p} - 1}{\frac{p}{q} - 1} = \rho_n .$$

Thus:

$$\sum_{i=0}^{n} \rho_i \cdot p^i \cdot w_{n-1-i} = 0 \qquad (2)$$

Because

$$c_{n-3} = \frac{1}{qp^{n-2}} \sum_{i=0}^{n-3} \rho_i \cdot p^i w_{n-2-i} = \rho_{n-1} + (p+q)\rho_n$$

it results also

$$c_{n-3} = \frac{1}{qp^{n-2}} \sum_{i=0}^{n} \rho_i p^i \cdot w_{n-2-i} - \frac{\rho_{n-1} p}{q} \cdot \frac{\frac{q}{p} - 1}{\frac{p}{q} - 1} - \frac{\rho_n p^2}{q} \cdot \frac{\frac{q^2}{p^2} - 1}{\frac{p}{q} - 1}$$

and thus:

$$\sum_{i=0}^{n} \rho_i \cdot p^i \cdot w_{n-2-i} = 0 \qquad (3)$$

From (2) and (3) we obtain:

$$\begin{cases} \sum_{i=0}^{n} \rho_i \cdot q^i = 0 \\ \sum_{i=0}^{n} \rho_i \cdot p^i = 0 \end{cases} \qquad p \neq q$$

or

$$\begin{cases} \sum_{i=0}^{n} \rho_i \cdot q^i = 0 \\ \sum_{i=0}^{n} \rho_i \cdot i \cdot p^i = 0 \end{cases} \qquad p = q$$

this means i).

From condition $c_k \geq 0$ for all k from 0 to n-2 we obtain ii).

References

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