ON UNIVALENT INTEGRAL OPERATOR

by
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Abstract. Let *S* be the class of regular and univalent function $f(z) = z + a_2 z^2 + \dots$, in the unit disc, $U = \{z : |z| < 1\}$.We prove new univalence criteria for the integral operator $F_{\alpha\beta}$.

Theorem 1. If the function *f* is regular in unit disc U, $f(z) = z + a_2 z^2 + \dots$, and

$$\left(1 - |z|^2\right) \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, (\forall) z \in U , \qquad (1)$$

then the function *f* is univalent in U.

Theorem 2. If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)} g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right|, \qquad (2)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \qquad (3)$$

the equalities hold only in case $g(z) = \varepsilon \frac{z + u}{1 + \overline{u} z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark A For *z=0* , from inequality (2) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)} g(\xi)}\right| \leq |\xi| , \qquad (4)$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}, \qquad (5)$$

Considering $g(0) = a$ and $\xi = z$ then $|g(z)| \leq \dfrac{|z| + |a|}{1 + |a||z|}$, (6)

for all $z \in U$.

Theorem 3. Let γ be a complex number and the function $h \in S$, $h(z) = z + a_2 z^2 + \ldots,$. If

$$\left| \frac{zh'(z) - h(z)}{zh(z)} \right| \leq 1, (\forall) z \in U , \qquad (7)$$

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right]}, \qquad (8)$$

then the function

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\gamma \cdot dt \in S , \qquad (10)$$

Theorem 4. Let $\alpha, \beta \in C,\ f, g \in S,\ f(z) = z + a_2 z^2 + \ldots, g(z) = z + b_2 z^2 + \ldots,$. If

$$\left| \frac{zf'(z) - f(z)}{zf(z)} \right| \leq 1, (\forall) z \in U , \qquad (11)$$

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1, (\forall) z \in U , \qquad (12)$$

$$\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1, \qquad (13)$$

$$|\alpha \cdot \beta| \leq \frac{1}{\max\limits_{|z|\leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right]}, \qquad (14)$$

where

$$|c| = \frac{|\alpha a_2 + \beta b_2|}{|\alpha \cdot \beta|}, \qquad (15)$$

then

$$F_{\alpha\beta}(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha \cdot \left(\frac{g(t)}{t} \right)^\beta dt \in S$$

Proof:

$$f, g \in S, \text{ and } \frac{f(z)}{z} \neq 0, \frac{g(z)}{z} \neq 0 .$$

For *z=0* we are $\left(\frac{f(z)}{z} \right)^\alpha \cdot \left(\frac{g(z)}{z} \right)^\beta = 1$.

We consider the function $h(z)=\frac{1}{|\alpha\cdot\beta|}\cdot\frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}$, where $|\alpha\cdot\beta|$ satisfy (14).

We calculate the derivative by order 1 and 2 for $F_{\alpha\beta}$.

We are: $F'_{\alpha\beta}(z)=\left(\frac{f(z)}{z}\right)^{\alpha}\cdot\left(\frac{g(z)}{z}\right)^{\beta}$

$$F''_{\alpha\beta}(z)=\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\cdot\frac{zf'(z)-f(z)}{z^2}\cdot\left(\frac{g(z)}{z}\right)^{\beta}+\beta\left(\frac{g(z)}{z}\right)^{\beta-1}\cdot\frac{zg(z)-g(z)}{z^2}\cdot\left(\frac{f(z)}{z}\right)^{\alpha}$$

Then *h(z)* are the form:

$$h(z)=\frac{1}{|\alpha\cdot\beta|}\cdot\frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}=\frac{1}{|\alpha\cdot\beta|}\cdot\frac{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\cdot\frac{zf'(z)-f(z)}{z^2}\cdot\left(\frac{g(z)}{z}\right)^{\beta}}{\left(\frac{f(z)}{z}\right)^{\alpha}\cdot\left(\frac{g(z)}{z}\right)^{\beta}}+$$

$$+\frac{1}{|\alpha\cdot\beta|}\cdot\frac{\beta\left(\frac{g(z)}{z}\right)^{\beta-1}\cdot\frac{zg(z)-g(z)}{z^2}\cdot\left(\frac{f(z)}{z}\right)^{\alpha}}{\left(\frac{f(z)}{z}\right)^{\alpha}\cdot\left(\frac{g(z)}{z}\right)^{\beta}}.$$

$$=\frac{1}{|\alpha\cdot\beta|}\cdot\alpha\cdot\frac{zf'(z)-f(z)}{zf(z)}+\frac{1}{|\alpha\cdot\beta|}\cdot\beta\cdot\frac{zg'(z)-g(z)}{zg(z)}.$$

We are $h(0)=\frac{1}{|\alpha\cdot\beta|}\cdot\alpha a_2+\frac{1}{|\alpha\beta|}\cdot\beta b_2$ and the condition (11) and (12)

But $|h(z)|=\left|\frac{1}{|\alpha\cdot\beta|}\cdot\alpha\cdot\frac{zf'(z)-f(z)}{zf(z)}+\frac{1}{|\alpha\cdot\beta|}\cdot\beta\cdot\frac{zg'(z)-g(z)}{zg(z)}\right|\le$

$\le\frac{|\alpha|}{|\alpha\cdot\beta|}\cdot\left|\frac{zf'(z)-f(z)}{zf(z)}\right|+\frac{|\beta|}{|\alpha\cdot\beta|}\cdot\left|\frac{zg'(z)-g(z)}{zg(z)}\right|\le\frac{1}{|\alpha|}+\frac{1}{|\beta|}<1$ from (13) and $|h(z)|<1$.

$$|h(0)|=\frac{|\alpha a_2+\beta b_2|}{|\alpha\beta|}=|c|$$

Applied Remark A for the function *h* obtained: $|h(z)|\le\frac{|z|+|c|}{1+|c|\cdot|z|},(\forall)z\in U$

But $|h(z)|=\frac{1}{|\alpha\cdot\beta|}\cdot\left|\frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right|$

And we have $\frac{1}{|\alpha \cdot \beta|} \cdot \left|\frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right| \leq \frac{|z|+|c|}{1+|c|\cdot|z|}, (\forall) z \in U \Leftrightarrow$

$$\Leftrightarrow \left|\frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right| \leq |\alpha \cdot \beta| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}, (\forall) z \in U \Leftrightarrow$$

$$\Leftrightarrow \left|\left(1-|z|^2\right) \cdot z \cdot \frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right| \leq |\alpha \cdot \beta| \cdot \left(1-|z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}, (\forall) z \in U \text{ .(applied th.1) } (16)$$

Let's consider the function $H : [0,1] \rightarrow R$, $H(x) = \left(1-x^2\right) \cdot x \cdot \frac{x+|c|}{1+|c|\cdot x}, \quad x = |z|$

$$H\left(\frac{1}{2}\right) = \left(1-\frac{1}{4}\right) \cdot \frac{1}{2} \cdot \frac{\frac{1}{2}+|c|}{1+|c|\cdot\frac{1}{2}} = \frac{3}{8} \cdot \frac{1+|c|}{2+|c|} > 0 \Rightarrow \max_{x\in[0,1]} H(x) > 0 .$$

Using this result in (16) we have:

$\left|\left(1-|z|^2\right) \cdot z \cdot \frac{F''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right| \leq |\alpha \cdot \beta| \cdot \max_{x\in[0,1]} \left[\left(1-|z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}\right], (\forall) z \in U$ and (14) implies

$\left(1-|z|^2\right) \cdot \left|\frac{zF''_{\alpha\beta}(z)}{F'_{\alpha\beta}(z)}\right| \leq 1, (\forall) z \in U$ and using the theorem 1 obtained $F \in S$.

Remark B. For $g(z) = z, \beta \in C, |\beta| > 1$, we obtained theorem 3.

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