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RESULTS ON MEROMORPHIC FUNCTIONS SHARING TWO SETS WITH REDUCED CARDINALITY AND WEIGHT

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Abstract. We prove two uniqueness theorems of two nonconstant meromorphic functions sharing two sets which improve results of H.X.Yi and W.R.Lu, I.Lahiri, Fang-Lahiri and Banerjee.

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1.Introduction and necessary background materials

Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities)and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [9]. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity then we replace the above set by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. When we let r, a real number, tend towards ∞ we will always assume that while approaching to ∞, r may avoid some subset E, say, of the real line of finite measure, not necessarily the same at every occurrence.

In 1976 F.Gross proposed the following question in [8].

Question A. *Can one find finite sets S_j,$j = 1, 2$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Gross also raised question about the cardinalities of such sets if it exist.

Yi[17] and independently Fang and Xu[5] gave the one and same positive answer to this question. Now it is natural to ask the following question.

Question B. *Can one find finite sets S_j, $j = 1, 2$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In 1994 Yi[15] gave an affirmative answer to Question B and proved that there exist two finite sets S_1(with two elements)and S_2(with nine elements) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

In 1996 Li and Yang [13] proved that there exist two finite sets S_1(with one element)and S_2(with fifteen elements) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

In 1997 Fang and Guo[4] obtained a better result than that of Li and Yang. They succeeded in establishing the above result with two sets with less cardinalities namely S_1 with one element and S_2 with nine elements.

Suppose that the polynomial $P(w)$is defined by

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \tag{1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying

$ab^{n-2} \neq 2$. We also define

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{2}$$

where $\alpha_1, \alpha_2$are two distinct roots of $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. It can be shown that $P(w)$ has only simple roots.{See [1,2].}

In 2002 Yi[19] proved the following result in which he not only reduced the cardinalities of the set S but also relaxed the sharing of the poles from CM to IM.

Theorem A.[19] *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$is given by (1)and $n(\geq 8)$. Suppose that f and g are two nonconstant meromorphic functions such that $E_f(S) = E_g(S)$ and $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$ then $f \equiv g$.*

As a consequence of Question B, Yi and Lü[20] raised the following question in 2004.

Question C. *Can one find finite sets S_j,$j = 1, 2$ such that any two nonconstant meromorphic functions f and g satisfying for $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ $j = 1, 2$ must be identical ?*

In this direction they established the following results which also improved results already obtained by Yi[16].

Theorem B.[20] *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$is given by (1)and $n(\geq 12)$. Suppose that f and g are two nonconstant meromorphic functions such*

that $\overline{E}_f(S) = \overline{E}_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$
then $f \equiv g$.

Theorem C.[20] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 13)$. *Suppose that* f *and* g *are two nonconstant meromorphic functions such that* $\overline{E}_f(S) = \overline{E}_g(S)$ *and* $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$
then $f \equiv g$.

In 2001 Lahiri introduced the notion of weighted sharing as follows.

Definition 1.[10,11] *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C}\cup\{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and*
$k + 1$*times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f *and* g *share the value* a *with weight* k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ of multiplicity
$m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and
only f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2.[11] *Let* S *be a set of distinct elements of* $\mathbb{C} \cup \{\infty\}$ *and* k *be a positive integer or* ∞. *We denote by* $E_f(S, k)$*the set* $\bigcup_{a\in S} E_k(a; f)$. *Clearly* $E_f(S) = E_f(S, \infty)$ *and* $\overline{E}_f(S) = E_f(S, 0)$.

Recently Banerjee[1] improved and supplemented Theorem A and Theorem B as follows.

Theorem D.[1] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 8)$. *Suppose that* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S, 2) = E_g(S, 2)$ *and*
$E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ *then* $f \equiv g$.

Theorem E.[1] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 9)$. *Suppose that* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S, 1) = E_g(S, 1)$ *and*
$E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ *then* $f \equiv g$.

Theorem F.[1] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 12)$. *Suppose that* f *and* g *are two nonconstant meromorphic functions such that*

$E_f(S,0) = E_g(S,0)$ *and*
$E_f(\{\infty\},3) = E_g(\{\infty\},3)$ *then* $f \equiv g$.

Note that none of the above mentioned theorems of Banerjee improves Theorem C, which has been claimed to be the best result till date in[20]. In a most recent paper Banerjee, however established the following result as a special case of which one can obtain Theorem C as well as Theorem F.

Theorem G.[2] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 9)$. *If* f *and* g *be two nonconstant meromorphic functions such that* $E_f(S,0) = E_g(S,0)$ *and* $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ *and* $\frac{11}{4} min\{\Theta_f, \Theta_g\} > \frac{9}{2} + \frac{2(n-3)}{(n-5)\{(n-2)k+(n-3)\}} + \frac{10}{n-5} - \frac{n}{2}$ *then* $f \equiv g$, *where* $\Theta_f = \Theta(0;f) + \Theta(b;f)$ *and* Θ_g *is defined similarly.*

Remark 1. In Theorem G when $n \geq 12$ and $k = 3$ we get Theorem F. Again when $n \geq 13$ and $k = 0$ we get Theorem C. Thus Theorem G improves both Theorems C and F.

Strictly speaking Theorem G is a generalization of Theorems C and F rather than direct improvements since it can neither reduce the cardinality of the shared set S in Theorem C nor it reduces the weight of the shared set $\{\infty\}$ in Theorem F. In this paper we propose our first theorem below as a corollary of which we may get the desired improvements of Theorem C and Theorem F.

Theorem 1. *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 9)$. *If* f *and* g *be two nonconstant meromorphic functions such that* $E_f(S,0) = E_g(S,0)$ *and* $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ *and* $min\{3\Theta(0;f) + 2\Theta(b;f), 3\Theta(0;g) + 2\Theta(b;g)\} > 4 + \frac{8}{n-5} + \frac{2n-6}{(n-5)\{(n-2)k+(n-3)\}} - \frac{n}{2}$ *then* $f \equiv g$.

Following corollary is a natural consequence of above theorem.

Corollary 1. *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 12)$. *If* f *and* g *be two nonconstant meromorphic functions such that* $E_f(S,0) = E_g(S,0)$ *and* $E_f(\{\infty\},0) = E_g(\{\infty\},0)$ *then* $f \equiv g$.

Recently Banerjee also obtained the following results in two different papers where he has considered the shared set S with less number of elements to obtain the uniqueness of functions under different conditions improving some previous results.

Theorem H.[2] *Let* $S = \{w \mid P(w) = 0\}$, *where* $P(w)$*is given by (1)and* $n(\geq 6)$. *If* f *and* g *be two nonconstant meromorphic functions such that* $E_f(S,2) = E_g(S,2)$ *and* $E_f(\{\infty\},0) = E_g(\{\infty\},0)$ *and* $2min\{\Theta_f, \Theta_g\} > 3 + \frac{3}{2(n-3)} + \frac{6}{3n-11} - \frac{n}{2}$ *then* $f \equiv g$, *where* $\Theta_f = \Theta(0;f) + \Theta(b;f)$ *and* Θ_g *is defined similarly.*

Theorem I.[3] *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$is given by (1)and $n(\geq 7)$. If f and g be two nonconstant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $min\{\Theta_f^1, \Theta_g^1\} > 7 + \frac{2}{n-3} - n$ then $f \equiv g$, where $\Theta_f^1 = 4\Theta(0; f) + 4\Theta(b; f) + \Theta(\infty; f)$ and Θ_g^1 is defined similarly.*

In our next Theorem we improve Theorem I by reducing the cardinality of the set S from 7 to 5 and extending the Theorem for any weight k, for the shared set $\{\infty\}$. Also we claim that our next result will also improve Theorem H.Thus our next result will combine both Theorems H and I in an improved result. Note that in the definition of the polynomial $P(w)$, we require $ab^{n-2} \neq 2$. For our purpose, in addition to it we assume $ab^{n-2} \neq 1$, by which the polynomial $P(w)$ will not lose any of its properties mentioned above. Thus from now on our set S is given by $S = \{w \mid P(w) = 0\}$ where $P(w)$ is given by (1) with $ab^{n-2} \neq 2, 1$.

We state below our next Theorem

Theorem 2. *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$is given by (1)and $n(\geq 5)$ and $ab^{n-2} \neq 2, 1$. Suppose that f and g are two nonconstant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$where k is a nonnegative integer or ∞.*
If $min\{\Theta_f^1, \Theta_g^1\} > 7 + \frac{2}{n-3} + \frac{8n-24}{(3n-11)\{(n-2)k+n-3\}} - n$, then $f \equiv g$ where Θ_f^1 and Θ_g^1 are same as Theorem I.

Remark 2. When $k = \infty$ in Theorem 2 we get the conclusion of Theorem I with the shared set S containing less number of elements(five elements). Thus Theorem 2 improves Theorem I.

When $n \geq 8$ in Theorem 2 we obtain Theorem D. Thus Theorem 2 improves Theorem D. Also it is easy to verify that the condition on ramification index in this theorem is weaker than the condition in the Theorem H for $n = 6$ and $n = 7$. Since when $n \geq 8$ the condition on ramification indices cease to exist both in Theorems H and 2, Theorem 2 improves Theorem H.

We close this section with a few more definitions.

Definition 3. *For $a \in \mathbb{C} \cup \{\infty\}$ For a positive integer m we denote by $N(r, a; f \mid \geq m)$ the counting function of those a-points of f whose multiplicities are not less than m where each a-point is counted according to its multiplicity. We agree to write $\overline{N}(r, a; f \mid \geq m)$ to denote the corresponding reduced counting function.*

Definition 4.[10,18,20] *Let f and g be two nonconstant meromorphic functions such that f and g share $(a, k)$where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the counting function of those a-points of $f$and g where $p > q (q > p)$, by $\overline{N}_E^{(k+1}(r, a; f)$ the counting functions of those a-points of f and g where $p = q \geq k + 1$ each point* *in these counting functions is counted only once. In the same way we can define* $\overline{N}_E^{(k+1}(r,a;g)$. *Clearly* $\overline{N}_E^{(k+1}(r,a;f) = \overline{N}_E^{(k+1}(r,a;g)$. *We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the corresponding* a*-points of* g. *Clearly* $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$. *We also denote by* $N_E^{1)}(r,a;f)$ *the counting function of those* a*-points of* f, g *for which* $p = q = 1$.

Definition 5.[1] *Let* f *and* g *share the value* 1 *IM.Let* z_0 *be a* 1*-point of* f *and* g *with multiplicities* p *and* q *respectively. Let* s *be a positive integer. We denote by* $\overline{N}_{f>s}(r,1;g)$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p > q = s$.

2.Lemmas

In this section we present some lemmas which will be required to establish our results. In the lemmas several times we use the function H defined by $H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$.

Let f and g be two nonconstant meromorphic functions and

$$F = R(f), G = R(g), \tag{3}$$

where $R(w)$ is given by (2). From (2) and (3) it is clear that

$$T(r,f) = \frac{1}{n}T(r,F) + S(r,f), T(r,g) = \frac{1}{n}T(r,G) + S(r,g). \tag{4}$$

Lemma 1.[2] *Let* F *and* G *be given by (3)where* $n \geq 3$ *is an integer and* $H \not\equiv 0$. *If* F, G *share* $(1,m)$*and* f,g *share* (∞,k), *where* $0 \leq m < \infty$. *Then* $\{\frac{n}{2}+1\}\{T(r,f)+T(r,g)\} \leq 2[\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)$ $+\overline{N}_*(r,\infty;f,g) - (m-\frac{3}{2})\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)$.

Lemma 2.[1] *Let* F *and* G *be given by (3) and* $H \not\equiv 0$. *If* F, G *share* $(1,m)$ *and* f,g *share* (∞,k), *where* $0 \leq m < \infty, 0 \leq k < \infty$, *then* $[(n-2)k+n-3]\overline{N}(r,\infty;f \mid\geq k+1) = [(n-2)k+n-3]\overline{N}(r,\infty;g \mid\geq k+1) \leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)$.

Lemma 3.[1] *Let* F *and* G *be given by (3) and* $H \not\equiv 0$. *If* F, G *share* $(1,m)$*and* f, g *share* (∞,k), *where* $0 \leq m < \infty, 0 \leq k < \infty$, *then* $[(n-2)k+n-3]\overline{N}(r,\infty;f \mid\geq k+1) = [(n-2)k+n-3]\overline{N}(r,\infty;g \mid\geq k+1) \leq \frac{m+2}{m+1}[\overline{N}(r,0;f)+\overline{N}(r,0;g)] + \frac{2}{m+1}\overline{N}(r,\infty;f) + S(r,f) + S(r,g)$.

Lemma 4.[2] *Let* F *and* G *be given by (3). Also let* S *be given as in Theorem 1, where* $n \geq 3$ *is an integer. If* $E_f(S,0) = E_g(S,0)$ *then* $S(r,f) = S(r,g)$.

Lemma 5. *If f and g share $(1,0)$ then* $N(r,1;g)-\overline{N}(r,1;g)\geq 2\overline{N}_L(r,1;g)+\overline{N}_L(r,1;f)+\overline{N}_E^{(2}(r,1;f)+\overline{N}_E^{(3}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)$. *Proof*: Let z_0 be a 1-point of f and g of respective multiplicities p and q.We denote by $N_2(r)$and $N_3(r)$the counting functions of those 1-points of f and g when $2\leq q=p$ and $1\leq p<q$ respectively where each point in these counting functions is counted $q-2$ times.
Since f,g share $(1,0)$ we have $N(r,1;g)-\overline{N}(r,1;g)$

$$\geq \overline{N}_L(r,1;g)+N_3(r)+N_2(r)+\overline{N}_E^{(2}(r,1;f)+\overline{N}_L(r,1;f)-\overline{N}_{f>1}(r,1;g).$$

Now observing $N_2(r)\geq \overline{N}_E^{(3}(r,1;f)$ and $N_3(r)\geq \overline{N}_L(r,1;g)-\overline{N}_{g>1}(r,1;f)$our lemma follows from above.

Lemma 6.[2] *Let F, G be given by (3). If F, G share $(1,m)$,where $0\leq m<\infty$,then*

(i) $\overline{N}_L(r,1;F)\leq \frac{1}{m+1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)]+S(r,f)$,

(ii) $\overline{N}_L(r,1;G)\leq \frac{1}{m+1}[\overline{N}(r,0;g)+\overline{N}(r,\infty;g)]+S(r,g)$

Lemma 7.[1] *Let F, G be given by (3)and $H\not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k),where $0\leq k\leq\infty$,then*

$$N_E^{1)}(r,1;F)\leq \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)+\overline{N}(r,b;f)$$

$$+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,F)+S(r,G)$$
where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f'which are not the zeros of

$f(f-b)$ and $F-1$, $\overline{N}_0(r,0;g')$is defined similarly.

Lemma 8. *Let F and G be given by (3). If F, G share $(1,0)$ and f, g share (∞,k) and $H\not\equiv 0$ then*

$$(n+1)T(r,f)+T(r,g)$$

$$\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;f)\}$$

$$+\overline{N}(r,\infty;f\mid\geq k+1)+2\overline{N}_L(r,1;F)+S(r,f)+S(r,g).$$

Proof. We denote by $N_0(r,0;f')$ the counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $F-1$. $N_0(r,0;g')$ is defined similarly. By the second fundamental theorem we get

$(n+1)T(r,f)+(n+1)T(r,g)$

$\leq \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)$

$-N_0(r,0;f')-N_0(r,0;g')+S(r,g)+S(r,f)$

$=\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)\}$

$+N_E^{1)}(r,1;F)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}(r,1;G)-N_0(r,0;f')-N_0(r,0;g')+S(r,g)+S(r,f)$
Note that since F, G share $(1,0)$ we have

$\overline{N}(r,1;F\mid\geq 2)=\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)-\overline{N}_{G>1}(r,1;F)$

Since f,g share (∞,k), $\overline{N}_*(r,\infty;f,g)\leq\overline{N}(r,\infty;f\mid\geq k+1)$, and hence using Lemma 7 with $m=0$ and Lemma 5 we obtain from above

$(n+1)T(r,f)+(n+1)T(r,g)$

$\leq\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)\}$

$+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)+\overline{N}(r,b;f)$

$+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')$

$+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)-\overline{N}_{G>1}(r,1;F)$

$+\overline{N}(r,1;G)-N_0(r,0;f')-N_0(r,0;g')+S(r,g)+S(r,f)$

$\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)\}$

$+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f\mid\geq k+1)+\overline{N}_L(r,1;F)+N(r,1;G)+\overline{N}_{F>1}(r,1;G)+S(r,f)+S(r,g)$

$\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;f)\}$

$+\overline{N}(r,\infty;f\mid\geq k+1)+2\overline{N}_L(r,1;F)+nT(r,g)-m(r,1;G)+S(r,f)+S(r,g).$

Therefore

$(n+1)T(r,f)+T(r,g)$

$\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;f)\}$

$+\overline{N}(r,\infty;f\mid\geq k+1)+2\overline{N}_L(r,1;F)+S(r,f)+S(r,g).$ This completes the proof.

Lemma 9.[2] *Let f, g be two non-constant meromorphic functions sharing $(\infty,0)$ and suppose that α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2-2n(n-2)bw+(n-1)(n-2)b^2=0$. Then $\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\cdot\frac{g^n}{(g-\alpha_1)(g-\alpha_2)}\not\equiv\frac{n^2(n-1)^2}{a^2}$, where $n\geq 3$ is an integer.*

Lemma 10.[7] *Let $Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2$, then $Q(w)=(w-1)^4(w-\beta_1)(w-\beta_2)..(w-\beta_{2n-6})$ where $\beta_j\in\mathbb{C}\backslash\{0,1\}$, $(j=1,2,..,2n-6)$ which are pairwise distinct.*

Lemma 11.*Let F, G be given by (5), where $n\geq 4$ is an integer. If f, g share $(\infty,0)$ then $F\equiv G\Rightarrow f\equiv g$.*
Proof.From the definitions of F, G we observe that $F\equiv G\Rightarrow\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\equiv\frac{g^n}{(g-\alpha_1)(g-\alpha_2)}$. Therefore f, g share $(0,\infty)$ and (∞,∞). Then from above and in view of the definition of $R(w)$ we obtain

$$n(n-1)f^2g^2(f^{n-2}-g^{n-2})-2n(n-2)bfg(f^{n-1}-g^{n-1})+(n-1)(n-2)b^2(f^n-g^n)=0. \tag{5}$$

Let $h=\frac{f}{g}$ that is $f=gh$ which on substitution in (5) yields

$$n(n-1)h^2g^2(h^{n-2}-1)-2n(n-2)bhg(h^{n-1}-1)+(n-1)(n-2)b^2(h^n-1)=0. \tag{6}$$

Note that since f and g share $(0,\infty)$ and (∞,∞),$0,\infty$ are the exceptional values of Picard of h. If h is non-constant then from Lemma 2.10 and (6) we have

$$\{n(n-1)h(h^{n-2}-1)g-n(n-2)b(h^{n-1}-1)\}^2=-n(n-2)b^2Q(h) \tag{7}$$

where $Q(h)=(h-1)^4(h-\beta_1)(h-\beta_2)...(h-\beta_{2n-6})$, $\beta_j\in\mathbb{C}\setminus\{0,1\}$,$j=1,2,..,2n-6$ which are pairwise distinct. From (7) we observe that each zero of $h-\beta_j$,$j=1,2,..,2n-6$ is of order at least two.Therefore by the second main theorem we obtain

$(2n-6)T(r,h) \leq \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\sum_{j=1}^{2n-6}\overline{N}(r,\beta_j;h)+S(r,h) \leq \frac{1}{2}(2n-6)T(r,h)+S(r,h)$, which is a contradiction for $n \geq 4$. Thus h must be a constant. From (7) it follows that $h^{n-2}-1=0$ and $h^{n-1}-1=0$ which implies that $h \equiv 1$. Therefore $f \equiv g$. This completes the proof.

Lemma 12.[2] *Let F, G be given by (3) and S be defined as in Theorem 1, where $n \geq 4$. If $E_f(S,0)=E_g(S,0)$ then $S(r,f)=S(r,g)$.*

3. Proof of theorems

Proof of Theorem 1. Since $E_f(S,0)=E_g(S,0)$, we see that F, G share $(1,0)$. We first suppose that $H \not\equiv 0$. From Lemma 3 we obtain for $m=0$ and $k=0$,

$\overline{N}(r,\infty;f) \leq \frac{2}{n-5}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}$
and for $m=0$ and $k=k$, $\overline{N}(r,\infty;f \mid \geq k+1) \leq \frac{2n-6}{(n-5)[(n-2)k+(n-3)]}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}$.

Hence using the above inequalities we obtain from Lemma 8 and Lemma 6 with $m=0$

$$\begin{aligned}(n+1)T(r,f)+T(r,g) &\leq 4\overline{N}(r,0;f)+4\overline{N}(r,\infty;f)+2\overline{N}(r,b;f)+2\overline{N}(r,0;g)\\ +2\overline{N}(r,b;g)+\overline{N}(r,\infty;f &\mid \geq k+1)+S(r,f)+S(r,g) \qquad (8)\end{aligned}$$

Similarly we obtain

$$\begin{aligned}(n+1)T(r,g)+T(r,f) &\leq 4\overline{N}(r,0;g)+4\overline{N}(r,\infty;f)+2\overline{N}(r,b;g)+2\overline{N}(r,0;f)\\ +2\overline{N}(r,b;f)+\overline{N}(r,\infty;f &\mid \geq k+1)+S(r,f)+S(r,g) \qquad (9)\end{aligned}$$

Combining (8) and (9)we obtain from above for $\epsilon > 0$
$(n+2)\{T(r,f)+T(r,g)\}$

$$\leq 6\overline{N}(r,0;f)+8\overline{N}(r,\infty;f)+4\overline{N}(r,b;f)$$

$$+6\overline{N}(r,0;g)+4\overline{N}(r,b;g)+2\overline{N}(r,\infty;f \mid \geq k+1)+S(r,f)+S(r,g)$$

$$\leq 6\overline{N}(r,0;f)+4\overline{N}(r,b;f)+6\overline{N}(r,0;g)+4\overline{N}(r,b;g)$$

$$+\frac{16}{n-5}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{4n-12}{(n-5)[(n-2)k+(n-3)]}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}$$

$$+S(r,f)+S(r,g)$$

$\leq \{10 - 6\Theta(0; f) - 4\Theta(b, f) + \epsilon\}T(r, f) + \{10 - 6\Theta(0; g) - 4\Theta(b, g) + \epsilon\}T(r, g)$

$+ \{\frac{16}{n-5} + \frac{4n-12}{(n-5)[(n-2)k+(n-3)]}\}\{T(r, f) + T(r, g)\}.$

and hence

$\{3\Theta(0; f) + 2\Theta(b, f) - 4 - \frac{8}{n-5} - \frac{2n-6}{(n-5)[(n-2)k+(n-3)]} + \frac{n}{2} - \frac{\epsilon}{2}\}T(r, f)$

$+ \{3\Theta(0; g) + 2\Theta(b, g) - 4 - \frac{8}{n-5} - \frac{2n-6}{(n-5)[(n-2)k+(n-3)]} + \frac{n}{2} - \frac{\epsilon}{2}\}T(r, g) \leq S(r, f) + S(r, g), r \notin E$. This leads to a contradiction for arbitrary $\epsilon > 0$. Hence $H \equiv 0$. We do not prove the rest of the part of the

Theorem as it is same as the proof of the corresponding part of Theorem 2.

Proof of Theorem 2. Since $E_f(S, 2) = E_g(S, 2)$ according to the definitions of F and G we observe that F, G share $(1, 2)$. If possible suppose that $H \not\equiv 0$. Since $n \geq 6$, using Lemma 1 for $m = 2$ and Lemma 2 for $k = 0$ and Lemma 3 for $m = 2$ we obtain for $\epsilon > 0$ $(\frac{n}{2} + 1)\{T(r, f) + T(r, g)\}$

$\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, b; f) + \overline{N}(r, b; g)\} + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, b; f) + \overline{N}(r, b; g)\} + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, \infty; f \mid\geq k+1) - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, b; f) + \overline{N}(r, b; g)\} + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \frac{4n-12}{(3n-11)\{(n-2)k+n-3\}}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)] - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g)$

$\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, b; f) + \overline{N}(r, b; g)\} + \frac{1}{2}\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\}$

$+ \frac{1}{n-3}\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + \frac{4n-12}{(3n-11)\{(n-2)k+n-3\}}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)] + S(r, f) + S(r, g)$

$\leq (\frac{9}{2} - 2\Theta(0, f) - 2\Theta(b, f) - \frac{1}{2}\Theta(\infty, f) + \frac{1}{n-3} + \frac{4n-12}{(3n-11)\{(n-2)k+n-3\}} + \epsilon)T(r, f)$ $+ (\frac{9}{2} - 2\Theta(0, g) - 2\Theta(b, g) - \frac{1}{2}\Theta(\infty, g) + \frac{1}{n-3} + \frac{4n-12}{(3n-11)\{(n-2)k+n-3\}} + \epsilon)T(r, g).$

Thus $\{\Theta_f - (7 + \frac{2}{n-3} + \frac{8n-24}{(3n-11)\{(n-2)k+n-3\}} - n) - 2\epsilon\}T(r, f) + \{\Theta_g - (7 + \frac{2}{n-3} + \frac{8n-24}{(3n-11)\{(n-2)k+n-3\}} - n) - 2\epsilon\}T(r, g)$

$\leq S(r, f) + S(r, g)$ which is a contradiction. Hence $H \equiv 0$. Then

$$F \equiv \frac{AG + B}{CG + D} \tag{10}$$

where A, B, C, D are constants such that $AD - BC \neq 0$. Also $T(r, F) = T(r, G) + O(1)$, and hence from (4)

$$T(r, f) = T(r, g) + O(1) \ . \tag{11}$$

Since $R(w) - c = \frac{a(w-b)^3 Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}$, where $c = \frac{ab^{n-2}}{2} \neq 1, \frac{1}{2}$ and $Q_{n-3}(w)$ is a polynomial in w of degree $n-3$, then in view of the definitions of F and G we notice that

$$\begin{aligned} \overline{N}(r, c; F) &\leq \overline{N}(r, b; f) + (n-3)T(r, f) \leq (n-2)T(r, f) + S(r, f), \\ \overline{N}(r, c; G) &\leq \overline{N}(r, b; g) + (n-3)T(r, g) \leq (n-2)T(r, g) + S(r, g). \end{aligned} \tag{12}$$

Now we consider the following cases.

Case 1. $C \neq 0$.
Since f, g share (∞, ∞) it follows from (10) that ∞ is an exceptional value of Picard of f and g. Therefore in view of the definitions of F and G it follows that

$$\begin{aligned} \overline{N}(r, \infty; F) &= \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) \\ \overline{N}(r, \infty; G) &= \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g). \end{aligned} \tag{13}$$

Subcase 1.1 $A \neq 0$.

Suppose $B \neq 0$. Then from (10)it follows that $\overline{N}(r, -\frac{B}{A}; G) = \overline{N}(r, 0; F)$. Thus from the second main theorem we have from (4) and (13)

$$\begin{aligned} nT(r, g) &\leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, -\frac{B}{A}; G) + S(r, G) \\ &\leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, 0; f) + S(r, g) \end{aligned} \tag{14}$$

Clearly (14) leads to a contradiction if $n \geq 5$.
Therefore $B = 0$. Then $F \equiv \frac{\frac{A}{C}.G}{G + \frac{D}{C}}$ and $\overline{N}(r, \frac{-D}{C}; G) = \overline{N}(r, \infty; F)$. We also note that $c = \frac{ab^{n-2}}{2} \neq 0$. If possible suppose $c = \frac{-D}{C}$. Also suppose that F has no 1-points. This amounts to saying that f has no w_i-points where $w_i \in S$ and $i = 1, 2, .., n(\geq 4)$, which is not possible. Therefore F must have some 1-points. Since F, G share 1-points, we have $A = C + D = C - cC$ and hence $F = \frac{(C - cC)G}{CG - cC} = \frac{(1-c)G}{G - c}$, since $C \neq 0$ by our assumption. Then since $c \neq \frac{1}{2}$,$\overline{N}(r, c; F) = \overline{N}(r, \frac{c^2}{2c-1}; G)$. Thus by the second main theorem and (12) we have $2nT(r, g) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, c; G) + \overline{N}(r, \frac{c^2}{2c-1}; G) + S(r, g) \leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) + (n-2)T(r, f) + S(r, g) \leq (5 + n - 2)T(r, g) + S(r, g)$ which leads to a contradiction for $n \geq 4$.

Next let $c \neq \frac{-D}{C}$. Hence as before by the second main theorem$2nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{-D}{C};G) + \overline{N}(r,c;G) + S(r,G)$

$\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + (n-2)T(r,g) + S(r,g)$

$\leq (5+n-2)T(r,g) + S(r,g).$
which leads to a contradiction for $n \geq 4$.

Subcase 1.2 $A = 0$. Then clearly $B \neq 0$ and $F \equiv \frac{1}{\gamma G+\delta}$ where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$.
Since F and G have some 1-points, then $\gamma + \delta = 1$ and so $F \equiv \frac{1}{\gamma G+1-\gamma}$. Suppose $\gamma \neq 1$. If $\frac{1}{1-\gamma} \neq c$ then by second main theorem

$2nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,\frac{1}{1-\gamma};F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,F)$

$\leq \overline{N}(r,0;f) + (n-2)T(r,f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f)$
$\Rightarrow (n+2)T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f)$,which is a contradiction for $n \geq 4$.

If $c = \frac{1}{1-\gamma}$, then $F \equiv \frac{c}{(c-1)G+1}$. If $c \neq \frac{1}{1-c}$, then by the second main theorem we obtain

$2nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}(r,\frac{1}{1-c};G) + \overline{N}(r,\infty;G) + S(r,g) \leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\infty;F) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g)$

$\leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g).$

Thus $(n+2)T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g)$, which leads to a contradiction for $n \geq 4$.

If $c = \frac{1}{1-c}$ then $G \equiv \frac{c(F-c)}{F}$ and as above we obtain

$nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,f)$

$\leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f).$

Above leads to a contradiction for $n \geq 5$. Therefore we must have $\gamma = 1$ and hence $FG \equiv 1$, which is impossible by lemma 9.

Case 2.$C = 0$.

Clearly $A \neq 0$ and $F \equiv \alpha G + \beta$, where $\alpha = \frac{A}{D}, \beta = \frac{B}{D}$. Since F and G must have some 1-points, $\alpha + \beta = 1$ and so $F \equiv \alpha G + 1 - \alpha$. Suppose $\alpha \neq 1$. If $1 - \alpha \neq c$, then by the second main theorem and (12) we obtain

$2nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1-\alpha;F) + S(r,f)$

$\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + (n-2)T(r,f) + \overline{N}(r,0;G) + S(r,f)$.

Thus$(n+2)T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,0;g) + S(r,f)$ which leads to a contradiction for $n \geq 4$.
If $1 - \alpha = c$,then $F \equiv (1-c)G + c$.Since $c \neq 1$we obtain from the second main theorem and (12)
$2nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\frac{c}{c-1};G) + S(r,g)$

$\leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\infty;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,0;F) + S(r,g)$.
Thus $(n+2)T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + S(r,f)$
which leads to a contradiction for $n \geq 4$. So $\alpha = 1$. Hence $F \equiv G$ and therefore by Lemma 11, $f \equiv g$. This completes the proof.

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