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SMARANDACHE $\mathbf{\Pi}_1\mathbf{\Pi}_2$ CURVES OF BIHARMONIC NEW TYPE CONSTANT $\mathbf{\Pi}_2-$SLOPE CURVES ACCORDING TO TYPE-2 BISHOP FRAME IN THE SOL SPACE $\mathfrak{SOL}^3$

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Abstract. In this paper, we study Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2$ curves of biharmonic new type constant $\mathbf{\Pi}_2-$ slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$.

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1. Introduction

Let γ be a unit speed regular curve in $\mathfrak{SOL}^3$ and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{\Pi}_1 &= -\epsilon_1\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{\Pi}_2 &= -\epsilon_2\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \epsilon_1\mathbf{\Pi}_1 + \epsilon_2\mathbf{\Pi}_2, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{B}) &= 1,\ g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_1) = 1,\ g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_2,\mathbf{\Pi}_2) = 1, \\ g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_1) &= g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_2) = g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_2) = 0. \end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet–Serret frame, first we write

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}.$$

The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\begin{aligned}
\mathbf{T} &= \sin \mathfrak{A}(s)\,\mathbf{\Pi}_1 - \cos \mathfrak{A}(s)\,\mathbf{\Pi}_2,\\
\mathbf{N} &= \cos \mathfrak{A}(s)\,\mathbf{\Pi}_1 + \sin \mathfrak{A}(s)\,\mathbf{\Pi}_2,\\
\mathbf{B} &= \mathbf{B}.
\end{aligned}$$

So by Frenet–Serret frame, we may express

$$\begin{aligned}
\epsilon_1 &= -\tau \cos \mathfrak{A}(s),\\
\epsilon_2 &= -\tau \sin \mathfrak{A}(s).
\end{aligned}$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathfrak{A}(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve γ. We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \epsilon_1, \epsilon_2\}$ as type-2 Bishop invariants of the curve γ, [22].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned}
\mathbf{\Pi}_1 &= \pi_1^1\mathbf{e}_1 + \pi_1^2\mathbf{e}_2 + \pi_1^3\mathbf{e}_3,\\
\mathbf{\Pi}_2 &= \pi_2^1\mathbf{e}_1 + \pi_2^2\mathbf{e}_2 + \pi_2^3\mathbf{e}_3.\\
\mathbf{B} &= B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3,
\end{aligned}$$

Theorem 1. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic new type constant $\mathbf{\Pi}_2-$slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the parametric equations of γ are*

$$\begin{aligned}
\boldsymbol{x}(s) &= \int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3}[\sin[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]\\
&\quad -\cos[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]ds, \qquad (1.2)\\
\boldsymbol{y}(s) &= \int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3}[\sin[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]\\
&\quad -\cos[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]ds,\\
\boldsymbol{z}(s) &= \frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3,
\end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are constants of integration.

2. Smarandache $\Pi_1\Pi_2$ Curves of Biharmonic Constant Π_2-Slope Curves according to New Type-2 Bishop Frame in Sol Space

Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed curve with constant curvatures in the Sol Space $\mathfrak{SOL}^3$ and $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ be its moving type-2 Bishop frame. Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2$ curves are defined by

$$\gamma_{\mathbf{\Pi}_1\mathbf{\Pi}_2} = \frac{1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}} \left(\mathbf{\Pi}_1 + \mathbf{\Pi}_2\right). \tag{2.1}$$

Theorem 2. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_2-$slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the equation of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2$ curves of biharmonic constant $\mathbf{\Pi}_2-$slope curves is given by*

$$\begin{aligned} \gamma_{\mathbf{\Pi}_1\mathbf{\Pi}_2}(s) &= \frac{1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}[\sin \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2] + \cos \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_1 \\ &+ \frac{1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}[\sin \mathfrak{E} \sin [\mathcal{R}_1 s + \mathcal{R}_2] + \cos \mathfrak{E} \sin [\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_2 \\ &+ \frac{1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}[\cos \mathfrak{E} - \sin \mathfrak{E}]\mathbf{e}_3, \end{aligned} \tag{2.2}$$

where $\mathcal{R}_1, \mathcal{R}_2$ are constants of integration.

Proof. We suppose that γ is a unit speed non-geodesic biharmonic new type-2 constant $\mathbf{\Pi}_2-$slope curve. Then,

$$\mathbf{\Pi}_2 = \sin \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2]\, \mathbf{e}_1 + \sin \mathfrak{E} \sin [\mathcal{R}_1 s + \mathcal{R}_2]\, \mathbf{e}_2 + \cos \mathfrak{E}\mathbf{e}_3, \tag{2.3}$$

where $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}$.

Then by type-2 Bishop formulas (2.1) and (1.1), we have

$$\mathbf{\Pi}_1 = \cos \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2]\, \mathbf{e}_1 + \cos \mathfrak{E} \sin [\mathcal{R}_1 s + \mathcal{R}_2]\, \mathbf{e}_2 - \sin \mathfrak{E}\mathbf{e}_3. \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1) we have (2.2), which completes the proof.

In terms of Eqs. (2.1) and (2.2), we may give:

Theorem 3. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_2-$slope curve according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the parametric equations of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2$ curve of biharmonic constant $\mathbf{\Pi}_2-$slope curve are given by*

$$x_{\mathbf{\Pi}_1\mathbf{\Pi}_2}(s) = \frac{e^{-\frac{1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\cos \mathfrak{E} - \sin \mathfrak{E}]}}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}[\sin \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2] + \cos \mathfrak{E} \cos [\mathcal{R}_1 s + \mathcal{R}_2]],$$

$$y_{\mathbf{\Pi}_1\mathbf{\Pi}_2}(s) = \frac{e^{\frac{1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\cos\mathfrak{E}-\sin\mathfrak{E}]}}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]+\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]],$$

$$z_{\mathbf{\Pi}_1\mathbf{\Pi}_2}(s) = \frac{1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\cos\mathfrak{E}-\sin\mathfrak{E}],$$

where $\mathcal{R}_1, \mathcal{R}_2$ *are constants of integration.*

Proof. Omitted.

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