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SUFFICIENT CONDITION FOR A CLASS OF ANALYTIC AND UNIVALENT FUNCTIONS OF COMPLEX ORDER

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Abstract. We establish a sufficient condition for a class $UCD(\gamma, \lambda, m, n, l, k, b)$ of complex order. Also, a subordination theorem and coefficient bounds for this class of functions are determined.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = f_z(0) - 1 = 0$.

A function $f(z)$ of the form (1) belongs to the $UCD(\gamma)$, $\gamma \geq 0$, if $Ref'(z) \geq \gamma |zf''(z)|$, $z \in U$ [3] and [14].

Furthermore a function $f(z)$ of the form (1) belongs to the class $UCD(\gamma, \lambda, m, n, l, k, b)$,
$\gamma \geq 0,\ \ l \geq 0, \lambda \geq 0\ \ m \in N$ of complex order $b \neq 0$ $(b \in \mathcal{C})$ if

$$Re\left[1 - \frac{2}{b} + \frac{2}{b}\frac{\Phi_n^{m+1}(\lambda, 1)f(z)}{\Phi_n^m(\lambda, l)f(z)}\right] \leq \beta(\gamma) \tag{2}$$

where all the parameters are as earlier defined and $\Phi_n^m(\lambda, l) := A \to A$ and

$$\Phi_n^m(\lambda, l)f(z) = D^m(\lambda, l)f(z) * R^n f(z)$$

where $D^m(\lambda,l)f(z)$ is the Catal et al derivative operator[5, 6, 7,10,11, 12] defined as follows:

$D^0(\lambda,l)f(z) = f(z) = z + \sum_{k=2}^{\infty} a_k z^k$

$D^{''}(\lambda,l)f(z) = D(\lambda,l)f(z) = D^0(\lambda,l)f(z)(\frac{1-\lambda+l}{1+l})f(D^0(\lambda,l)f(z))'\frac{\lambda z}{1+l}$

$= z + \sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})a_k z^k$

and in general

$$D^m(\lambda,l) = D(D^{m-1}(\lambda,l)f(z) = z + \sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})^m a_k z^k$$

With different choices of the parametrics derivative operators associated to Salagean [13] and Al- Oboudi [1] and the classes of functions studied by Oladipo in [10] could be derived.

Also, R^n is the Ruscheweyh derivative operartor and it is defined as

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k)a_k z^k,$$

where

$$c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$$

Therefore, $\Phi_n^m(\lambda,l)f(z)$ can be written as

$$\Phi_n^m(\lambda,l)f(z) = z + \sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})^m c(n,k)a_k z^k, \quad z \in U \tag{3}$$

Let $p(z)$ (class of Caratheodory functions) be analytic and such that $Rep(z) > 0$ and $p(0) = 1$. $p(z)$ is written in the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{4}$$

and that $|p_k| \leq 2. \quad k \geq 1$

Furthermore, let $f(z)$ and $g(z)$ be analytic functions in U, $f(z)$ is said to be subordinate to $g(z)$ in U, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwartz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) \quad (z \in U)$

It is well known that, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(u) \subset g(u) \quad$ (cf [4,13]).

2. Coefficient Inequalities

Theorem 1. *Let* $F(z) = z+\sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)a_k z^k$*, where* $F(z) = \Phi_n^m(\lambda,l)f(z)$*. If* $F(z)$ *belongs to the class* $UCD(\gamma,\lambda,m,n,l,k,b)$ *of complex order* b $(b \in \mathbb{C}\setminus\{0\})$*, then*

$$\sum_{k=2}^{\infty} k(\gamma(k-1)+b)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)\,|a_k| \le b$$

and all other parameters remained as earlier defined.

Proof. Suppose $F(z) \in UCD(\gamma,\lambda,m,n,l,k,b)$, then

$$\left|\frac{1}{b}z\frac{F''(z)}{F'(z)}\right| = \left|\frac{\frac{1}{b}\sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)k(k-1)a_k z^{k-1}}{1+\sum_{n=2}^{\infty} k a_k(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)z^{k-1}}\right|$$

Since $F(z) \in UCD(\gamma,\lambda,m,n,l,k,b)$,

$$Re(\frac{1}{b}z\frac{F''(z)}{F'(z)}) \le \frac{1}{\gamma} \Rightarrow \left|\frac{1}{b}z\frac{F''(z)}{F'(z)}\right| \le \frac{1}{\gamma}.$$

Therefore,
$\left|\frac{\frac{1}{b}\sum_{k=2}^{\infty} k(k-1)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)a_k z^{k-1}}{1+\sum_{k=2}^{\infty} k(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)a_k z^{k-1}}\right|$

$\le \frac{\frac{1}{b}\sum_{k=2}^{\infty} k(k-1)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)|a_k|}{1-\sum_{k=2}^{\infty} k(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)|a_k|} \le \frac{1}{\gamma}$

$\gamma\sum_{k=2}^{\infty} k(k-1)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)\,|a_k|$

$\le b-\sum_{k=2}^{\infty} bk(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)\,|a_k|$

$\sum_{k=2}^{\infty}(\gamma k(k-1)+bk)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)\,|a_k| \le b$

$\sum_{k=2}^{\infty} k(\gamma(k-1)+b)(\frac{1+\lambda(k-1)+l}{1+l})^m C(n,k)\,|a_k| \le b$

Corollary 2. *If* $b=1$*,* $m=0$*,* $C(n,k)=1$ *then* $f \in UCD(\gamma)$ *i.e.,*

$$\sum_{k=2}^{\infty} k(\gamma(k-1)+1)\,|a_k| \le 1.$$

Corollary 3. *If* $b = 1$*,* $\gamma = 0$*,* $m = 0$ $C(n,k) = 1$*, then* f *is starlike. i.e.,* $\sum_{k=2}^{\infty} k\,|a_k| \le 1$*.*

Corollary 4. *If* $b = 1$, $\gamma = 1$, $m = 0$ $C(n,k) = 1$, *then* f *is convex i.e.,* $\sum_{k=2}^{\infty} k^2 |a_k| \leq 1$.

Theorem 5. *Let* $f \in \mathcal{A}$ *and suppose*

$$G(z,t) = \frac{1}{b} f(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) t f(ze^t) \prec f(z)$$

and

$$Re\left[lim_{t \to 0} \frac{G_{tt}(z,t)}{zf'(z)}\right] \leq \beta(\gamma)$$

then $f \in UCD(\gamma)$ *of complex order* b.

Proof. Suppose $G \prec f$ then $G(0,0) = f(0) = 0$

$G(U) = G(z,0) = \frac{1}{b} f(z) \subset f(z) \Rightarrow G(U) \subset f(U)$

$G(0,t) \equiv 0$

Now, $G(z,t) = \frac{1}{b} f(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) t f(ze^t)$

$G_t(z,t) = \frac{1}{b} f'(ze^t)(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) f(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) t f'(ze^t)(ze^t)$

$G_{tt}(z,t) = \frac{1}{b} f''(ze^t)(ze^t)^2 + \frac{1}{b} f'(ze^t)(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) f'(ze^t)(ze^t)$
$+ \frac{1}{2}(1 - \frac{1}{b}) f'(ze^t)(ze^t) + \frac{1}{2}(1 - \frac{1}{b}) t f''(ze^t)(ze^t)^2 + \frac{1}{2}(1 - \frac{1}{b}) t f'(ze^t)(ze^t)$

$\lim_{t \to 0} \frac{G_{tt}(z,t)}{zf'(z)} = \frac{1}{b} \frac{f''(z)z^2}{zf'(z)} + \frac{1}{b} \frac{f'(z)z}{zf'(z)} + \frac{1}{2}(1 - \frac{1}{b}) \frac{f'(z)^2}{zf'(z)} + \frac{1}{2}(1 - \frac{1}{b}) \frac{f'(z)}{zf'(z)}$

$= \frac{1}{b} \frac{f''(z)}{f'(z)} + \frac{1}{b} + \frac{1}{2}(1 - \frac{1}{b}) + \frac{1}{2}(1 - \frac{1}{b})$

$= \frac{1}{b} z \frac{f''(z)}{f'(z)} + 1$

Therefore, $Re\left[\lim_{t \to 0} \frac{G_{tt}(z,t)}{zf'(z)}\right] = Re\left[\frac{1}{b} z \frac{f''(z)}{f'(z)} + 1\right]$

Hence,

$$Re\left[1 + \frac{1}{b} \frac{zf''(z)}{zf'(z)}\right] \leq \beta(\gamma)$$

This gives the desired result.

3. Coefficient bounds

Our next result is on coefficient bounds for the function in the class $UCD(\gamma,\lambda,m,n,k,b,l)$

Theorem 6. *Let $f(z)$ belong to the class $UCD(\gamma,\lambda,m,n,k,b,l)$ of compleex order b $(b\in\mathbb{C}\{0\})$, then*

$$|a_2|\leq\frac{b}{(\frac{1+\lambda+l}{1+l})^m\left[\gamma(\frac{1+\lambda+1}{1+l})-\gamma-b\right]C(n,2)}$$

$$|a_3|\leq\frac{b}{(\frac{1+2\lambda+l}{1+l})^m\left[\gamma(\frac{1+2\lambda+1}{1+l})-\gamma-b\right]C(n,3)}$$

$$|a_4|\leq\frac{b\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]+b^2}{(\frac{1+3\lambda+1}{1+l})^m\left[\gamma(\frac{1+3\lambda+l}{1+l})-\gamma-b\right]\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]C(n,4)}$$

Proof. Let $\frac{1-\frac{2}{b}+\frac{2}{b}\frac{\phi^{m+1}(\lambda,l)f(z)}{\phi^m(\lambda,l)f(z)}-\beta}{\beta(\gamma)-1}=p(z)$

$b\phi^m(\lambda,l)f(z)-2\phi^m(\lambda,l)f(z)+2\phi^{m+1}(\lambda,l)f(z)-\beta b\phi^m(\lambda,l)f(z)=(\beta(\gamma)-1)b\phi^m(\lambda,l)f(z)(p(z))$

$$2\left[\phi^{m+1}(\lambda,l)f(z)-\phi^m(\lambda,l)f(z)\right]=b(\beta(\gamma)-1)\phi^m(\lambda,l)f(z)\left[p(z)+1\right]$$

$$2\left[\phi^{m+1}(\lambda,l)f(z)-\phi^m(\lambda,l)f(z)\right]=b(\beta(\gamma)-1)\phi^m(\lambda,l)f(z)\left[2+\sum_{k=1}^{\infty}p_kz^k\right]$$

$$2\left[\phi^{m+1}(\lambda,l)f(z)-\phi^m(\lambda,l)f(z)\right]=\frac{b}{\gamma}\phi^m(\lambda,l)f(z)\left[2+\sum_{k=1}^{\infty}p_kz^k\right]$$

$$2\gamma\left[\phi^{m+1}(\lambda,l)f(z)-\phi^m(\lambda,l)f(z)\right]=b\phi^m(\lambda,l)f(z)\left[2+\sum_{k=1}^{\infty}p_kz^k\right]$$

$$2\gamma\left[\sum_{k=2}^{\infty}(\tfrac{1+\lambda(k-1)+l}{1+l})^{m+1}C(n,k)-\sum_{k=2}^{\infty}(\tfrac{1+\lambda(k-1)+l}{1+l})^{m}C(n,k)\right]a_kz^k$$
$$-b\phi^m(\lambda,l)f(z)\sum_{k=1}^{\infty}p_kz^k=2b\phi^m(\lambda,l)f(z)$$

$$2\gamma(\tfrac{1+\lambda+l}{1+l})^{m+1}C(n,2)a_2z^2+(\tfrac{1+2\lambda+l}{1+l})^{m+1}C(n,3)a_3z^3+(\tfrac{1+3\lambda+l}{1+l})^{m+1}C(n,4)a_4z^2$$
$$+(\tfrac{1+4\lambda+l}{1+l})^{m+1}C(n,5)a_5z^5+\ldots-(\tfrac{1+\lambda+l}{1+l})^{m}C(n,2)a_2z^2-(\tfrac{1+2\lambda+l}{1+l})^{m}C(n,3)a_3z^3$$
$$-(\tfrac{1+3\lambda+l}{1+l})^{m}C(n,4)a_4z^4-(\tfrac{1+4\lambda+l}{1+l})^{m}C(n,5)a_5z^5-\ldots-b\left[z+\sum_{k=2}^{\infty}(\tfrac{1+\lambda(k-1)+l}{1+l})^{m}a_kz^k\right]\cdot$$
$$\sum_{k=1}^{\infty}p_kz^k=2b\left[z+\sum_{k=2}^{\infty}(\tfrac{1+\lambda(k-1)+l}{1+l})^{m}C(n,k)a_kz^k\right]$$

$$2\gamma\left[(\tfrac{1+\lambda+l}{1+l})^{m+1}C(n,2)a_2-(\tfrac{1+\lambda+l}{1+l})^{m}C(n,2)a_2\right]-bp_1=2b(\tfrac{1+\lambda+l}{1+l})^{m}C(n,2)a_2$$

$$2\gamma\left[(\tfrac{1+\lambda+l}{1+l})^{m+1}C(n,2)a_2-(\tfrac{1+\lambda+l}{1+l})^{m}C(n,2)a_2\right]-2b(\tfrac{1+\lambda+l}{1+l})^{m}C(n,2)a_2=bp_1$$

$$a_2=\frac{bp_1}{2(\frac{1+\lambda+l}{1+l})^m\left[\gamma(\frac{1+\lambda+1}{1+l})-\gamma-b\right]^{n+1}}$$

$|a_2| \leq \frac{2b}{2(\frac{1+\lambda+l}{1+l})^m\left[\gamma(\frac{1+\lambda+1}{1+l})-\gamma-b\right]C(n,2)}$

$2\gamma\left[(\frac{1+2\lambda+l}{1+l})^{m+1}C(n,3)a_3-(\frac{1+2\lambda+l}{1+l})^m C(n,3)a_3\right]-bp_2=2b(\frac{1+2\lambda+l}{1+l})^m C(n,3)a_3$

$2\gamma\left[(\frac{1+2\lambda+l}{1+l})^{m+1}C(n,3)a_3-(\frac{1+2\lambda+l}{1+l})^m C(n,3)a_3\right]-2b(\frac{1+2\lambda+l}{1+l})^m C(n,3)a_3=bp_2$

$a_3=\frac{bp_2}{2(\frac{1+2\lambda+l}{1+l})^m\left[\gamma(\frac{1+2\lambda+1}{1+l})-\gamma-b\right]^{n+1}}$

$|a_3| \leq \frac{b}{(\frac{1+2\lambda+l}{1+l})^m\left[\gamma(\frac{1+2\lambda+1}{1+l})-\gamma-b\right]C(n,3)}$

$2\gamma\left[(\frac{1+3\lambda+l}{1+l})^{m+1}C(n,4)a_4-(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4\right]-bp_3-b(\frac{1+\lambda+l}{1+l})^m C(n,2)p_1a_2=2b(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4$

$2\gamma\left[(\frac{1+3\lambda+l}{1+l})^{m+1}C(n,4)a_4-(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4\right]-2b(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4-\frac{b^2p_1^2}{2\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]}=bp_3$

$2\gamma\left[(\frac{1+3\lambda+l}{1+l})^{m+1}C(n,4)a_4-(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4\right]-2b(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4=bp_3+\frac{b^2p_1^2}{2\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]}$

$2\gamma\left[(\frac{1+3\lambda+l}{1+l})^{m+1}C(n,4)a_4-(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4\right]-2b(\frac{1+3\lambda+l}{1+l})^m C(n,4)a_4=\frac{2bp_3\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]+b^2p_1^2}{2\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]}$

$2(\frac{1+3\lambda+l}{1+l})^m\left[\gamma(\frac{1+3\lambda+l}{1+l})-\gamma-b\right]C(n,4)a_4=\frac{2bp_3\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]+b^2p_1^2}{2\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]}$

$a_4=\frac{2bp_3\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]+b^2p_1^2}{4(\frac{1+3\lambda+l}{1+l})^m\left[\gamma(\frac{1+3\lambda+1}{1+l})-\gamma-b\right]\left[\gamma(\frac{1+\lambda+1}{1+l})-\gamma-b\right]C(n,4)}$

$|a_4| \leq \frac{b\left[\gamma(\frac{1+2\lambda+l}{1+l})-\gamma-b\right]+b^2}{(\frac{1+3\lambda+1}{1+l})^m\left[\gamma(\frac{1+3\lambda+l}{1+l})-\gamma-b\right]\left[\gamma(\frac{1+\lambda+l}{1+l})-\gamma-b\right]C(n,4)}$

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