Acta Universitatis Apulensis
ISSN: 1582-5329 No. 36/2013
pp. 203-208

ON A FAMILY OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper, we introduce and study a new subclass $M(w, \alpha, \beta, \gamma)$ of meromorphically univalent functions with positive coefficients. We first obtained a necessary and sufficient conditions for a function to be in the class $M(w, \alpha, \beta, \gamma)$, we then investigated the convex combination of certain meromorphic functions as well as the distortion and convolution properties.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: Meromorphic, Distortion, Convolution.

1. Introduction

Let M_p denote the class of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \ (a_n \geq 0, p \in N = \{1, 2, 3, ...\}) \tag{1}$$

which are analytic and univalent in the punctured unit disk

$$D = \{z : z \in C \ and \ 0 < |z| < 1\}$$

and which have a simple pole at the origin ($z = 0$) with residue 1 there. Altintas et al [1] defines the function $M(p, \alpha, \beta)$ as the function $f(z) \in M_p$ satisfying the inequality:

$$\text{Re}\{zf(z) - \alpha z^2 f'(z)\} > \beta \tag{2}$$

for some $\alpha(\alpha > 1)$ and $\beta(0 \leq \beta < 1)$, for all $z \in D$. Other subclasses of the class M_p were studied recently by Cho et al [2,3]
Let $A_w(z)$ denote the set of functions analytic in D given by

$$f(z) = \frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \tag{3}$$

which have a simple pole at $(z = w)$ with residue $1-\alpha$, $0 \leq \alpha < 1$ there, $z_1 \in D$ and w is an arbitrary fixed point in D.
Firas Ghanim and Maslina Darus obtained various properties of the function of the form

$$f(z) = \frac{1}{z-p} + \sum_{n=1}^{\infty} a_n z^n$$

with fixed second coefficient. We define the function $f(z)$ in $A_w(z)$ to be in the class $M(w, \alpha, \beta, \gamma)$ if it satisfies the inequality:

$$\text{Re}\{(z-w)f(z) - \beta(z-w)^2 f'(z)\} > \gamma \tag{4}$$

for some $\beta(\beta > 1)$, $\alpha(\alpha < 1)$ and $\gamma(0 \leq \gamma < 1)$, for all $z, w \in D$.
The purpose of this paper is to investigate some properties of the functions belonging to the class $M(w, \alpha, \beta, \gamma)$.

2. Main Results

Theorem 1. *Let the function $f(z)$ be in the class $A_w(z)$. Then $A_w(z)$ belong to the class $M(w, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{n=1}^{\infty} (n\beta - 1)a_n \leq (1-\alpha)(1+\beta) - \gamma \tag{5}$$

Proof. Let $f(z)$ be as in (3), suppose that

$$f(z) \in M(w, \alpha, \beta, \gamma)$$

Then we have from (4) that

$$\text{Re}\{(z-w)(\frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_n(z-w)^n) - \beta(z-w)^2(-\frac{1-\alpha}{(z-w)^2} + \sum_{n=1}^{\infty} na_n(z-w)^{n-1})\}$$

$$= \text{Re}\{1-\alpha + \sum_{n=1}^{\infty} a_n(z-w)^{n+1} + \beta(1-\alpha) - \sum_{n=1}^{\infty} \beta n a_n(z-w)^{n+1}\}$$

$$= \text{Re}\{(1-\alpha)(1+\beta) - \sum_{n=1}^{\infty} (n\beta - 1)a_n(z-w)^{n+1}\} > \gamma \quad (z, w \in D)$$

If we choose $z - w$ to be real and let $z - w \to 1^-$, we get

$$(1-\alpha)(1+\beta) - \sum_{n=1}^{\infty} (n\beta - 1)a_n \geq \gamma \quad (\beta > 1; 0 \leq \gamma < 1)$$

which is equivalent to (5) Conversely, let us suppose that the inequality (5) holds. Then we have:

$$|(z-w)f(z)-\beta(z-w)^2f'(z)-(1-\alpha)(1+\alpha)|$$
$$=|-\sum_{n=1}^{\infty}(n\beta-1)a_n(z-w)^{n+1}|\leq\sum_{n=1}^{\infty}(n\beta-1)a_n|(z-w)|^{n+1}$$
$$\leq(1-\alpha)(1+\beta)-\gamma,\qquad(z,w\in D,\ \beta>1,\ 0\leq\alpha<1,\ 0\leq\gamma<1)$$

which implies that $f(z)\in M(w,\alpha,\beta,\gamma)$ Finally, we note that the assertion (5) of theorem 1 is sharp, the external function being:

$$f(z)=\frac{1-\alpha}{z-w}+\frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}z^n,\qquad n\in N=\{1,2,3,...\}$$

Corollary 2. *Let $f(z)$ be in $A_w(z)$. If $f(z)\in M(w,\alpha,\beta,\gamma)$, then*

$$a_n\leq\frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}\quad n\geq1$$

Theorem 3. *Let the function $f(z)$ be in $A_w(z)$ and the function $g(z)$ defined by*

$$\frac{1-\alpha}{z-w}+\sum_{n=1}^{\infty}b_n(z-w)^n,\quad b_n\geq0\tag{6}$$

be in the same class $M(w,\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=\frac{1-\alpha}{z-w}+\sum_{n=1}^{\infty}c_n(z-w)^n$$

also in the class $M(w,\alpha,\beta,\gamma)$, where $c_n=(1-\lambda)a_n+\lambda b_n$; $\quad 0\leq\lambda\leq1$

Proof. Suppose that each of $f(z)$, $g(z)$ is in the class $M(w,\alpha,\beta,\gamma)$ hen by (5) we have

$$\begin{aligned}\sum_{n=1}^{\infty}(n\beta-1)c_n &= \sum_{n=1}^{\infty}(n\beta-1)[(1-\lambda)a_n+\lambda b_n]\\ &= (1-\lambda)\sum_{n=1}^{\infty}(n\beta-1)a_n+\sum_{n=1}^{\infty}(n\beta-1)b_n\\ &\leq (1-\lambda)(1-\alpha)(1+\beta)-\gamma+\lambda)(1-\alpha)(1+\beta)-\gamma\\ &= (1-\alpha)(1+\beta)-\gamma\ \ (0\leq\alpha<1,\ \beta>1,\ 0\leq\gamma,0\leq\lambda\leq1)\end{aligned}$$

This completes the proof of Theorem 2.

Distortion Theorem

Theorem 4. *Let $f(z) \in M(w, \alpha, \beta, \gamma)$, then*

$$\frac{1-\alpha}{|z-w|} - \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}|z-w|^n \leq |f(z)| \leq |f(w)| \leq$$
$$\leq \frac{1-\alpha}{|z-w|} + \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}|z-w|^n \quad (7)$$

The result is sharp for the function $f(z)$ given by

$$|f(z)| = \frac{1-\alpha}{|z-w|} - \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}|z-w|^n \leq |, \ \ 0 \leq \alpha < 1, \ \beta > 1, \ 0 \leq \gamma < 1, \ n \in N \quad (8)$$

Proof. Since

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}, \quad 0 \leq \alpha < 1, \ \beta > 1, \ 0 \leq \gamma < 1, \ n \in N \quad (9)$$

and

$$\sum_{n=1}^{\infty} n a_n \leq n[\frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}], \quad 0 \leq \alpha < 1, \ \beta > 1, \ 0 \leq \gamma < 1, \ n \in N \quad (10)$$

For $|f(z)| \in M(w, \alpha, \beta, \gamma)$, we have

$$\begin{aligned} |f(z)| &\geq \frac{1-\alpha}{|z-w|} - |z-w|^n \sum_{n=1}^{\infty} a_n \\ &\geq \frac{1-\alpha}{|z-w|} - \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}|z-w|^n \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\leq \frac{1-\alpha}{|z-w|} + |z-w|^n \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1-\alpha}{|z-w|} + \frac{(1-\alpha)(1+\beta)-\gamma}{n\beta-1}|z-w|^n \end{aligned}$$

which complete the proof of the theorem 3.

Theorem 5. *Let $f(z) \in M(w, \alpha, \beta, \gamma)$, then*

$$\frac{1-\alpha}{|z-w|^2} - \frac{n[(1-\alpha)(1+\beta)-\gamma]}{n\beta - 1}|z-w|^{n-1} \leq |f'(z)| \leq |f(w)| \leq$$
$$\leq \frac{1-\alpha}{|z-w|^2} + \frac{n[(1-\alpha)(1+\beta)-\gamma]}{\beta - 1}|z-w|^{n-1}$$

Proof. We find from (3) and (10) that

$$\begin{aligned} |f'(z)| &\geq \frac{1-\alpha}{|z-w|^2} - n|z-w|^{n-1}\sum_{n=1}^{\infty} a_n \\ &\geq \frac{1-\alpha}{|z-w|^2} - \frac{n[(1-\alpha)(1+\beta)-\gamma]}{n\beta - 1}|z-w|^{n-1} \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\leq \frac{1-\alpha}{|z-w|^2} + |z-w|^{n-1}\sum_{n=1}^{\infty} a_n \\ &\geq \frac{1-\alpha}{|z-w|^2} + \frac{n[(1-\alpha)(1+\beta)-\gamma]}{n\beta - 1}|z-w|^{n-1} \end{aligned}$$

which complete the proof of the theorem 4.

Convolution Properties

Let the convolution of two complex-valued meromorphic functions

$$f_1(z) = \frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_{n1}(z-w)^n \text{ and } f_1(z) = \frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_{n1}(z-w)^n$$

be defined by

$$F(z) = (f_1(z) * f_2(z)) = (f_1 * f_2)(z)\frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_{n1}a_{n2}(z-w)^n$$

Theorem 6. *Let the function $F(z)$ be in the class $A_w(z)$. Then $F(z)$ belong to the class $M(w, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{n=1}^{\infty}(n\beta - 1)a_{n1}a_{n2} \leq (1-\alpha)(1+\beta) - \gamma$$

Proof. Following the procedure inthe proof of the Theorem 1, we obtain the result.

Theorem 7. *Let the function $F(z)$ be in the class $A_w(z)$ and the function $G(z)$ be defined by*
$\frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} b_{n1}b_{n2}(z-w)^n, \quad b_{n1}b_{n2} \geq 0$
$\frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} a_{n1}a_{n2}(z-w)^n$ *be in the same class $M(w,\alpha,\beta,\gamma)$. Then the function $H(z)$ defined by*

$$H(z) = (1-\lambda)F(z) + \lambda G(z) = \frac{1-\alpha}{z-w} + \sum_{n=1}^{\infty} c_n(z-w)^n$$

is also in the class $M(w,\alpha,\beta,\gamma)$, where $c_n = (1-\lambda)a_{n1}a_{n2} + \lambda b_{n1}b_{n2}; \quad 0 \leq \lambda \leq 1$

Proof. Suppose that each of $F(z)$, $G(z)$ is in the class $M(w,\alpha,\beta,\gamma)$. Then by (5) we have

$$\begin{aligned} \sum_{n=1}^{\infty}(n\beta-1) &= \sum_{n=1}^{\infty}(n\beta-1)[(1-\lambda)a_{n1}a_{n2} + \lambda b_{n1}b_{n2}] \\ &= (1-\lambda)\sum_{n=1}^{\infty}(n\beta-1)a_{n1}a_{n2} + \lambda\sum_{n=1}^{\infty}(n\beta-1)b_{n1}b_{n2}] \\ &\leq (1-\lambda)(1-\alpha)(1+\beta) - \gamma + \lambda(1-\alpha)(1+\beta) - \gamma \\ &= (1-\alpha)(1+\beta) - \gamma \ \ (0 \leq \alpha < 1,\ \beta > 1,\ 0 \leq \gamma < 1,\ 0 \leq \lambda \leq 1) \end{aligned}$$

This completes the proof of Theorem 6.

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