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# ON CERTAIN SUBCLASSES OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING THE CATAS OPERATOR

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Abstract. We introduce a certain subclass of multivalent analytic functions by making use of Cătas operator. Further, we determine coefficient estimates, distortion bounds, radii of starlikeness and convexity for the analytic functions belong to the class. Also, subordination theorems and integral means inequalities of functions $f$ in the class $T\mathcal{S}(\alpha, \beta, p, n)$ are obtained.

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## 1. Introduction

Let $\mathcal{A}(p, n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n, p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disc $\mathcal{U} = \{z \ : \ z \in \mathbb{C}, \ |z| < 1\}$. We write $\mathcal{A}(1, 1) = \mathcal{A}$.

A function $f \in \mathcal{A}(p, n)$ is said to be in the class $S(p, n, \alpha)$ of $p-$valently star-like functions of order $\alpha$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U}; \ 0 \leq \alpha < p). \tag{2}$$

Furthermore, a function $f \in \mathcal{A}(p, n)$ is said to be in the class $K(p, n, \alpha)$ of $p-$valently convex functions of order $\alpha$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U}; \ 0 \leq \alpha < p). \tag{3}$$

The classes $S(p, n, \alpha)$ and $K(p, n, \alpha)$ were studied by Owa [12]. The class $S^*(p, \alpha) := S(p, 1, \alpha)$ was considered by Patil and Thakare [14].

We denote by $T(p, n)$ the subclass of the class $\mathcal{A}(p, n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0;\ n, p \in \mathbb{N} = \{1, 2, \dots\}) \tag{4}$$

and define two further classes $T^*(p, n, \alpha)$ and $C(p, n, \alpha)$ by

$$T^*(p, n, \alpha) := S(p, n, \alpha) \cap T(p, n), \quad C(p, n, \alpha) := K(p, n, \alpha) \cap T(p, n).$$

Further the classes $T^*(p, \alpha) := S^*(p, \alpha) \cap T(p), \quad C(p, \alpha) := K(p, \alpha) \cap T(p)$.

A function $f \in \mathcal{A}(p, n)$ is said to be $\beta-$uniformly starlike of order $\alpha$ $(-p \leq \alpha \leq p)$ and $\beta \geq 0$ denoted by $\beta - UST(\alpha, p, n)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - p\right| \quad (z \in \mathcal{U}). \tag{5}$$

Also, a function $f \in \mathcal{A}(p, n)$ is said to be $\beta-$uniformly convex of order $\alpha$ $(-p \leq \alpha \leq p)$ and $\beta \geq 0$ denoted by $\beta - UCV(\alpha, p, n)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta \left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \quad (z \in \mathcal{U}). \tag{6}$$

We observe that, the classes $\beta - UST(\alpha, \beta, 1, 1) = UST(\alpha, \beta)$ and $\beta - UCV(\alpha, \beta, 1, 1) = UCV(\alpha, \beta)$ are $\beta-$uniformly starlike of order $\alpha$ $(-1 \leq \alpha \leq 1)$ and $\beta-$uniformly convex of order $\alpha$ $(-1 \leq \alpha \leq 1)$ introduced and studied by Bharathi et al [4]. In particular, the classes $UCV(0, 1)$ and $UCV(0, \beta)$ were introduced by Goodman [9] and Kanas and Wisniowska [10], respectively.

Let $f, g \in \mathcal{A}(p, n)$, where $f(z)$ is given by (1) and $g(z)$ is defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}. \tag{7}$$

Then the Hadamard product (or convolution) $f * g$ of the functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) := z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} := (g * f)(z). \tag{8}$$

We consider the following multiplier transformations.

**Definition 1.** *[5] Let $f \in \mathcal{A}(p,n)$. For $p,n \in \mathbb{N}$, $\eta,\lambda \geq 0$, $\ell \geq 0$, define the multiplier transformations $\mathcal{I}_p(\eta,\lambda,\ell)$ on $\mathcal{A}(p,n)$ by the following infinite series:*

$$\mathcal{I}_p(\eta,\lambda,\ell) := z^p + \sum_{k=n}^{\infty} \left[\frac{p+\lambda k+\ell}{p+\ell}\right]^{\eta} a_{p+k} z^{p+k}. \tag{9}$$

It should be remarked that the class of multiplier transforms $\mathcal{I}_p(\eta,\lambda,\ell)$ is a generalization of several other linear operators considered, in earlier investigations [1, 2, 3, 5, 6, 7, 8, 13, 15, 16, 17] and [18].

If $f(z)$ is given by (1), then we have

$$\mathcal{I}_p(\eta,\lambda,\ell)f(z) = (f * \varphi^{\eta}_{p,\lambda,\ell})(z),$$

where

$$\varphi^{\eta}_{p,\lambda,\ell}(z) = z^p + \sum_{k=n}^{\infty} \left[\frac{p+\lambda k+\ell}{p+\ell}\right]^{\eta} z^{p+k}.$$

In particular, we set

$$\mathcal{I}_1(\eta,\lambda,\ell)f(z) := \mathcal{I}(\eta,\lambda,\ell)f(z).$$

Motivated by the earlier works of [4, 9] and [10] we introduce a new subclass of p-valent functions with negative coefficients and discuss some interesting properties of this generalized function class.

**Definition 2.** *A function $f \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{S}(\alpha,\beta,p,n)$ if it satisfies the inequality*

$$\Re\left(\frac{z(\mathcal{I}_p(\eta,\lambda,\ell)f(z))'}{\mathcal{I}_p(\eta,\lambda,\ell)f(z)} - \alpha\right) > \beta\left|\frac{z(\mathcal{I}_p(\eta,\lambda,\ell)f(z))'}{\mathcal{I}_p(\eta,\lambda,\ell)f(z)} - p\right|, \qquad z \in \mathcal{U} \tag{10}$$

*for some $-p \leq \alpha \leq p$, $\beta \geq 0$. Furthermore, we define the class $T\mathcal{S}(\alpha,\beta,p,n)$ by $\mathcal{S}(\alpha,\beta,p,n) \cap T(p,n)$.*

The main object of this work is to determine coefficient estimates, distortion bounds, radii of starlikeness and convexity for the analytic functions to belong to this general class. Also, subordination theorems and integral means inequalities of functions $f$ in the class $T\mathcal{S}(\alpha,\beta,p,n)$ are obtained.

## 2. Coefficient Inequalities

First we give a coefficient inequality for the class $T\mathcal{S}(\alpha, \beta, p, n)$.

**Theorem 1.** *A sufficient condition for a function $f(z)$ of the form (1) to be in $\mathcal{S}(\alpha, \beta, p, n)$ is*

$$\sum_{k=n}^{\infty} \left( \frac{p + \lambda k + \ell}{p + \ell} \right)^{\eta} [k(1 + \beta) + p - \alpha] |a_{k+p}| \leq p - \alpha \tag{11}$$

*for $z \in \mathcal{U}$, $-p \leq \alpha \leq p$ and $\beta \geq 0$.*

*Proof.* It is suffices to show that

$$\beta \left| \frac{z\mathcal{I}_p(\eta, \lambda, \ell) f(z)'}{\mathcal{I}_p(\eta, \lambda, \ell) f(z)} - p \right| - \Re \left( \frac{z\mathcal{I}_p(\eta, \lambda, \ell) f(z)'}{\mathcal{I}_p(\eta, \lambda, \ell) f(z)} - p \right) \leq p - \alpha, \quad z \in \mathcal{U}.$$

We have

$$\begin{aligned} & \beta \left| \frac{z\mathcal{I}_p(\eta, \lambda, \ell) f(z)'}{\mathcal{I}_p(\eta, \lambda, \ell) f(z)} - p \right| - \Re \left( \frac{z\mathcal{I}_p(\eta, \lambda, \ell) f(z)'}{\mathcal{I}_p(\eta, \lambda, \ell) f(z)} - p \right) \\ & \leq (1 + \beta) \left| \frac{z\mathcal{I}_p(\eta, \lambda, \ell) f(z)'}{\mathcal{I}_p(\eta, \lambda, \ell) f(z)} - p \right| \\ & \leq (1 + \beta) \left| \frac{p z^p + \sum\limits_{k=n}^{\infty} \left( \frac{p+\lambda k+\ell}{p+\ell} \right)^{\eta} (k + p) a_{k+p} z^{k+p}}{z^p + \sum\limits_{k=n}^{\infty} \left( \frac{p+\lambda k+\ell}{p+\ell} \right)^{\eta} a_{k+p} z^{k+p}} - p \right| \\ & \leq \frac{(1 + \beta) \sum\limits_{k=n}^{\infty} k \left( \frac{p+\lambda k+\ell}{p+\ell} \right)^{\eta} a_{k+p}}{1 - \sum\limits_{k=n}^{\infty} \left( \frac{p+\lambda k+\ell}{p+\ell} \right)^{\eta} a_{k+p}} . \end{aligned}$$

The last expression is bounded above by $(p - \alpha)$ if

$$\sum_{k=n}^{\infty} \left( \frac{p + \lambda k + \ell}{p + \ell} \right)^{\eta} [k(1 + \beta) + p - \alpha] |a_{k+p}| \leq p - \alpha \ .$$

**Theorem 2.** *A necessary and sufficient condition for a function $f(z)$ of the form (4) to be in $T\mathcal{S}(\alpha, \beta, p, n)$ is*

$$\sum_{k=n}^{\infty} \left( \frac{p + \lambda k + \ell}{p + \ell} \right)^{\eta} [k(1 + \beta) + p - \alpha] |a_{k+p}| \leq p - \alpha \ . \tag{12}$$

*Proof.* In view of Theorem 1, we need only to prove the necessity. If $f \in T\mathcal{S}(\alpha, \beta, p, n)$ and $z$ is real then

$$\frac{p - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} (k+p) a_{k+p} z^k}{1 - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p} z^k} - \alpha \geq \beta \left| \frac{\sum\limits_{k=n}^{\infty} k \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p} z^k}{1 - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p} z^k} \right|$$

If we choose $z$ to be real and let $z \to 1^-$, we get

$$\frac{p - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} (k+p) a_{k+p}}{1 - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p}} - \alpha \geq \frac{\sum\limits_{k=n}^{\infty} k\beta \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p}}{1 - \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p}}$$

or, equivalently,

$$p - \alpha - \sum_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} (k+p-\alpha) a_{k+p} \geq k\beta \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} a_{k+p}.$$

Thus, we have desired inequality

$$\sum_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha] |a_{k+p}| \leq p - \alpha \ .$$

**Corollary 3.** *If $f(z)$ of the form (4) is in $T\mathcal{S}(\alpha, \beta, p, n)$, then*

$$a_{k+p} \leq \frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]} \quad (k \geq n; n \in \mathbb{N}) \tag{13}$$

*with the equality only for the function*

$$f(z) = z^p - \frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]} z^{k+p}, \quad (k \geq n, n \in \mathbb{N}). \tag{14}$$

## 3. Extreme Points

**Theorem 4.** *Let $f_p(z) = z^p$ and $f_{k+p}(z) = z^p - \frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]} z^{k+p}$, $(k \geq n, n \in \mathbb{N})$. Then $f \in T\mathcal{S}(\alpha, \beta, p, n)$ if and only if $f(z)$ can be expressed in the form $f(z) = \mu_p z^p + \sum\limits_{k=n}^{\infty} \mu_{k+p} f_{k+p}(z)$, where $\mu_{k+p} \geq 0$ and $\mu_p + \sum\limits_{k=n}^{\infty} \mu_{k+p} = 1$.*

*Proof.* Suppose that $f(z)$ is given by

$$f(z) \quad = \quad \mu_p z^p + \sum_{k=n}^{\infty} \mu_{k+p} f_{k+p}(z), \tag{15}$$

so that we find from the hypothesis of the theorem

$$f(z) \quad = \quad z^p - \sum_{k=n}^{\infty} \frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]} \mu_{k+p} z^{k+p}, \tag{16}$$

where the coefficients $\mu_{k+p}$ are given with $\mu_p + \sum\limits_{k=n}^{\infty} \mu_{k+p} = 1$, $\mu_{k+p} \geq 0$. Then, since

$$\sum_{k=n}^{\infty} \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{(p-\alpha)} \frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]} \mu_{k+p}$$
$$= \sum_{k=n}^{\infty} \mu_{k+p} = 1 - \mu_{n+p} \leq 1, \quad (n \in \mathbb{N}).$$

Therefore $f \in T\mathcal{S}(\alpha, \beta, p, n)$.

Conversely, suppose that $f \in T\mathcal{S}(\alpha, \beta, p, n)$. Since

$$a_{k+p} \leq \frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}.$$

Setting

$$\mu_{k+p} = \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{(p-\alpha)} a_{k+p}$$

and

$$\mu_p = 1 - \sum_{k=n}^{\infty} \mu_{k+p} \ .$$

Then we have

$$f(z) \quad = \quad \sum_{k=n}^{\infty} \mu_{k+p} f_{k+p}(z) \ .$$

Hence proved.

## 4. Distortion Bounds

In this section, we obtain bounds for functions in the class $T\mathcal{S}(\alpha,\beta,p,n)$.

**Theorem 5.** *Let the function $f$ defined by (4)be in the class $T\mathcal{S}(\alpha,\beta,p,n)$. Then for $|z|=r$ we have*

$$r^p-\frac{(p-\alpha)r^{n+p}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}\le|f(z)|\le r^p+\frac{(p-\alpha)r^{n+p}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]} \quad (17)$$

*and*

$$pr^{p-1}-\frac{(p-\alpha)(p+n)r^{p+n-1}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}\le|f^{'}(z)|\le pr^{p-1}+\frac{(p-\alpha)(p+n)r^{p+n-1}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}. \quad (18)$$

*Proof.* In view of Theorem 2, we have

$$\begin{aligned}&\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]\sum_{k=n}^{\infty}|a_{k+p}|\\&\le\sum_{k=n}^{\infty}\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]|a_{k+p}|\le(p-\alpha),\end{aligned}$$

which is equivalent to

$$\sum_{k=n}^{\infty}|a_{k+p}|\le\frac{(p-\alpha)}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}\ . \quad (19)$$

Using (4) and (19), we obtain

$$\begin{aligned}|f(z)|&\le|z|^p+|z|^{n+p}\sum_{k=n}^{\infty}|a_{k+p}|\\&\le r^p+r^{n+p}\sum_{k=n}^{\infty}|a_{k+p}|\\&\le r^p+\frac{(p-\alpha)r^{n+p}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]} \quad (20)\end{aligned}$$

and similarly

$$f(z)\ \ge\ r^p-\frac{(p-\alpha)r^{n+p}}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}. \quad (21)$$

Again using (4) and (19), we have,

$$\begin{aligned} |f'(z)| &\geq p|z|^{p-1} - \sum_{k=n}^{\infty}(k+p)|a_{k+p}||z|^{k+p-1} \\ &\geq p|z|^{p-1} - (p+n)|z|^{p+n-1}\sum_{k=n}^{\infty}|a_{k+p}| \\ &\geq p|z|^{p-1} - \frac{(p+n)(p-\alpha)}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}|z|^{p+n-1} \end{aligned} \tag{22}$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{(p+n)(p-\alpha)}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}|z|^{p+n-1}. \tag{23}$$

From (22) and (23) with $|z| = r$, we have (18). Hence the proof is complete.

## 5. Radius of Close-to-convexity, starlikeness and convexity

**Theorem 6.** *Let the function $f(z)$ defined by (4) be in the class $T\mathcal{S}(\alpha,\beta,p,n)$ then $f(z)$ is close-to-convex of order $\delta$ for $0 \leq \delta < p$ in $|z| = r_1$, where*

$$r_1 = \inf_{k\geq n}\left\{\left(\frac{p-\delta}{k+p}\right)\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}\right\}^{\frac{1}{k}}$$

*where .*

*Proof.* It is sufficient to prove that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p-\delta, \quad (|z| \leq r_1). \tag{24}$$

Now,

$$\begin{aligned} \left|\frac{f'(z)}{z^{p-1}} - p\right| &= \left|\frac{pz^{p-1} - \sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}{z^{p-1}} - p\right| \\ &\leq \sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^{k}. \end{aligned}$$

The above expression is less than $p-\delta$ if

$$\sum_{k=n}^{\infty}\frac{k+p}{p-\delta}a_{k+p}|z|^k<1.$$

Using the fact that $f\in T\mathcal{S}(\alpha,\beta,p,n)$ if and only if

$$\sum_{k=n}^{\infty}\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}|a_{k+p}|<1.$$

we say (24) is true if

$$\frac{k+p}{p-\delta}|z|^k<\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}.$$

Or, equivalently,

$$|z|^k<\left(\frac{p-\delta}{k+p}\right)\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}$$

which yields the close-to-convexity of the family.

**Theorem 7.** *Let the function $f(z)$ defined by (4) be in the class $T\mathcal{S}(\alpha,\beta,p,n)$ then $f(z)$ is starlike of order $\delta$ for $0\leq\delta<p$ in $|z|=r_2$, where*

$$r_2=\inf_{k\geq n}\left\{\left(\frac{p-\delta}{k+p-\delta}\right)\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}\right\}^{\frac{1}{k}}$$

*where .*

*Proof.* To find the required result it is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\delta,\quad (|z|\leq r_2). \tag{25}$$

Now,

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum\limits_{k=n}^{\infty}ka_{k+p}|z|^k}{1-\sum\limits_{k=n}^{\infty}a_{k+p}|z|^k}.$$

The above expression is less than $p-\delta$ if

$$\sum_{k=n}^{\infty}\left(\frac{k+p-\delta}{p-\delta}\right)a_{k+p}|z|^k<1.$$

Using the fact, that $f\in T\mathcal{S}(\alpha,\beta,p,n)$ if and only if

$$\sum_{k=n}^{\infty}\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}|a_{k+p}|<1.$$

We say (25) is true if

$$\left(\frac{k+p-\delta}{p-\delta}\right)|z|^k<\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}$$

Or, equivalently,

$$|z|^k<\left(\frac{p-\delta}{k+p-\delta}\right)\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}$$

which yields the starlikeness of the family.

**Theorem 8.** *Let the function $f(z)$ defined by (4) be in the class $T\mathcal{S}(\alpha,\beta,p,n)$ then $f(z)$ is convex of order $\delta$ for $0\leq\delta<p$ in $|z|=r_3$, where*

$$r_3=\inf_{k\geq n}\left\{\left(\frac{p(p-\delta)}{(k+p)(k+p-\delta)}\right)\frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}{(p-\alpha)}\right\}^{\frac{1}{k}}$$

*where .*

*Proof.* It is sufficient to prove that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|\leq p-\delta,\quad (|z|\leq r_3). \tag{26}$$

Now,

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| \leq \frac{\sum_{k=n}^{\infty}k(k+p)a_{k+p}|z|^k}{p-\sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^k}.$$

The above expression is less than $p-\delta$ if

$$\sum_{k=n}^{\infty} \frac{(k+p)(k+p-\delta)}{p(p-\delta)} a_{k+p}|z|^k < 1.$$

Using the fact, that $f \in T\mathcal{S}(\alpha, \beta, p, n)$ if and only if

$$\sum_{k=n}^{\infty} \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{(p-\alpha)} |a_{k+p}| < 1.$$

We say (26) is true if

$$\frac{(k+p)(k+p-\delta)}{p(p-\delta)} |z|^k < \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{(p-\alpha)}$$

Or, equivalently,

$$|z|^k < \frac{p(p-\delta)}{(k+p)(k+p-\delta)} \quad \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{(p-\alpha)}$$

which yields the convexity of the family.

## 6. Subordination Results

In this section we obtain subordination theorem for the class $T\mathcal{S}(\alpha, \beta, p, n)$. To prove our result we need the following definition and lemma.

**Definition 3.** *[19] A sequence $\{b_k\}$ of complex numbers is said to be subordinating sequence if for each function $f$ of the form (1) from the class $\mathcal{C}$ we have*

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad (a_1 = 1). \tag{27}$$

**Lemma 9.** *[19] The sequence $\{b_k\}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{k=1}^{\infty} b_k z^k\right) > 0 \quad (z \in \mathcal{U}). \tag{28}$$

**Theorem 10.** *Let $f \in T\mathcal{S}(\alpha, \beta, p, n)$ and $g \in \mathcal{C}$ then*

$$\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}\frac{f(z)}{z^{p-1}} * g(z) \prec g(z) \tag{29}$$

*and*

$$\Re\left(\frac{f(z)}{z^{p-1}}\right) > -\frac{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]} \quad (z \in \mathcal{U}). \tag{30}$$

*The constant factor* $\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}$ *cannot be replaced by a larger number.*

*Proof.* Let a function $f$ of the form (4) belong to the class $T\mathcal{S}(\alpha, \beta, p, n)$ and suppose that a function $g$ of the form

$$g(z) = \sum_{k=1}^{\infty} c_k z^k \quad (c_1 = 1; z \in \mathcal{U})$$

belongs to the class $\mathcal{C}$. Then

$$\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}\frac{f(z)}{z^{p-1}} * g(z) = \sum_{k=1}^{\infty} b_k c_k z^k \quad (z \in \mathcal{U}),$$

where

$$b_k = \begin{cases} \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} & \text{if } k = 1, \\ 0 & \text{if } 2 \leq k \leq n \\ \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} a_{k+p-1} & \text{if } k > n. \end{cases}$$

Thus, by Definition 3 the subordination result (29) holds true if $\{b_k\}$ is the subordinating factor sequence. Since $\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha] \geq \left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]$ for $k \geq n$, we have:

$$\Re\left(1 + 2\sum_{k=1}^{\infty} b_k z^k\right) = \Re\left(1 + \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} z\right.$$

$$
\begin{aligned}
& \left. + \sum_{k=n}^{\infty} \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} a_{k+p} z^{k+1} \right) \\
\geq \quad & 1 - \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} r \\
& - \frac{r \sum\limits_{k=n}^{\infty} \left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]|a_{k+p}|}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}\ .
\end{aligned}
$$

Thus, by using Theorem 2 we obtain

$$
\begin{aligned}
\Re\left(1+2\sum_{k=1}^{\infty} b_k z^k\right) \quad \geq \quad & 1 - \frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} r \\
& - \frac{p-\alpha}{[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} r > 0.
\end{aligned}
$$

This evidently proves the inequality (28) and hence the subordination result (29). The inequality (30) follows from (29) by taking

$$
g(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k \quad (z \in \mathcal{U}).
$$

Next, we observe that the function

$$
F(z) := z^p - \frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]} z^{k+p} \quad (z \in \mathcal{U}; k \geq n; n \in \mathbb{N}).
$$

clearly $F(z)$ belongs to the class $T\mathcal{S}(\alpha, \beta, p, n)$. For this function (29) becomes

$$
\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} \frac{F(z)}{z^{p-1}} \prec \frac{z}{1-z}.
$$

it is easily verified that

$$
\min\left\{\Re\left(\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]} \frac{F(z)}{z^{p-1}}\right)\right\} = -\ \frac{1}{2}
$$

and the constant $\frac{\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]}{2[p-\alpha+\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\eta}[n(1+\beta)+p-\alpha]]}$ can not be replaced by any larger one.

## 7. Integral Mean Inequalities

Due to Littlewood [11] we obtain integral means inequalities for the functions from the class $T\mathcal{S}(\alpha,\beta,p,n)$.

**Lemma 11.** *Let $f,g\in\mathcal{A}$. If $f\prec g$, then for $z=re^{i\theta}$ $(0<r<1)$ and $\delta>0$, we have*

$$\int_0^{2\pi}|f(z)|^{\delta}\,d\theta\leq\int_0^{2\pi}|g(z)|^{\delta}\,d\theta\ . \tag{31}$$

Applying Lemma 11 and Theorem 2 we prove the following result.

**Theorem 12.** *Suppose $f\in T\mathcal{S}(\alpha,\beta,p,n)$, then*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\delta}d\theta\leq\int_0^{2\pi}\left|f_{k+p}(re^{i\theta})\right|^{\delta}d\theta\qquad(0<r<1;\delta>0), \tag{32}$$

*where*

$$f_{k+p}(z)=z^p-\frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}z^{k+p}\quad(z\in\mathcal{U};k\geq n+1;n\in\mathbb{N}).$$

*Proof.* For function $f(z)$ of the form (4), the inequality (32) is equivalent to the following

$$\int_0^{2\pi}\left|1-\sum_{k=n}^{\infty}a_{k+p}z^k\right|^{\delta}d\theta\leq\int_0^{2\pi}\left|1-\frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}z^k\right|^{\delta}d\theta\ .$$

By Lemma 11, it suffices to show that

$$1-\sum_{k=n}^{\infty}a_{k+p}z^k\prec 1-\frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}z^k\ .$$

Thus by setting

$$1-\sum_{k=n}^{\infty}a_{k+p}z^k=1-\frac{p-\alpha}{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta}[k(1+\beta)+p-\alpha]}w(z)^k$$

and using Theorem 2 we obtain

$$|w(z)|^k = \left| \sum_{k=n}^{\infty} \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{p-\alpha} a_{k+p} z^k \right|$$
$$\leq |z|^n \sum_{k=n}^{\infty} \frac{\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\eta} [k(1+\beta)+p-\alpha]}{p-\alpha} |a_{k+p}| \leq |z|^n < 1.$$

This completes the proof.

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