

SOME IMPLICIT SUMMATION FORMULAS AND SYMMETRIC IDENTITIES FOR THE GENERALIZED HERMITE-BASED POLYNOMIALS

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ABSTRACT. In this paper, we introduce a unified family of Hermite-based Apostol Bernoulli, Euler and Genocchi polynomials. We shall show that there is an intimate connection between these polynomials and a new class of generalized polynomials associated with the modified Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x, \nu)$ of degree n and order α introduced by Derre and Simsek. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Bernoulli, Euler and Genocchi numbers and polynomials.

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1. INTRODUCTION

Recently, Ozarslan [15] introduced the following unification of the Apostol Bernoulli, Apostol Euler and Apostol Genocchi polynomials. Explicitly Ozarslan studied a generating function of the form

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x; k, a, b) \frac{t^n}{n!} \quad (1.1)$$

$$(t + b \ln(\frac{\beta}{\alpha})) < 2\pi, k \in N_0; a, b \in \mathbb{R}^+; \alpha \in \mathbb{R}, \beta \in \mathbb{C}$$

We notice that for $\alpha = 1$,

$$P_{n,\beta}^{(1)}(x; k, a, b) = P_{n,\beta}(x; k, a, b), n \in N$$

and then (1.1) reduces to

$$\frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}(x; k, a, b) \frac{t^n}{n!} \quad (1.2)$$

which is defined by Ozden [17]. Ozden et al in [19] introduced many properties of these polynomials. We give some specific special cases

1. By substituting $a = b = k = 1$ and $\beta = \lambda$ into (1.1), one has the Apostol-Bernoulli polynomials $P_{n,\beta}^{(1)}(x; 1, 1, 1) = B_n^{(\alpha)}(x; \lambda)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, (|t + \log \lambda| < 2\pi) \quad (1.3)$$

(see for details [6] [9] [13] [15] [18] [23] see also the references cited in each of these earlier works)

For $\lambda = \alpha = 1$ in (1.3), the result reduces to

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, |t| < 2\pi \quad (1.4)$$

where $B_n(x)$ denotes the classical Bernoulli polynomials (see from example [2]-[27]; see also the references cited in each of these earlier works)

2. If we substitute $b = \alpha = 1, k=0, a=-1$ and $\beta = \lambda$ into (1.1), we have the Apostol-Euler polynomials $P_{n,\lambda}^{(1)}(x; 0, -1, 1) = E_n^1(x, \lambda)$

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, (|t + \log \lambda| < 2\pi) \quad (1.5)$$

(see for details [6] [9] [13] [15] [18] [23] see also the references cited in each of these earlier works)

For $\lambda = 1$ in (1.5), the result reduces to

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, |t| < 2\pi \quad (1.6)$$

where $E_n(x)$ denotes the classical Euler polynomials (see from example [5] [6] [7] [9] [10] [24] [25] [27] [29]; see also the references cited in each of these earlier works)

3. By substituting $b = \alpha = 1, k=1, a=-1$ and $\beta = \lambda$ into (1.1), one has the Apostol-Genocchi polynomials $P_{n,\beta}^{(1)}(x; 1, -1, 1) = \frac{1}{2}G_n(x; \lambda)$, which is defined by means of

the following generating function

$$\frac{2t}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!}, (|t + \log \lambda| < 2\pi) \quad (1.7)$$

(see for details [6] [9] [13] [15] [18] [23] see also the references cited in each of these earlier works)

4. By substituting $x=0$ in the generating function (1.1), we obtain the corresponding unification of the generating functions of Bernoulli, Euler and Genocchi numbers of higher order. Thus we have

$$P_{n,\beta}^{(\alpha)}(0; k, a, b) = P_{n,\beta}^{(\alpha)}(k, a, b), n \in \mathbb{N}$$

Derre and Simsek [3] modified the Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x)$ (see for detail [12]) as $\Phi_n^{(\alpha)}(x, \nu)$ of degree n and order α by the means of the following generating function:

$$g_1(t, x; \alpha, \nu) = f(t, \alpha) e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, \nu) \frac{t^n}{n!} \quad (1.8)$$

where $f(t, \alpha)$ is a function of t and integer α

Observe that $\Phi_n^{(\alpha)}(x, 0) = \Phi_n^{(\alpha)}(x)$ (cf. [12]).

Setting $f(t, \alpha) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha$ in (1.8), we obtain the following polynomials given by the generating function

$$g_2(t, x; \alpha, \nu) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \frac{P_{n,\beta}^{(\alpha)}(x, k, a, b, \nu) t^n}{n!} \quad (1.9)$$

Observe that the polynomials $P_{n,\beta}^{(\alpha)}(x, k, a, b, \nu)$ are related to not only generalized polynomials but also the Hermite-based polynomials. For example, if $h(t, 0) = 0$ in (1.9), we have

$$P_{n,\beta}^{(\alpha)}(x, k, a, b, 0) = P_{n,\beta}^{(\alpha)}(x, k, a, b)$$

where $P_{n,\beta}^{(\alpha)}(x, k, a, b)$ denotes the generalized polynomials of higher order which is defined by means of the following generating function

$$F_B(t, x; \alpha) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, k, a, b) \frac{t^n}{n!} \quad (1.10)$$

One can easily see that

$$P_{n,\beta}^{(\alpha)}(0, k, a, b) = P_{n,\beta}^{(\alpha)}(k, a, b)$$

that is

$$F_B(t; \alpha) = \left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^\alpha = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(k, a, b) \frac{t^n}{n!} \quad (1.11)$$

where $P_n^{(\alpha)}(k, a, b)$ are generalized numbers.

If we take $h(t, \nu) = h(t, y) = yt^2$ in (1.1), we get generalized Hermite-based polynomials of two variables ${}_H P_{n,\beta}^{(\alpha)}(x, y)$ introduced by Ozarslan [15] in the form

$$\left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b) \frac{t^n}{n!} \quad (1.12)$$

which is essentially a generalization of Hermite-based Bernoulli, Euler and Genocchi polynomials. We use the following definitions, relations and identities to derive our main results.

For each integer $k \geq 0$, $S_k(n) = \sum_{i=0}^n i^k$ is called sum of integer powers or simply power sum. The exponential generating function for $S_k(n)$ is

$$\sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!} = 1 + e^t + e^{2t} + \dots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1} \quad (1.13)$$

For each $k \in N_0$, the sum $M_k(n) = \sum_{i=0}^n (-1)^k i^k$ is known as the sum of alternative integer powers. The following generating relation is straight forward for

$$\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} = 1 - e^t + e^{2t} - \dots - (-1)^n e^{nt} = \frac{1 - (-e^t)^{(n+1)}}{e^t + 1} \quad (1.14)$$

Definition 1.1 Let $c > 0$. The generalized 2-variable 1-parameter Hermite Kamp'e de Feriet polynomials $H_n(x, y, c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y, c) \frac{t^n}{n!} \quad (1.15)$$

This is an extended 2-variable Hermite Kamp'e de Feriet polynomials $H_n(x, y)$ (see[1]) defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} \quad (1.16)$$

Note that

$$H_n(x, y, e) = H_n(x, y)$$

In order to collect the powers of t we expand the left hand side of (1.15) to get the representation

$$H_n(x, y, c) = n! \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\ln c)^{n-j} x^{n-2j} y^j}{j!(n-2j)!} \quad (1.17)$$

In this paper, we first give definitions of the generalized polynomials $P_{n,\beta}^{(\alpha)}(x, y; k, a, b)$ which generalize the concepts stated above and then research their basic properties and relationships with Bernoulli numbers $B_n(k, a, b)$, Bernoulli polynomials $B_n(x, k, a, b)$, Euler numbers $E_n(k, a, b)$ Euler polynomials $E_n(x; k, a, b)$, the generalized Euler numbers $E_n(a, b)$ and generalized Euler polynomials. The remainder of this paper is organized as follows. We modify generating functions for the Milne-Thomson's polynomials [13] and derive some identities related to Hermite-based generalized Apostol-Bernoulli, Euler and Genocchi (ABEG) polynomials ${}_H P_{n,\beta}^\alpha(x, y; k, a, b, c)$ are defined. Some implicit summation formulae and general symmetry identities of (ABEG) polynomials ${}_H P_{n,\beta}^\alpha(x, y; k, a, b, c)$ are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al, Natalini et al, Zhang et al, Yang, Ozarslan, Pathan, Pathan et al.

2. DEFINITIONS AND PROPERTIES OF THE HERMITE-BASED GENERALIZED APOSTOL-BERNOULLI, EULER AND GENOCCHI (ABEG) POLYNOMIALS

$${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c)$$

In the modified Milne Thomson's polynomials due to Derre and Simsek [3,12] defined by (1.8) if we set $f(t, \alpha) = \left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha$, we obtain the following generalized polynomials $P_{n,\beta}^{(\alpha)}(x; k, \nu; a, b, c)$

Definition 2.1 Let $a, b, c > 0$ and $a \neq b$. The generalized Euler polynomials $E_n^{(\alpha)}(x, \nu; a, b, c)$ for nonnegative integer n are defined by

$$\left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, k, \nu; a, b, c) \frac{t^n}{n!}, \quad x \in \mathfrak{R}, k \in N_0, a, b \in \mathfrak{R} \setminus \{0\} \quad (2.1)$$

Setting $h(t, \nu) = h(t, y) = yt^2$ in (2.1), we get

Definition 2.2 Let $a, b, c > 0$ and $a \neq b$. The generalized polynomials $P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c)$ for nonnegative integer n are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b c^t - a^b}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!}, \quad k \in N_0, a, b \in \mathbb{R} \setminus \{0\}, \alpha, \beta \in C \quad (2.2)$$

For $\alpha = 1$, we obtain from (2.2) the generating function

$$\left(\frac{2^{1-k}t^k}{\beta^b c^t - a^b}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}(x, y; k, a, b, c) \frac{t^n}{n!}, \quad (2.3)$$

whereas for $y = 0$ and $c = e$, (2.2) reduces to known result Ozarslan [15].

Definition 2.3 Let $a, b, c > 0$ and $a \neq b$. The generalized Apostol Bernoulli, Euler and Genocchi (ABEG) numbers $P_{n,\beta}^{(\alpha)}(k, a, b, c)$ for nonnegative integer n are defined by

$$\Phi(t; k, \alpha, a, b, c) = \left(\frac{2^{1-k}t^k}{\beta^b c^t - a^b}\right)^\alpha = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(k, a, b, c) \frac{t^n}{n!}, \quad k \in N_0, a, b \in \mathbb{R} \setminus \{0\}, \alpha, \beta \in C \quad (2.4)$$

It is easy to prove that

$$P_{n,\beta}^{\alpha+\gamma}(k, a, b, c) = \sum_{m=0}^n \binom{n}{m} P_{m,\beta}^{(\alpha)}(k, a, b, c) P_{n-m,\beta}^{(\gamma)}(k, a, b, c) \quad (2.5)$$

Further setting $c = e$ in (2.2), we get

Definition 2.4 Let $a, b, c > 0$ and $a \neq b$. The generalized Hermite-based polynomials ${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, e)$ for nonnegative integer n are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, e) \frac{t^n}{n!}, \quad k \in N_0, a, b \in \mathbb{R} \setminus \{0\}, \alpha, \beta \in C \quad (2.6)$$

For the existence of the expansion, we need

- (i) $|t| < 2\pi$ where $\alpha \in N_0, k=1$ and $\left(\frac{\beta}{a}\right)^b = 1$; $|t| < 2\pi$ when $\alpha \in N_0, k=2,3,..$ and $\left(\frac{\beta}{a}\right)^b = 1$; $|t| < |b \log\left(\frac{\beta}{a}\right)|$ when $\alpha \in N_0, k \in N$ and $\left(\frac{\beta}{a}\right)^b \neq 1$ or $(\neq -1)$; $x, y \in \mathbb{R}, \beta \in C \setminus \{0\}, 1^\alpha = 1$
- (ii) $|t| < 2\pi$ when $\left(\frac{\beta}{a}\right)^b = -1$; $|t| < |b \log\left(\frac{\beta}{a}\right)|$ when $\left(\frac{\beta}{a}\right)^b \neq -1, x, y \in \mathbb{R}, k=0, \alpha, \beta \in C, a, b, c \in C, 1^\alpha = 1$

(iii) $|t| < 2\pi$ when $\alpha \in N_0$ and $\left(\frac{\beta}{a}\right)^b = -1$, $x, y \in R$, $k \in N$, $\beta \in C$, $a, b, c \in C/\{0\}$ $1^\alpha = 1$ where $w = |w| e^{i\theta}$, $-\pi \leq \theta < \pi$.

For $k=a=b=1$ and $\beta = \lambda$ in (2.6), we define the following

Definition 2.5 Let $\alpha \in N_0$, λ be an arbitrary (real or complex) parameter and $x, y \in R$. The Hermite-based generalized Apostol-Bernoulli polynomials are defined by

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(\alpha)}(x, y; \lambda) \frac{t^n}{n!}, \quad (2.7)$$

($|t| < 2\pi$, when $\lambda = 1$; $|t| < |\log(-\lambda)|$, when $\lambda \neq 1$, $x, y \in C$, $\alpha \in C$, $1^\alpha = 1$)

For $k+1=-a=b=1$ and $\beta = \lambda$ in (2.6), we define the following

Definition 2.6 Let $\alpha \in N_0$, λ be an arbitrary (real or complex) parameter and $x, y \in R$. The Hermite-based generalized Apostol-Euler polynomials are defined by

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; \lambda) \frac{t^n}{n!}, \quad (2.8)$$

($|t| < 2\pi$, when $\lambda = 1$; $|t| < |\log(-\lambda)|$, when $\lambda \neq 1$, $x, y \in C$, $\alpha \in C$, $1^\alpha = 1$)

For $k+1=-2a=b=1$ and $2\beta = \lambda$ in (2.6), we define the following

Definition 2.7 Let $\alpha \in N_0$, λ be an arbitrary (real or complex) parameter and $x, y \in R$. The Hermite-based generalized Apostol-Genocchi polynomials are defined by

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H G_n^{(\alpha)}(x, y; \lambda) \frac{t^n}{n!}, \quad (2.9)$$

($|t| < 2\pi$, when $\lambda = 1$; $|t| < |\log(-\lambda)|$, when $\lambda \neq 1$, $x, y \in C$, $\alpha \in C$, $1^\alpha = 1$)

The generalized polynomials $P_{n,\beta}^\alpha(x, y; k, a, b, c)$ defined by (2.2) have the following properties which are stated as theorem below.

Theorem 2.1 Let $a, b, c > 0$ and $a \neq b$. Then $x \in R$ and $n \geq 0$. Then

$$\begin{aligned} P_{n,\beta}^{(\alpha)}(x, y; k, 1, 1, e) &= {}_H P_{n,\beta}^{(\alpha)}(x, y; k), P_{n,\lambda}^{(\alpha)}(x, y; k, 1, 1, e) = {}_H B_n^{(\alpha)}(x, y; \lambda), \\ P_{n,\beta}^{(\alpha)}(x; k, a, b, e) &= P_{n,\beta}^{(\alpha)}(x; k, a, b), P_{n,\beta}^{(\alpha)}(x, 0; 1, 1, e) = B_n^{(\alpha)}(x; \lambda) \end{aligned} \quad (2.10)$$

$$P_{n,\beta}^{(\alpha+\gamma)}(x+y, z+u; k, a, b, c) = \sum_{m=0}^{\infty} \binom{n}{m} P_{m,\beta}^{(\gamma)}(z, u; a, b, c) P_{n-m,\beta}^{(\alpha)}(x, y; k, a, b, c) \quad (2.11)$$

$$P_{n,\beta}^{(\alpha)}(x+z, y; a, b, c) = \sum_{m=0}^n \binom{n}{m} P_{n-m,\beta}^{(\alpha)}(x; k, a, b, c) H_m(z, y; c) \quad (2.12)$$

Proof. The formula in (2.10) are obvious. Applying Definition (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha+\gamma)}(x+y, z+u; k, a, b, c) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} \sum_{m=0}^{\infty} P_{m,\beta}^{(\gamma)}(z, u; k, a, b, c) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n P_{m,\beta}^{(\gamma)}(z, u; k, a, b, c) \frac{t^m}{m!} P_{n-m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^{n-m}}{(n-m)!} \end{aligned}$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.11). Again by Definition (2.2) of generalized polynomials, we have

$$\left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{(x+z)t+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x+z, y; k, a, b, c) \frac{t^n}{n!} \quad (2.13)$$

which can be written as

$$\left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{xt} c^{zt+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x; k, a, b, c) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(z, y, c) \frac{t^m}{m!} \quad (2.14)$$

Replacing n by n-m in (2.14), comparing with (2.13) and equating their coefficients of t^n leads to formula (2.12).

3. IMPLICIT SUMMATION FORMULAE INVOLVING GENERALIZED HERMITE-BASED POLYNOMIALS

For the derivation of implicit formulae involving generalized polynomials

$P_{n,\beta}^\alpha(x, y; k, a, b, c)$ and generalized Hermite-based polynomials ${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, e)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [8] and Hermite-based polynomials in Ozarslan [15] holds as well. First we prove the following results involving generalized polynomials $P_{n,\beta}^\alpha(x, y; k, a, b, c)$.

Theorem 3.1 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, The following implicit summation formulae for generalized polynomials $P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c)$ holds true:

$$P_{n+m,\beta}^{(\alpha)}(z, y; k, a, b, c) = \sum_{p,q=0}^{n,m} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} P_{n+m-p-q,\beta}^{(\alpha)}(x, y; k, a, b, c) \quad (3.1)$$

Proof. We replace t by $t + u$ and rewrite the generating function (2.2) as

$$\left(\frac{2^{1-k}(t+u)^k}{\beta^b c^{t+u} - a^b}\right)^\alpha e^{y(t+u)^2} = c^{-x(t+u)} \sum_{k,l=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n u^m}{n! m!} \quad (3.2)$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+u)} \sum_{n,m=0}^{\infty} P_{n+m}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(z, y; k, a, b, c) \frac{t^n u^m}{n! m!} \quad (3.3)$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(z-x)(t+u)]^N}{N!} \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(z, y; k, a, b, c) \frac{t^n u^m}{n! m!} \quad (3.4)$$

which on using formula [28,p.52(2)]

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n y^m}{n! m!} \quad (3.5)$$

in the left hand side becomes

$$\sum_{p,q=0}^{\infty} \frac{(z-x)^{p+q}}{p!q!} \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(z, y; k, a, b, c) \frac{t^n u^m}{n! m!} \quad (3.6)$$

Now replacing n by $n-p$, m by $m-q$ and using the lemma [28,p.100(1)] in the left hand side of (3.6), we get

$$\begin{aligned} \sum_{n,m=0}^{\infty} \sum_{p,q=0}^{n,m} \frac{(z-x)^{p+q}}{p!q!} P_{n+m-p-q,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n u^m}{(n-p)! (m-q)!} \\ = \sum_{n,m=0}^{\infty} P_{n+m,\beta}^{(\alpha)}(z, y; k, a, b, c) \frac{t^n u^m}{n! m!} \end{aligned} \quad (3.7)$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Corollary 1 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$, the following implicit summation formula for generalized Apostol Bernoulli polynomials $B_n^\alpha(x, y; \lambda, a, b, c)$

$$B_{n+m}^{(\alpha)}(z, y; \lambda, a, b, c) = \sum_{p,q=0}^{n,m} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} B_{n+m-p-q}^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.8)$$

Corollary 2 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$, the following implicit summation formula for generalized Apostol Euler polynomials $E_n^\alpha(x, y; \lambda, a, b, c)$

$$E_{n+m}^{(\alpha)}(z, y; \lambda, a, b, c) = \sum_{p,q=0}^{n,m} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} E_{n+m-p-q}^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.9)$$

Corollary 3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$, the following implicit summation formula for generalized Apostol Genocchi polynomials $G_n^\alpha(x, y; \lambda, a, b, c)$

$$G_{n+m}^{(\alpha)}(z, y; \lambda, a, b, c) = \sum_{p,q=0}^{n,m} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} G_{n+m-p-q}^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.10)$$

Theorem 3.2 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) = \sum_{m=0}^n \binom{n}{m} P_{n-m,\beta}^{(\alpha)}(k, a, b) H_m(x, y, c) \quad (3.11)$$

Proof. By the definition of generalized polynomials and the definition (1.1), we have

$$\left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(k, a, b) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} H_m(x, y; c) \frac{t^m}{m!} \right)$$

Now replacing n by $n-m$ and comparing the coefficients of t^n , we get the result (3.11).

Remark. For $c = e$, (3.11) yields

$${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, e) = \sum_{m=0}^n \binom{n}{m} P_{n-m,\beta}^{(\alpha)}(k, a, b) H_m(x, y)$$

Corollary 1 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$B_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{m} B_{n-m}^{(\alpha)}(\lambda, a, b) H_m(x, y, c) \quad (3.12)$$

Corollary 2 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$E_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{m} E_{n-m}^{(\alpha)}(\lambda, a, b) H_m(x, y, c) \quad (3.13)$$

Corollary 3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$G_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{m} G_{n-m}^{(\alpha)}(\lambda, a, b) H_m(x, y, c) \quad (3.14)$$

Theorem 3.3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\ln c)^{n-m-j} P_{m,\beta}^{(\alpha)}(k, a, b) x^{n-2j-m} y^j \frac{n!}{m!j!(n-2j-m)!} \quad (3.15)$$

Proof. Applying the definition (2.2) to the term $\left(\frac{2^{1-k}t^k}{\beta^b c^t - a^b}\right)^\alpha$ and expanding the exponential function c^{xt+yt^2} at $t = 0$ yields

$$\begin{aligned} \left(\frac{2^{1-k}t^k}{\beta^b c^t - a^b}\right)^\alpha c^{xt+yt^2} &= \left(\sum_{m=0}^{\infty} P_{m,\beta}^{(\alpha)}(k, a, b) \frac{t^m}{m!}\right) \left(\sum_{n=0}^{\infty} x^n (\ln c)^n \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} (\ln c)^{n-k} P_{m,\beta}^{(\alpha)}(k, a, b) x^{n-k}\right) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!}\right) \end{aligned}$$

Replacing n by $n-2j$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2j}{m} (\ln c)^{n-m-j} P_{m,\beta}^{(\alpha)}(k, a, b) x^{n-m-2j} y^j\right) \frac{t^n}{(n-2j)!j!} \quad (3.16) \end{aligned}$$

Combining (3.16) and (2.2) and equating their coefficients of t^n produce the formula (3.15).

Corollary 1 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$B_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\ln c)^{n-m-j} B_m^{(\alpha)}(\lambda, a, b) x^{n-2j-m} y^j \frac{n!}{m!j!(n-2j-m)!} \quad (3.17)$$

Corollary 2 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$E_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\ln c)^{n-m-j} E_m^{(\alpha)}(\lambda, a, b) x^{n-2j-m} y^j \frac{n!}{m!j!(n-2j-m)!} \quad (3.18)$$

Corollary 3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$G_n^{(\alpha)}(x, y; \lambda, a, b, c) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\ln c)^{n-m-j} G_m^{(\alpha)}(\lambda, a, b) x^{n-2j-m} y^j \frac{n!}{m!j!(n-2j-m)!} \quad (3.19)$$

Theorem 3.4 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$P_{n,\beta}^{(\alpha)}(x+1, y; k, a, b, c) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} (\ln c)^{n-m-j} P_{m,\beta}^{(\alpha)}(x; k, a, b, c) \quad (3.20)$$

Proof. By the definition of generalized polynomials, we have

$$\left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x+1, y; k, a, b, c) \frac{t^n}{n!} \quad (3.21)$$

which can be written as

$$\begin{aligned} & \left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{xt} c^t c^{yt^2} \\ &= \left(\sum_{m=0}^{\infty} P_{m,\beta}^{(\alpha)}(x; k, a, b, c) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} x^n (\ln c)^n \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} (\ln c)^{n-m} P_{m,\beta}^{(\alpha)}(x; k, a, b, c) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!} \right) \end{aligned} \quad (3.22)$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x+1, y; k, a, b, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} (\ln c)^{n-m-j} P_{m,\beta}^{(\alpha)}(x; k, a, b, c) \frac{t^n}{(n-2j)!j!} \quad (3.23)$$

Combining (3.21) and (3.23) and equating their coefficients of t^n leads to formula (3.20).

Corollary 1 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$B_n^{(\alpha)}(x+1, y; \lambda, a, b, c) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} (\ln c)^{n-m-j} B_m^{(\alpha)}(x; \lambda, a, b, c) \quad (3.24)$$

Corollary 2 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$E_n^{(\alpha)}(x+1, y; \lambda, a, b, c) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} (\ln c)^{n-m-j} E_m^{(\alpha)}(x; \lambda, a, b, c) \quad (3.25)$$

Corollary 3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$$G_n^{(\alpha)}(x+1, y; \lambda, a, b, c) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} (\ln c)^{n-m-j} G_m^{(\alpha)}(x; \lambda, a, b, c) \quad (3.26)$$

Theorem 3.5 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b, e) = \sum_{m=0}^n \binom{m}{n} P_{n-m,\beta}^{(\alpha-1)}(k, a, b) {}_H P_{m,\beta}^{(\alpha)}(x, y; k, a, b, e) \quad (3.27)$$

Proof. By the definition of generalized Hermite-based polynomials, we have

$$\begin{aligned} & \left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right) \left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^\alpha e^{xt+yt^2} \\ &= \left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right) \sum_{m=0}^{\infty} {}_H P_{m,\beta}^{(\alpha)}(x, y; k, a, b, e) \frac{t^m}{m!} \end{aligned}$$

Now proceeding on the lines of Theorem 3.2, replacing n by $n-m$ and equating the coefficients of t^n leads to formula (3.27)

Corollary 1 Let $a, b > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$${}_H B_n^{(\alpha)}(x, y; \lambda, a, b, e) = \sum_{m=0}^n \binom{n}{m} B_{n-m}^{(\alpha-1)}(\lambda, a, b) {}_H B_m^{(\alpha)}(x, y; \lambda, a, b, e) \quad (3.28)$$

Corollary 2 Let $a, b > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$${}_H E_n^{(\alpha)}(x, y; \lambda, a, b, e) = \sum_{m=0}^n \binom{n}{m} E_{n-m}^{(\alpha-1)}(\lambda, a, b) {}_H E_m^{(\alpha)}(x, y; \lambda, a, b, e) \quad (3.29)$$

Corollary 3 Let $a, b > 0$ and $a \neq b$. Then for $x, y \in R$ and $n \geq 0$. Then

$${}_H G_n^{(\alpha)}(x, y; \lambda, a, b, e) = \sum_{m=0}^n \binom{n}{m} G_{n-m}^{(\alpha-1)}(\lambda, a, b) {}_H G_m^{(\alpha)}(x, y; \lambda, a, b, e) \quad (3.30)$$

Theorem 3.6 For arbitrary real or complex parameter α , the following implicit summation formula involving generalized Euler polynomials $P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c)$ holds true

$$P_{n,\beta}^{(\alpha)}(x+1, y; k, a, b, c) = \sum_{m=0}^n \binom{n}{k} (\ln c)^{n-m} P_{m,\beta}^{(\alpha)}(x, y; k, a, b, c) \quad (3.31)$$

Proof. By the definition of generalized polynomials, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x+1, y; k, a, b, c) \frac{t^n}{n!} - \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} \\ &= \left(\frac{2^{1-k} t^k}{\beta^b c^t - a^b} \right)^\alpha c^{xt+yt^2} (c^t - 1) \\ &= \left(\sum_{m=0}^{\infty} P_{m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!} \right) - \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n (\ln c)^{n-m} P_{m,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{(n-m)!} - \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x, y; k, a, b, c) \frac{t^n}{n!} \end{aligned}$$

Finally, equating the coefficients of the like powers of t^n , we get (3.31).

Corollary 1 For arbitrary real or complex parameter α , the following implicit summation formula involving generalized Apostol Bernoulli polynomials $B_n^{(\alpha)}(x, y; \lambda, a, b, c)$ holds true:

$$B_n^{(\alpha)}(x+1, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{k} (\ln c)^{n-m} B_m^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.32)$$

Corollary 2 For arbitrary real or complex parameter α , the following implicit summation formula involving generalized Apostol Euler polynomials $E_n^{(\alpha)}(x, y; \lambda, a, b, c)$ holds true:

$$E_n^{(\alpha)}(x + 1, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{k} (\ln c)^{n-m} E_m^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.33)$$

Corollary 3 For arbitrary real or complex parameter α , the following implicit summation formula involving generalized Apostol Genocchi polynomials $G_n^{(\alpha)}(x, y; \lambda, a, b, c)$ holds true:

$$G_n^{(\alpha)}(x + 1, y; \lambda, a, b, c) = \sum_{m=0}^n \binom{n}{k} (\ln c)^{n-m} G_m^{(\alpha)}(x, y; \lambda, a, b, c) \quad (3.34)$$

4. GENERAL SYMMETRY IDENTITIES

In this section, we give general symmetry identities for the generalized polynomials $P_{n,\beta}^{(\alpha)}(x, y; k, a, b)$ and $P_{n,\beta}^{(\alpha)}(k, a, b)$ by applying the generating functions (1.1) and (2.6). The results extend some known identities of Zhang et al [31], Yang et al [30], Pathan [20], Pathan and Khan [21] and Ozarslan [15]. Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 4.1 Let $\alpha, k \in N_0, a, b \in R/\{0\}; \beta \in C, x, y \in R$ and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} d^m c^{n-m} {}_H P_{n-m,\beta}^{(\alpha)}(dx, d^2z; k, a, b) {}_H P_{m,\beta}^{(\alpha)}(cy, c^2z; k, a, b) \\ &= \sum_{m=0}^n \binom{n}{m} c^m d^{n-m} {}_H P_{n-m,\beta}^{(\alpha)}(cx, c^2z; k, a, b) {}_H P_{m,\beta}^{(\alpha)}(dy, d^2z; k, a, b) \end{aligned} \quad (4.1)$$

Proof. Start with

$$g(t) = \left(\frac{c^k d^k 2^{2(1-k)} t^{2k}}{(\beta^b e^{ct} - a^b)(\beta^b e^{dt} - a^b)} \right)^\alpha e^{cd(x+y)t + (c^2+d^2)zt^2} \quad (4.2)$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t) = \sum_{n=0}^{\infty} {}_H P_{n,\beta}^{(\alpha)}(dx, d^2z; k, a, b) \frac{(ct)^n}{n!} \sum_{m=0}^{\infty} {}_H P_{m,\beta}^{(\alpha)}(cy, c^2z; k, a, b) \frac{(dt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^n {}_H P_{n-m, \beta}^{(\alpha)}(dx, d^2 z; k, a, b) \frac{(c)^{n-m}}{(n-m)!} {}_H P_{m, \beta}^{(\alpha)}(cy, c^2 z; k, a, b) \frac{(d)^m (t)^n}{m! n!}$$

On the similar lines we can show that

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_H P_{n, \beta}^{(\alpha)}(cx, c^2 z; k, a, b) \frac{(dt)^n}{n!} \sum_{m=0}^{\infty} {}_H P_{m, \beta}^{(\alpha)}(dy, d^2 z; k, a, b) \frac{(ct)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n {}_H P_{n-m, \beta}^{(\alpha)}(cx, c^2 z; k, a, b) \frac{(d)^{n-m}}{(n-m)!} {}_H P_{m, \beta}^{(\alpha)}(dy, d^2 z; k, a, b) \frac{(c)^m (t)^n}{m! n!} \end{aligned}$$

by comparing the coefficients of t^n on the right hand sides of the last two equations we arrive at the desired result.

For $k=a=b=1$ and $\beta = \lambda$ in Theorem 4.1, we get the following corollary

Corollary 1 For all $c, d, m \in \mathbb{N}, n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the Hermite-based generalized Apostol-Bernoulli polynomials

$$\begin{aligned} &\sum_{m=0}^n \binom{n}{m} d^m c^{n-m} {}_H B_{n-m}^{(\alpha)}(dx, d^2 z; \lambda) {}_H B_m^{(\alpha)}(cy, c^2 z; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} c^m d^{n-m} {}_H B_{n-m}^{(\alpha)}(cx, c^2 z; \lambda) {}_H B_m^{(\alpha)}(dy, d^2 z; \lambda) \end{aligned} \quad (4.3)$$

Remark For $\lambda = 1$ in equation (4.3), the result reduces to known result of Pathan and Khan [21] and further by taking $\alpha = 1, \lambda = 1$ in equation (4.3), the result reduces to another known result of Pathan [20]

For $k+1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 4.1, we get the corollary

Corollary 2 For all $m \in \mathbb{N}, n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} &\sum_{m=0}^n \binom{n}{m} d^m c^{n-m} {}_H E_{n-m}^{(\alpha)}(dx, d^2 z; \lambda) {}_H E_m^{(\alpha)}(cy, c^2 z; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} c^m d^{n-m} {}_H E_{n-m}^{(\alpha)}(cx, c^2 z; \lambda) {}_H E_m^{(\alpha)}(dy, d^2 z; \lambda) \end{aligned} \quad (4.4)$$

Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in Theorem 4.1, we get the corollary

Corollary 3 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} d^m c^{n-m} {}_H G_{n-m}^{(\alpha)}(dx, d^2 z; \lambda) {}_H G_m^{(\alpha)}(cy, c^2 z; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} c^m d^{n-m} {}_H G_{n-m}^{(\alpha)}(cx, c^2 z; \lambda) {}_H G_m^{(\alpha)}(dy, d^2 z; \lambda) \end{aligned} \quad (4.5)$$

Theorem 4.2 Let $\alpha, k \in N_0; a, b \in R/\{0\}; \beta \in C$ and $x, y \in R$ and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H P_{n-m, \beta}^{(\alpha)} \left(dx + \frac{d}{c} i + j, d^2 z; k, a, b \right) P_{m, \beta}^{\alpha}(cy; k, a, b) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H P_{n-m, \beta}^{(\alpha)} \left(cx + \frac{c}{d} i + j, c^2 z; k, a, b \right) P_{m, \beta}^{\alpha}(dy; k, a, b) \end{aligned} \quad (4.6)$$

Proof. Let

$$\begin{aligned} g(t) &= \left(\frac{2^{2(1-k)} c^k d^k t^{2k}}{(\beta^b e^{ct} - a^b)(\beta^b e^{dt} - a^b)} \right)^{\alpha} \frac{(e^{cdt} - 1)^2}{(e^{ct} - 1)(e^{dt} - 1)} e^{cd(x+y)t + c^2 d^2 z t^2} \\ g(t) &= \left(\frac{2^{(1-k)} c^k t^k}{(\beta^b e^{ct} - a^b)} \right)^{\alpha} e^{cdxt + c^2 d^2 z t^2} \left(\frac{e^{cdt} - 1}{e^{ct} - 1} \right) \left(\frac{2^{(1-k)} d^k t^k}{(\beta^b e^{dt} - a^b)} \right)^{\alpha} e^{cdyt} \left(\frac{e^{cdt} - 1}{e^{dt} - 1} \right) \\ &= \left(\frac{2^{(1-k)} c^k t^k}{(\beta^b e^{ct} - a^b)} \right)^{\alpha} e^{cdxt + c^2 d^2 z t^2} \sum_{i=0}^{c-1} e^{dti} \left(\frac{2^{(1-k)} d^k t^k}{(\beta^b e^{dt} - a^b)} \right)^{\alpha} e^{cdyt} \sum_{j=0}^{d-1} e^{ctj} \\ &= \left(\frac{2^{(1-k)} c^k t^k}{(\beta^b e^{ct} - a^b)} \right)^{\alpha} e^{c^2 d^2 z t^2} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} e^{(dx + \frac{d}{c} i + j)ct} \sum_{m=0}^{\infty} P_{m, \beta}^{\alpha}(cy; k, a, b) \frac{(dt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H P_{n-m, \beta}^{(\alpha)} \left(dx + \frac{d}{c} i + j, d^2 z; k, a, b \right) P_{m, \beta}^{\alpha}(cy; k, a, b) \end{aligned} \quad (4.7)$$

On the other hand

$$g(t) = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} d^{n-m} c^m {}_H P_{n-m, \beta}^{(\alpha)} \left(cx + \frac{c}{d} i + j, c^2 z; k, a, b \right) P_{m, \beta}^{\alpha}(dy; k, a, b) \quad (4.8)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

For $k=a=b=1$ and $\beta = \lambda$ in Theorem 4.2, we get the following corollary at once

Corollary 1 For all $c, d, m \in N, n \in N_0, \lambda \in C$, we have the following symmetry identity for the Hermite-based generalized Apostol-Bernoulli polynomials

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H B_{n-m}^{(\alpha)} \left(dx + \frac{d}{c}i + j, d^2 z; \lambda \right) B_m^\alpha(cy; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H B_{n-m}^{(\alpha)} \left(cx + \frac{c}{d}i + j, c^2 z; \lambda \right) B_m^\alpha(dy; \lambda) \end{aligned} \quad (4.9)$$

Remark For $\lambda = 1$ in equation (4.9), the result reduces to known result of Pathan and Khan [21] and further by taking $\alpha = 1, \lambda = 1$ in equation (4.9), the result reduces to known result of Pathan [20]

For $k + 1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 4.2, we get the corollary

Corollary 2 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H E_{n-m}^{(\alpha)} \left(dx + \frac{d}{c}i + j, d^2 z; \lambda \right) E_m^\alpha(cy; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H E_{n-m}^{(\alpha)} \left(cx + \frac{c}{d}i + j, c^2 z; \lambda \right) E_m^\alpha(dy; \lambda) \end{aligned} \quad (4.10)$$

Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in Theorem 4.2, we get the corollary

Corollary 3 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H G_{n-m}^{(\alpha)} \left(dx + \frac{d}{c}i + j, d^2 z; \lambda \right) G_m^\alpha(cy; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H G_{n-m}^{(\alpha)} \left(cx + \frac{c}{d}i + j, c^2 z; \lambda \right) G_m^\alpha(dy; \lambda) \end{aligned} \quad (4.11)$$

Theorem 4.3 For each pair of integers a and b and all integers and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H P_{n-m,\beta}^{(\alpha)} \left(dx + \frac{d}{c}i, d^2z; k, a, b \right) P_{m,\beta}^\alpha \left(cy + \frac{c}{d}j; k, a, b \right) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H P_{n-m,\beta}^{(\alpha)} \left(cx + \frac{c}{d}i, c^2z; k, a, b \right) P_{m,\beta}^\alpha \left(dy + \frac{d}{c}j; k, a, b \right) \end{aligned} \quad (4.12)$$

Proof. The proof is analogous to Theorem 4.2 but we need to write equation (4.6) in the form

$$g(t) = \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H P_{n-m,\beta}^{(\alpha)} \left(dx + \frac{d}{c}i, d^2z; k, a, b \right) P_{m,\beta}^\alpha \left(cy + \frac{c}{d}j; k, a, b \right) \quad (4.13)$$

On the other hand equation (4.6) can be shown equal to

$$g(t) = \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H P_{n-m,\beta}^{(\alpha)} \left(cx + \frac{c}{d}i, c^2z; k, a, b \right) P_{m,\beta}^\alpha \left(dy + \frac{d}{c}j; k, a, b \right) \quad (4.14)$$

Next making change of index and by equating the coefficients of t^n to zero in (4.13) and (4.14), we get the theorem.

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

For $k=a=b=1$ and $\beta = \lambda$ in Theorem 4.3, we get the following corollary at once

Corollary 1 For all $c, d, m \in \mathbb{N}, n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the Hermite-based generalized Apostol-Bernoulli polynomials

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H B_{n-m}^{(\alpha)} \left(dx + \frac{d}{c}i, d^2z; \lambda \right) B_m^\alpha \left(cy + \frac{c}{d}j; \lambda \right) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H B_{n-m}^{(\alpha)} \left(cx + \frac{c}{d}i, c^2z; \lambda \right) B_m^\alpha \left(dy + \frac{d}{c}j; \lambda \right) \end{aligned} \quad (4.15)$$

Remark For $\lambda = 1$ in equation (4.15), the result reduces to known result of Pathan and Khan [21] and further by taking $\alpha = 1, \lambda = 1$ in equation (4.15), the result reduces to known result of Pathan [20]

For $k+1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 4.3, we get the corollary

Corollary 2 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H E_{n-m}^{(\alpha)} \left(dx + \frac{d}{c} i, d^2 z; \lambda \right) E_m^\alpha \left(cy + \frac{c}{d} j; \lambda \right) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H E_{n-m}^{(\alpha)} \left(cx + \frac{c}{d} i, c^2 z; \lambda \right) E_m^\alpha \left(dy + \frac{d}{c} j; \lambda \right) \end{aligned} \quad (4.16)$$

Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in Theorem 4.3, we get the corollary

Corollary 3 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m {}_H G_{n-m}^{(\alpha)} \left(dx + \frac{d}{c} i, d^2 z; \lambda \right) G_m^\alpha \left(cy + \frac{c}{d} j; \lambda \right) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^m d^{n-m} {}_H G_{n-m}^{(\alpha)} \left(cx + \frac{c}{d} i, c^2 z; \lambda \right) G_m^\alpha \left(dy + \frac{d}{c} j; \lambda \right) \end{aligned} \quad (4.17)$$

Now, we prove the following symmetric identities involving sum of integer powers $S_k(n)$ and $M_k(n)$ given by equation (1.13) and (1.14), Hermite-based polynomials ${}_H P_{n,\beta}^{(\alpha)}(x, y; k, a, b)$ and $P_{n,\beta}^{(\alpha)}(x; k, a, b)$

Theorem 4.4 For all integers $a > 0, b > 0$ and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} c^{n-m} d^{m+1} {}_H P_{n-m,\beta}^{(\alpha)}(dx, d^2 z; k, a, b) \sum_{i=0}^m \binom{m}{i} S_i \left(c-1; \left(\frac{\beta}{a}\right)^b \right) P_{m-i,\beta}^\alpha(cy; k, a, b) \\ &= \sum_{m=0}^n \binom{n}{m} c^{m+1} d^{n-m} {}_H P_{n-m,\beta}^{(\alpha)}(cx, c^2 z; k, a, b) \sum_{i=0}^m \binom{m}{i} S_i \left(d-1; \left(\frac{\beta}{a}\right)^b \right) P_{m-i,\beta}^\alpha(dy; k, a, b) \end{aligned} \quad (4.18)$$

Proof. We now use

$$\begin{aligned} g(t) &= \frac{(2^{2(1-k)} c^k d^k t^{2k})^\alpha (\beta^b e^{cdt} - a^b) e^{cd(x+y)t + c^2 d^2 z t^2}}{(\beta^b e^{cct} - a^b)^\alpha (\beta^b e^{dct} - a^b)^{\alpha+1}} \\ g(t) &= \left(\frac{2^{(1-k)} c^k t^k}{\beta^b e^{cct} - a^b} \right)^\alpha e^{cdxt + c^2 d^2 z t^2} \left(\frac{\beta^b e^{cdt} - a^b}{\beta^b e^{dct} - a^b} \right) \left(\frac{2^{(1-k)} d^k t^k}{(\beta^b e^{dct} - a^b)} \right)^\alpha e^{cdyt} \end{aligned}$$

$$= \sum_{n=0}^{\infty} {}_H P_{n,\beta}^{(\alpha)}(dx, d^2z; k, a, b) \frac{(ct)^n}{n!} \sum_{n=0}^{\infty} S_n \left(c-1; \left(\frac{\beta}{a}\right)^b \right) \frac{(dt)^n}{n!} \sum_{n=0}^{\infty} P_{n,\beta}^{\alpha}(cy; k, a, b) \frac{(dt)^n}{n!}$$

Using a similar plan, we get

$$g(t) = \sum_{n=0}^{\infty} {}_H P_{n,\beta}^{(\alpha)}(cx, c^2z; k, a, b) \frac{(dt)^n}{n!} \sum_{n=0}^{\infty} S_n \left(d-1; \left(\frac{\beta}{a}\right)^b \right) \frac{(ct)^n}{n!} \sum_{n=0}^{\infty} P_{n,\beta}^{\alpha}(dy; k, a, b) \frac{(ct)^n}{n!}$$

Finally, (4.18) follows after an appropriate change of summation index and comparison of the coefficients of t .

For $k=a=b=1$ and $\beta = \lambda$ in Theorem 4.4, we get the following corollary at once

Corollary 1 For all $c, d, m \in N, n \in N_0, \lambda \in C$, we have the following symmetry identity for the Hermite-based generalized Apostol-Bernoulli polynomials

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} c^{n-m} d^{m+1} {}_H B_{n-m}^{(\alpha)}(dx, d^2z; \lambda) \sum_{i=0}^m \binom{m}{i} S_i(c-1; \lambda) B_{m-i}^{\alpha}(cy; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} c^{m+1} d^{n-m} {}_H B_{n-m}^{(\alpha)}(cx, c^2z; \lambda) \sum_{i=0}^m \binom{m}{i} S_i(d-1; \lambda) B_{m-i}^{\alpha}(dy; \lambda) \end{aligned} \quad (4.19)$$

Remark For $\lambda = 1$ in equation (4.19), the result reduces to known result of Pathan and Khan [21] and further by taking $\alpha = 1, \lambda = 1$ in equation (4.19), the result reduces to known result of Pathan [20]

For $k+1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 4.4, we get the corollary

Corollary 2 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} c^{n-m} d^{m+1} {}_H E_{n-m}^{(\alpha)}(dx, d^2z; \lambda) \sum_{i=0}^m \binom{m}{i} M_i(c-1; \lambda) E_{m-i}^{\alpha}(cy; \lambda) \\ &= \sum_{m=0}^n \binom{n}{m} c^{m+1} d^{n-m} {}_H E_{n-m}^{(\alpha)}(cx, c^2z; \lambda) \sum_{i=0}^m \binom{m}{i} M_i(d-1; \lambda) E_{m-i}^{\alpha}(dy; \lambda) \end{aligned} \quad (4.20)$$

Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in Theorem 4.4, we get the corollary

Corollary 3 For all $m \in N, n \in N_0, \lambda \in C$, we have for each pair of positive even integers c and d or for each pair of positive odd integers c and d .

$$\sum_{m=0}^n \binom{n}{m} c^{n-m} d^{m+1} {}_H G_{n-m}^{(\alpha)}(dx, d^2z; \lambda) \sum_{i=0}^m \binom{m}{i} M_i(c-1; \lambda) G_{m-i}^{\alpha}(cy; \lambda)$$

$$= \sum_{m=0}^n \binom{n}{m} c^{m+1} d^{n-m} {}_H G_{n-m}^{(\alpha)}(cx, c^2z; \lambda) \sum_{i=0}^m \binom{m}{i} M_i(d-1; \lambda) G_{m-i}^{\alpha}(dy; \lambda) \quad (4.21)$$

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