ON A SYSTEM OF FOURTH-ORDER RATIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the qualitative behavior of a system of fourth-order rational difference equations. More precisely, we study the local asymptotic stability and global asymptotic character of the unique equilibrium point of a fourth-order discrete dynamical system of rational form. Moreover, boundedness behavior and the rate of convergence of the positive solutions which converge to equilibrium at origin are investigated. Some numerical example are given to verify our theoretical results.

1. Introduction and preliminaries

Recently, studying the qualitative behavior of difference equations and systems is a topic of a great interest. Applications of discrete dynamical systems and difference equations have appeared recently in many areas such as ecology, population dynamics, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, neural networks, quanta in radiation, genetics in biology, economics, psychology, sociology, physics, engineering, economics, probability theory and resource management. Unfortunately, these are only considered as the discrete analogs of differential equations. It is a well-known fact that difference equations appeared much earlier than differential equations and were instrumental in paving the way for the development of the latter. It is only recently that difference equations have started receiving the attention they deserve. Perhaps this is largely due to the advent of computers where differential equations are solved by using their approximate difference equation formulations. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. The theory of difference equations

occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems. For more results for qualitative behavior of difference equations, we refer the interested reader to [3, 4, 6, 7, 10, 11, 12].

Zhang *et al.* [5] studied the dynamics of a system of rational third-order difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \ y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}, \ n = 0, 1, \cdots.$$

Din *et al.* [10] investigated the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \ n = 0, 1, \cdots,$$

Our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations:

$$x_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3}}, \ y_{n+1} = \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3}}, \quad (1)$$

$n = 0, 1, \cdots$, where the parameters α_1, β_1, γ_1, α_2, β_2, γ_2 and initial conditions x_0, x_{-1}, x_{-2}, x_{-3}, y_0, y_{-1}, y_{-2}, y_{-3} are positive real numbers.

Let us consider eighth-dimensional discrete dynamical system of the form:

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}, x_{n-3}, y_{n-3}), \quad (2) \\
y_{n+1} &= g(x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}, x_{n-3}, y_{n-3}),
\end{aligned}
$$

$n = 0, 1, \cdots$, where $f : I^4 \times J^4 \to I$ and $g : I^4 \times J^4 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$ of system (2) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-3, -2, -1, 0\}$. Along with system (2) we consider the corresponding vector map $F = (f, g, x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2})$. An equilibrium point of (2) is a point $(\bar{x}, \bar{y})$ that satisfies

$$
\begin{aligned}
\bar{x} &= f(\bar{x}, \bar{y}, \bar{x}, \bar{y}, \bar{x}, \bar{y}, \bar{x}, \bar{y}) \\
\bar{y} &= g(\bar{x}, \bar{y}, \bar{x}, \bar{y}, \bar{x}, \bar{y}, \bar{x}, \bar{y})
\end{aligned}
$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2).*

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-3, -2, -1, 0\}$ if

$$\left\| \sum_{i=-3}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \delta \text{ implies } \|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon \text{ for all } n > 0, \text{ where } \|.\| \text{ is}$$

usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists

$$\eta > 0 \text{ such that } \left\| \sum_{i=-3}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \eta \text{ and } (x_n, y_n) \to (\bar{x}, \bar{y}) \text{ as } n \to \infty.$$

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of a map*

$$F = (f, g, x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is:

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ x_{n-2} \\ y_{n-2} \\ x_{n-3} \\ y_{n-3} \end{pmatrix}$ and F_J is Jacobian matrix of system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

To construct corresponding linearized form of system (1) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}, x_{n-3}, y_{n-3}) \mapsto (f, g, f_1, g_1, f_2, g_2, f_3, g_3), \quad (3)$$

where $f = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3}}$, $g = \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3}}$, $f_1 = x_n$, $g_1 = y_n$, $f_2 = x_{n-1}$, $g_2 = y_{n-1}$, $f_2 = x_{n-2}$ and $g_2 = y_{n-2}$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (3) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & A_1 & 0 & A_1 & B_1 & 0 & C_1 & 0 \\ A_2 & 0 & A_2 & 0 & 0 & B_2 & 0 & C_2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where $A_1 = -\frac{\alpha_1 \gamma_1 \bar{x}^3 \bar{y}}{(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)^2}$, $B_1 = -\frac{\alpha_1 \gamma_1 \bar{x}^2 \bar{y}^2}{(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)^2}$, $C_1 = \frac{\alpha_1 \beta_1}{(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)^2}$, $A_2 = -\frac{\alpha_2 \gamma_2 \bar{x} \bar{y}^3}{(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)^2}$, $B_2 = -\frac{\alpha_2 \gamma_2 \bar{x}^2 \bar{y}^2}{(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)^2}$ and $C_2 = \frac{\alpha_2 \beta_2}{(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)^2}$. The characteristic polynomial of $F_J(\bar{x}, \bar{y})$ about equilibrium point $(\bar{x}, \bar{y})$ is given by

$$P(\lambda) = \lambda^8 - A_1 A_2 \lambda^6 - A\lambda^5 - B\lambda^4 + B_1 B_2 \lambda^2 + (B_1 C_2 + B_2 C_1)\lambda + C_1 C_2, \quad (4)$$

where $A = B_1 + B_2 + 2A_1 A_2$ and $B = C_1 + C_2 + A_1 A_2$.

Lemma 1. *[1] Suppose that the system $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, of difference equations has $\bar{X}$ as a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

2. Main results

Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then system (1) has only one equilibrium point which is $(0, 0)$. In this section, we will show that the unique equilibrium point $(0, 0)$ of system (1) is locally asymptotically stable as well as globally asymptotically stable under certain parametric conditions. Moreover, boundedness character and rate of convergence of positive solutions of (1) are also investigated.

Theorem 2. *Assume that $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then every positive solution of system (1) is bounded.*

Proof. It follows from system (1) that

$$x_{n+1} \leq \frac{\alpha_1 x_{n-3}}{\beta_1}, \ y_{n+1} \leq \frac{\alpha_2 y_{n-3}}{\beta_2}, \ n = 0, 1, \cdots.$$

Consider the following linear fourth-order difference equations

$$u_{n+1} = \frac{\alpha_1 u_{n-3}}{\beta_1}, \ n = 0, 1, \cdots, \tag{5}$$

and

$$v_{n+1} = \frac{\alpha_2 v_{n-3}}{\beta_2}, \ n = 0, 1, \cdots. \tag{6}$$

Then solution of (5) is given by

$$u_n = c_1 \left(\frac{\iota\alpha_1}{\beta_1}\right)^{\frac{n}{4}} + c_2 \left(-\frac{\alpha_1}{\beta_1}\right)^{\frac{n}{4}} + c_3 \left(-\frac{\iota\alpha_1}{\beta_1}\right)^{\frac{n}{4}} + c_4 \left(\frac{\alpha_1}{\beta_1}\right)^{\frac{n}{4}},$$

where c_1, c_2, c_3 and c_4 depend on initial values u_{-3}, u_{-2}, u_{-1} and u_0. Similarly, solution of (6) is given by

$$v_n = r_1 \left(\frac{\iota\alpha_2}{\beta_2}\right)^{\frac{n}{4}} + r_2 \left(-\frac{\alpha_2}{\beta_2}\right)^{\frac{n}{4}} + r_3 \left(-\frac{\iota\alpha_2}{\beta_2}\right)^{\frac{n}{4}} + r_4 \left(\frac{\alpha_2}{\beta_2}\right)^{\frac{n}{4}},$$

where r_1, r_2, r_3 and r_4 depend on initial values v_{-3}, v_{-2}, v_{-1} and v_0. Suppose that $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, $u_i = x_i$ and $v_i = y_i$ for $i \in \{-3, -2, -1, 0\}$, then by comparison method we obtain that $x_n \leq \sum\limits_{i=-3}^{0} x_i$ and $y_n \leq \sum\limits_{i=-3}^{0} y_i$ for all $n = 1, 2, \cdots$.

Theorem 3. *If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then the equilibrium $(0,0)$ of (1) is locally asymptotically stable.*

Proof. It is easy to see that the linearized system of (1) about the equilibrium point $(0,0)$ is given by

$$\Phi_{n+1} = D\Phi_n, \tag{7}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ x_{n-2} \\ y_{n-2} \\ x_{n-3} \\ y_{n-3} \end{pmatrix},$$

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and

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_2}{\beta_2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of D is given by

$$P(\lambda) = \lambda^8 - \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} \right) \lambda^4 + \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2}.$$

Then roots of this characteristic polynomial are given by

$$\lambda_{1,2} = \pm \frac{\alpha_1}{\beta_1}, \ \ \lambda_{3,4} = \pm \iota \frac{\alpha_1}{\beta_1}, \ \ \lambda_{5,6} = \pm \frac{\alpha_2}{\beta_2}, \ \ \lambda_{7,8} = \pm \iota \frac{\alpha_2}{\beta_2}.$$

Now, it is easy to see that $|\lambda_k| < 1$ for all $k = 1, 2, \cdots, 8$. Since all eigenvalues of Jacobian matrix $F_J(0,0)$ about $(0,0)$ lie in open unit disk $|\lambda| < 1$. Hence from Lemma 1 the equilibrium point $(0,0)$ is locally asymptotically stable.

Theorem 4. *If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then the equilibrium $(0,0)$ of system (1) is globally asymptotically stable.*

Proof. We know from Theorem 3 that the equilibrium point $(0,0)$ of system (1) is locally asymptotically stable, and so it suffices to show that $(0,0)$ is a global attractor. It follows from Theorem 2 that

$$0 \leq |x_n| \leq (x_{-3} + x_{-2} + x_{-1} + x_0) \left(\frac{\alpha_1}{\beta_1} \right)^{\frac{n}{4}},$$

and

$$0 \leq |y_n| \leq (y_{-3} + y_{-2} + y_{-1} + y_0) \left(\frac{\alpha_2}{\beta_2} \right)^{\frac{n}{4}},$$

for all $n = 1, 2, \cdots$. Assume that $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then we obtain that

$$\lim_{n \to \infty} x_n = 0 \ \text{ and } \ \lim_{n \to \infty} y_n = 0.$$

Thus equilibrium point $(0,0)$ is a global attractor. Using this and the local asymptotic stability proven in Theorem 3 the proof follows.

2.1. Rate of convergence

In this subsection we will determine the rate of convergence of solutions of system (1) which converge to the equilibrium point $(0, 0)$. The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B(n)) X_n, \tag{8}$$

where X_n is an m-dimensional vector, $A \in C^{m \times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m \times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{9}$$

as $n \to \infty$, where $\| \cdot \|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Lemma 5. *(Perron's Theorem)[2] Suppose that condition (9) holds. If X_n is a solution of (8), then either $X_n = 0$ for large n or*

$$\rho = \lim_{n \to \infty} (\|X_n\|)^{1/n}, \tag{10}$$

or

$$\rho = \lim_{n \to \infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{11}$$

exists and is equal to the norm of one of the eigenvalues of the matrix A.

Assume that $\lim_{n \to \infty} x_n = \bar{x}$ and $\lim_{n \to \infty} x_n = \bar{y}$. First we will find a system of limiting equations for system (1). The error terms are given as

$$
\begin{aligned}
x_{n+1} - \bar{x} &= \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3}} - \frac{\alpha_1 \bar{x}}{\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2} \\
&= -\frac{\alpha_1 \gamma_1 \bar{x} y_{n-1} x_{n-2} x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)}(y_n - \bar{y}) \\
&\quad - \frac{\alpha_1 \gamma_1 \bar{x} \bar{y} x_{n-2} x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)}(y_{n-1} - \bar{y}) \\
&\quad - \frac{\alpha_1 \gamma_1 \bar{x} \bar{y}^2 x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)}(x_{n-2} - \bar{x}) \\
&\quad + \frac{\alpha_1 \gamma_1}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)}(x_{n-3} - \bar{x}),
\end{aligned}
$$

and

$$
\begin{aligned}
y_{n+1} - \bar{y} &= \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3}} - \frac{\alpha_2 \bar{y}}{\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2} \\
&= -\frac{\alpha_2 \gamma_2 \bar{y} x_{n-1} y_{n-2} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)}(x_n - \bar{x}) \\
&\quad - \frac{\alpha_2 \gamma_2 \bar{x} \bar{y} y_{n-2} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)}(x_{n-1} - \bar{x}) \\
&\quad - \frac{\alpha_2 \gamma_2 \bar{x}^2 \bar{y} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)}(y_{n-2} - \bar{y}) \\
&\quad + \frac{\alpha_2 \beta_2}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)}(y_{n-3} - \bar{y}).
\end{aligned}
$$

We let $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, then one has

$$
e_{n+1}^1 = F_1 e_n^2 + F_2 e_{n-1}^2 + F_3 e_{n-2}^1 + F_4 e_{n-3}^1,
$$

and

$$
e_{n+1}^2 = G_1 e_n^1 + G_2 e_{n-1}^1 + G_3 e_{n-2}^2 + G_4 e_{n-3}^2,
$$

where

$$
F_1 = -\frac{\alpha_1 \gamma_1 \bar{x} y_{n-1} x_{n-2} x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)},
$$

$$
F_2 = -\frac{\alpha_1 \gamma_1 \bar{x} \bar{y} x_{n-2} x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)},
$$

$$
F_3 = -\frac{\alpha_1 \gamma_1 \bar{x} \bar{y}^2 x_{n-3}}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)},
$$

$$
F_4 = \frac{\alpha_1 \gamma_1}{(\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3})(\beta_1 + \gamma_1 \bar{x}^2 \bar{y}^2)},
$$

$$
G_1 = -\frac{\alpha_2 \gamma_2 \bar{y} x_{n-1} y_{n-2} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)},
$$

$$
G_2 = -\frac{\alpha_2 \gamma_2 \bar{x} \bar{y} y_{n-2} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)},
$$

$$
G_3 = -\frac{\alpha_2 \gamma_2 \bar{x}^2 \bar{y} y_{n-3}}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)},
$$

$$
G_4 = \frac{\alpha_2 \beta_2}{(\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3})(\beta_2 + \gamma_2 \bar{x}^2 \bar{y}^2)}.
$$

Now, it is easy to see that

$$\lim_{n\to\infty} F_1 = \lim_{n\to\infty} F_2 = -\frac{\alpha_1\gamma_1\bar{x}^3\bar{y}}{(\beta_1 + \gamma_1\bar{x}^2\bar{y}^2)^2},$$

$$\lim_{n\to\infty} F_3 = -\frac{\alpha_1\gamma_1\bar{x}^2\bar{y}^2}{(\beta_1 + \gamma_1\bar{x}^2\bar{y}^2)^2}, \quad \lim_{n\to\infty} F_4 = \frac{\alpha_1\beta_1}{(\beta_1 + \gamma_1\bar{x}^2\bar{y}^2)^2},$$

$$\lim_{n\to\infty} G_1 = \lim_{n\to\infty} G_2 = -\frac{\alpha_2\gamma_2\bar{x}\bar{y}^3}{(\beta_2 + \gamma_2\bar{x}^2\bar{y}^2)^2},$$

$$\lim_{n\to\infty} G_3 = -\frac{\alpha_2\gamma_2\bar{x}^2\bar{y}^2}{(\beta_2 + \gamma_2\bar{x}^2\bar{y}^2)^2}, \quad \lim_{n\to\infty} G_4 = \frac{\alpha_2\beta_2}{(\beta_2 + \gamma_2\bar{x}^2\bar{y}^2)^2}.$$

Hence, the limiting system of error terms at $(\bar{x}, \bar{y}) = (0,0)$ can be written as

$$E_{n+1} = KE_n, \tag{12}$$

where

$$E_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \\ e_{n-2}^1 \\ e_{n-2}^2 \\ e_{n-3}^1 \\ e_{n-3}^2 \end{pmatrix},$$

and

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_2}{\beta_2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

One can observe that (12) is similar to linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y}) = (0,0)$. Using Lemma 5, one has following result.

Theorem 6. *Assume that $\{(x_n, y_n)\}$ be a positive solution of system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x}, \bar{y}) = (0,0)$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{x}, \bar{y})|,$$

where $\lambda F_J(\bar{x}, \bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(0,0)$.

3. Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider some interesting numerical examples in this section. These examples show that the unique equilibrium point $(0,0)$ of system (1) is globally asymptotically stable if and only if $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$.

Example 1. Let $\alpha_1 = 17$, $\beta_1 = 18$, $\gamma_1 = 1.5$, $\alpha_2 = 15$, $\beta_2 = 16$ and $\gamma_2 = 1.2$. Then, system (1) can be written as:

$$x_{n+1} = \frac{17x_{n-3}}{18 + 1.5y_n y_{n-1} x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{15y_{n-3}}{16 + 1.2x_n x_{n-1} y_{n-2} y_{n-3}}, \quad (13)$$

with initial conditions $x_{-3} = 1.3$, $x_{-2} = 1.8$, $x_{-1} = 1.6$, $x_0 = 1.1$, $y_{-3} = 0.1$, $y_{-2} = 1.2$, $y_{-1} = 0.8$, $y_0 = 0.5$. Furthermore, in Fig. 1 plot of x_n is shown in Fig. 1a, plot of y_n is shown in Fig. 1b and global attractor of system (13) is shown in Fig. 1c.

Example 2. Let $\alpha_1 = 1.7$, $\beta_1 = 1.74$, $\gamma_1 = 1.1$, $\alpha_2 = 1.5$, $\beta_2 = 1.54$ and $\gamma_2 = 1.2$. Then, system (1) can be written as:

$$x_{n+1} = \frac{1.7x_{n-3}}{1.74 + 1.1y_n y_{n-1} x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{1.5y_{n-3}}{1.54 + 1.2x_n x_{n-1} y_{n-2} y_{n-3}}, \quad (14)$$

with initial conditions $x_{-3} = 0.3$, $x_{-2} = 0.8$, $x_{-1} = 0.6$, $x_0 = 0.1$, $y_{-3} = 0.4$, $y_{-2} = 0.2$, $y_{-1} = 0.7$, $y_0 = 0.5$. Furthermore, in Fig. 2 plot of x_n is shown in Fig. 2a, plot of y_n is shown in Fig. 2b and global attractor of system (14) is shown in Fig. 2c.

Example 3. Let $\alpha_1 = 80$, $\beta_1 = 84$, $\gamma_1 = 7.5$, $\alpha_2 = 150$, $\beta_2 = 153$ and $\gamma_2 = 4.5$. Then, system (1) can be written as:
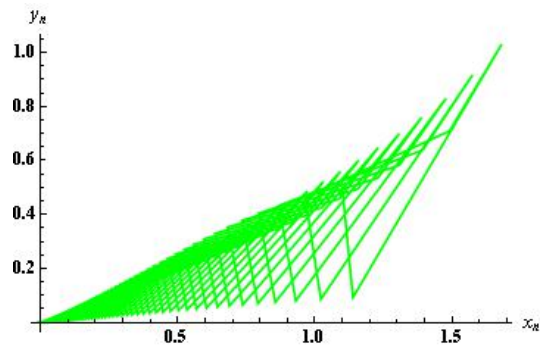
$$x_{n+1} = \frac{80x_{n-3}}{84 + 7.5y_n y_{n-1} x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{150y_{n-3}}{153 + 4.5x_n x_{n-1} y_{n-2} y_{n-3}}, \quad (15)$$

with initial conditions $x_{-3} = 0.5$, $x_{-2} = 0.3$, $x_{-1} = 0.1$, $x_0 = 0.2$, $y_{-3} = 0.1$, $y_{-2} = 0.4$, $y_{-1} = 0.6$, $y_0 = 0.8$. Furthermore, in Fig. 3 plot of x_n is shown in Fig. 3a, plot of y_n is shown in Fig. 3b and global attractor of system (15) is shown in Fig. 3c.
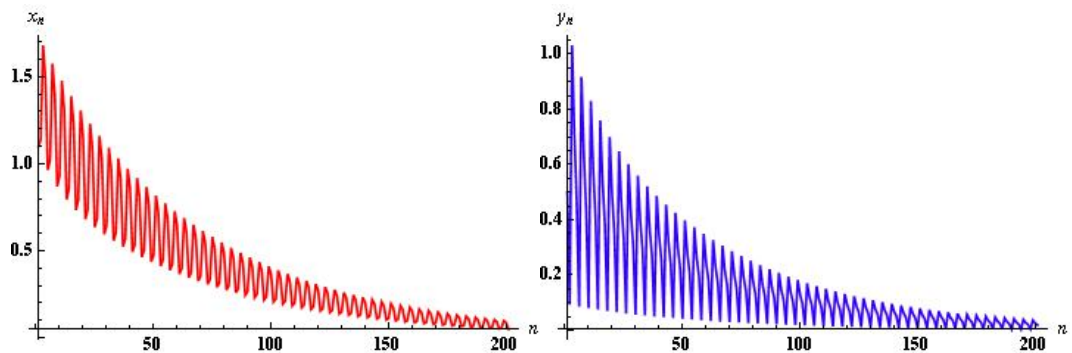
(a) Plot of x_n for the system (13)

(b) Plot of y_n for the system (13)

(c) An attractor of the system (13)

Figure 1: Plots for the system (13)

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(a) Plot of x_n for the system (14)

(b) Plot of y_n for the system (14)
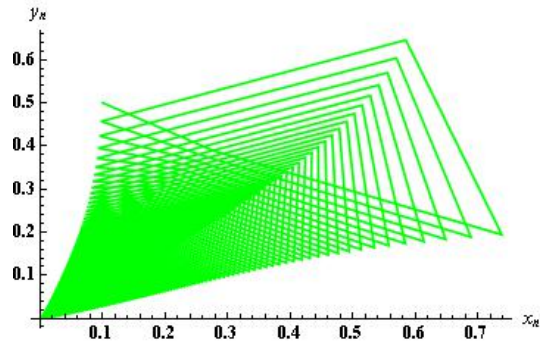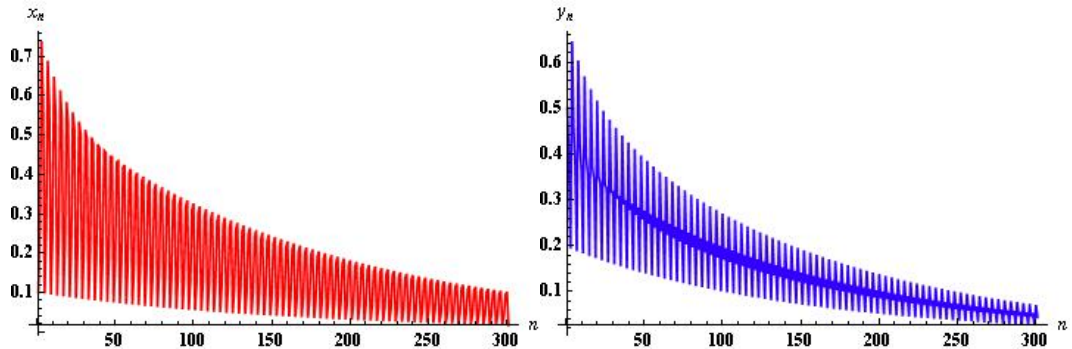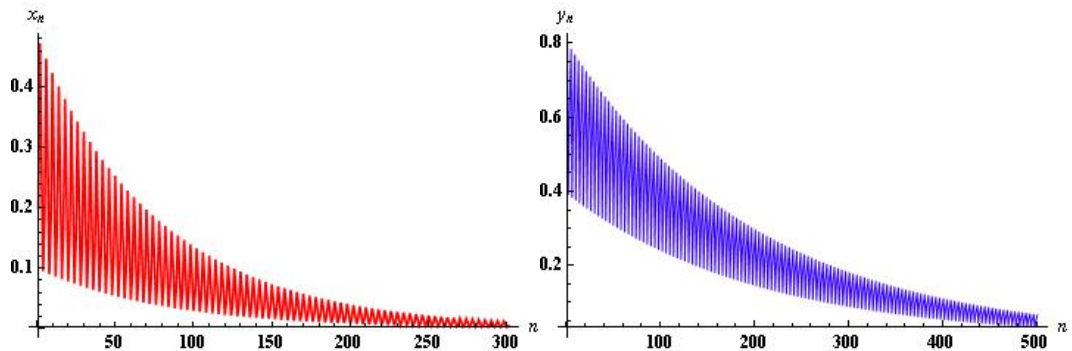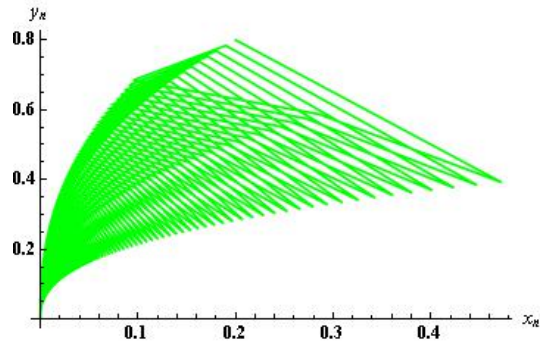

(c) An attractor of the system (14)

Figure 2: Plots for the system (14)

(a) Plot of x_n for the system (15)

(b) Plot of y_n for the system (15)

(c) An attractor of the system (15)

Figure 3: Plots for the system (15)

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