

## COMPLEX WAVE BEHAVIOR TO THE TDB AND (2 +1)-DZ EQUATIONS

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**ABSTRACT.** In this present study by means of our method we extract new application of the homogeneous balance method for obtaining the new complex solutions to the (2 + 1)-dimensional Zoomeron equation and the Tzitzeica–Dodd–Bullough (TDB) equation. Under some parameter conditions, exact solitary wave solutions are obtained. Note that it is always useful and desirable to construct exact solutions especially soliton-type (dark, bright, kink, anti-kink, etc.) envelope for the understanding of most nonlinear physical phenomena.

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### 1. INTRODUCTION

The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of NPDEs. Also, explicit formulas may provide physical information and help us to understand the mechanism of related physical models. A large number of such equations have been studied in these contexts, and numerous analytic and computational effective techniques have been proposed to investigate these types of equations.

The aim of this article is to look for new study relating to the homogeneous balance method for solving the renowned Tzitzeica–Dodd–Bullough equation

$$u_{xy} - e^{-u} - e^{-2u} = 0,$$

and the (2 + 1)-dimensional Zoomeron equation

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0,$$

to demonstrate the suitability and straightforwardness of the method.

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as inverse scattering method [1,9], Hirota bilinear transformation[5,12], the tanh–sech method [6,11,13,8], sine–cosine method [10,2] and Exp-function method [3,7,14,4] were used to develop nonlinear dispersive and dissipative problems.

## 2. AN ANALYSIS OF THE METHOD

For a given partial differential equation

$$G(u, u_x, u_t, u_{xx}, u_{tt}, \dots), \tag{1}$$

our method mainly consists of four steps:

**Step 1:** We seek complex solutions of Eq. (1) as the following form:

$$u = u(\xi), \quad \xi = ik(x - ct), \tag{2}$$

Where  $k$  and  $c$  are real constants. Under the transformation (2), Eq. (1) becomes an ordinary differential equation

$$N(u, iku', -ikcu', -k^2u'', \dots), \tag{3}$$

Where  $u' = \frac{du}{d\xi}$ .

**Step 2:** We assume that the solution of Eq. (3) is of the form

$$u(\xi) = \sum_{i=0}^n a_i \phi^i(\xi), \tag{4}$$

Where  $a_i (i = 1, 2, \dots, n)$  are real constants to be determined later and  $\phi$  satisfy the Riccati equation

$$\phi' = a\phi^2 + b\phi + c \tag{5}$$

Eq. (5) admits the following solutions:

**Case1:** when  $a = 1, b = 0$ , the Riccati Eq. (5) has the following solutions

$$\begin{aligned} \phi &= -\sqrt{-c} \tanh(\sqrt{-c}\xi), & c < 0 \\ \phi &= -\frac{1}{\xi}, & c < 0 \\ \phi &= \sqrt{c} \tan(\sqrt{c}\xi), & c > 0. \end{aligned} \tag{6}$$

**Case2:** Let  $\phi = \sum_{i=0}^n b_i \tanh^i \xi$ , Balancing  $\phi'$  with  $\phi^2$  in Eq.(5) gives  $m = 1$  so

$$\phi = b_0 + b_1 \tanh \xi, \tag{7}$$

Substituting Eq. (7) into Eq. (5), we obtain the following solution of Eq. (5)

$$\phi = -\frac{1}{2a} (b + 2 \tanh \xi), \quad ac = \frac{b^2}{4} - 1. \tag{8}$$

**Case3:** We suppose that the Riccati Eq. (5) have the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^n \sinh^{i-1} (A_i \sinh \omega + B_i \cosh \omega), \tag{9}$$

Where  $\frac{d\omega}{d\xi} = \sinh \omega$  or  $\frac{d\omega}{d\xi} = \cosh \omega$ . It is easy to find that  $m = 1$  by balancing  $\phi'$  with  $\phi^2$ . So we choose

$$\phi = A_0 + A_1 \sinh \omega + B_1 \cosh \omega, \tag{10}$$

Where  $\frac{d\omega}{d\xi} = \sinh \omega$ , we substitute (10) and  $\frac{d\omega}{d\xi} = \sinh \omega$ , into (5) and set the coefficients of  $\sinh^i \omega, \cosh^i \omega$  ( $i = 0, 1, 2; j=0, 1$ ) to zero. We obtain a set of algebraic equations and solving these equations we have the following solutions

$$A_0 = -\frac{b}{2a}, A_1 = 0, B_1 = \frac{1}{2a} \tag{11}$$

Where  $c = \frac{b^2-4}{4a}$  and

$$A_0 = -\frac{b}{2a}, A_1 = \pm \sqrt{\frac{1}{2a}}, B_1 = \frac{1}{2a} \tag{12}$$

Where  $c = \frac{b^2-1}{4a}$ . To  $\frac{d\omega}{d\xi} = \sinh \omega$  we have

$$\sinh \omega = -\csc h \xi, \cosh \omega = -\coth \xi \tag{13}$$

From (11)–(13), we obtain

$$\phi = -\frac{b + 2 \coth \xi}{2a} \tag{14}$$

Where  $c = \frac{b^2-4}{4a}$  and

$$\phi = -\frac{b \pm \csc h \xi + \coth \xi}{2a} \tag{15}$$

Where  $c = \frac{b^2-1}{4a}$ .

**Step3.** Substituting (6-15) into (3) along with (5), then the left hand side of Eq. (3) is converted into a polynomial in  $F(\xi)$ ; equating each coefficient of the polynomial to zero yields a set of algebraic equations.

**Step4.** Solving the algebraic equations obtained in step 3, and substituting the results into (4), then we obtain the exact traveling wave solutions for Eq. (1).

### 3. THE TZITZEICA–DODD–BULLOUGH (TDB) EQUATION

In this sub-section, we will exert the MSE method to obtain new and more general exact solutions and then the solitary wave solutions of the Tzitzeica–Dodd–Bullough equation,

$$u_{xy} - e^{-u} - e^{-2u} = 0 \tag{16}$$

Using the transformation  $v = e^{-u}$  Eq. (16) transforms into the following partial differential equation,

$$vv_{xt} - v_x v_t + v^3 + v^4 = 0. \tag{17}$$

We use the wave transformation  $v = v(\xi)$ , with wave complex variable  $\xi = ik(x - ct)$ , where  $k$  and  $c$  are real constants. System (17) takes the form as

$$ck^2 vv'' - ck^2 (v')^2 + v^3 + v^4 = 0. \tag{18}$$

Considering the homogeneous balance between  $vv''$  and  $v^4$  in (18), we required that  $3m = m + 2 \Rightarrow m = 1$ . So the solution takes the form

$$v = a_1 F + a_0, \tag{19}$$

Substituting (19) into Eq. (18) yields a set of algebraic equations for  $a_1, a_0, k, c$  and solving these equations with Maple package we have

$$\begin{aligned} a_1 &= \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} \\ a_0 &= \pm \frac{1}{2} \sqrt{-cb} (3-2k) \\ a &= \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^3 \sqrt{-c}} \end{aligned} \tag{20}$$

From (5),(19) and (20), we obtain the complex travelling wave solutions of (6) as follows

$$v_1 = \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} [-\sqrt{-c} \tanh(\sqrt{-c} ik(x - ct))] \pm \frac{1}{2} \sqrt{-cb} (3-2k),$$

So we have

$$u_1 = -\ln \left[ \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} [-\sqrt{-c} \tanh(\sqrt{-c} ik(x - ct))] \pm \frac{1}{2} \sqrt{-cb} (3-2k) \right]$$

Where  $c < 0$  and  $k$  is an arbitrary real constant. And

$$u_2 = -\ln \left[ \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} \left[ -\frac{1}{ik(x - ct)} \right] \pm \frac{1}{2} \sqrt{-cb} (3-2k) \right]$$

Where  $c < 0$  and  $k$  is an arbitrary real constant.

$$u_3 = -\ln \left[ \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} \left[ \sqrt{c} \tan(\sqrt{c}ik(x-ct)) \right] \pm \frac{1}{2} \sqrt{-cb}(3-2k) \right]$$

Where  $c > 0$  and  $k$  is an arbitrary real constant and from (8),(19) and (20) we have the complex travelling wave solutions of (6) as follows

$$u_4 = -\ln \left[ \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} \left[ -\frac{1}{2a} (b + 2 \tanh ik(x-ct)) \right] \pm \frac{1}{2} \sqrt{-cb}(3-2k) \right]$$

From (14),(19) and (20) we obtain the complex travelling wave solutions of (6) as follows

$$u_5 = -\ln \left[ \pm \frac{2(3-2k)^2 + \sqrt{-c}(3-2k)}{8ck^2} \left[ -\frac{b + 2 \coth ik(x-ct)}{2a} \right] \pm \frac{1}{2} \sqrt{-cb}(3-2k) \right]$$

In these cases if assume  $u_{1,2,3,4,5} = \ln [D]$ ,  $D$  Must be greater than zero (or  $D > 0$ ).

#### 4. THE (2 + 1)-DIMENSIONAL ZOOMERON EQUATION

Let us consider the Zoomeron equation

$$\left( \frac{u_{xy}}{u} \right)_{tt} - \left( \frac{u_{xy}}{u} \right)_{xx} + 2(u^2)_{xt} = 0 \tag{21}$$

where  $u(x, y, t)$  is the amplitude of the relative wave mode. The traveling wave transformation

$$u = u(\xi), \quad \xi = ik(x + y - \omega t) \tag{22}$$

Reduces Eq. (21) into the following ODE:

$$k^2(1 - \omega^2)u'' - 2\omega u^3 + R = 0. \tag{23}$$

where  $R$  is a constant of integration. Balancing the highest order derivative  $u''$  and nonlinear term of the highest order  $u^3$ , yields  $m = 1$ . So the solution takes the form

$$u = a_1 F + a_0, \tag{24}$$

Substituting (22) into Eq. (23) yields a set of algebraic equations for  $a_1, a_0, k, c$  and solving these equations with Maple package we have

$$\begin{aligned} a_1 &= \pm k \sqrt{\frac{1-\omega^2}{\omega}} \\ a_0 &= \pm k \frac{(1-\omega^2)ab}{2\sqrt{\omega(1-\omega)}} \end{aligned} \tag{25}$$

From (5),(24) and (25), we obtain the complex travelling wave solutions of (21) as follows

So we have

$$u_1 = \pm k \sqrt{\frac{1-\omega^2}{\omega}} [-\sqrt{-c} \tanh(\sqrt{-c} ik(x+y-\omega t))] \pm k \frac{(1-\omega^2) ab}{2\sqrt{\omega(1-\omega)}}$$

Where  $c < 0$  and  $k$  is an arbitrary real constant. And

$$u_2 = \pm k \sqrt{\frac{1-\omega^2}{\omega}} \left[ -\frac{1}{ik(x+y-\omega t)} \right] \pm k \frac{(1-\omega^2) ab}{2\sqrt{\omega(1-\omega)}}$$

Where  $c < 0$  and  $k$  is an arbitrary real constant.

$$u_3 = \pm k \sqrt{\frac{1-\omega^2}{\omega}} [\sqrt{c} \tan(\sqrt{c} ik(x+y-\omega t))] \pm k \frac{(1-\omega^2) ab}{2\sqrt{\omega(1-\omega)}}$$

Where  $c > 0$  and  $k$  is an arbitrary real constant and from (8), (24) and (25) we have

$$u_4 = \pm k \sqrt{\frac{1-\omega^2}{\omega}} \left[ -\frac{1}{2a} (b + 2 \tanh ik(x+y-\omega t)) \right] \pm k \frac{(1-\omega^2) ab}{2\sqrt{\omega(1-\omega)}}$$

Finally we have the complex travelling wave solutions of (6) from (14), (24) and (25) as follows

$$u_5 = \pm k \sqrt{\frac{1-\omega^2}{\omega}} \left[ -\frac{b + 2 \coth ik(x+y-\omega t)}{2a} \right] \pm k \frac{(1-\omega^2) ab}{2\sqrt{\omega(1-\omega)}}$$

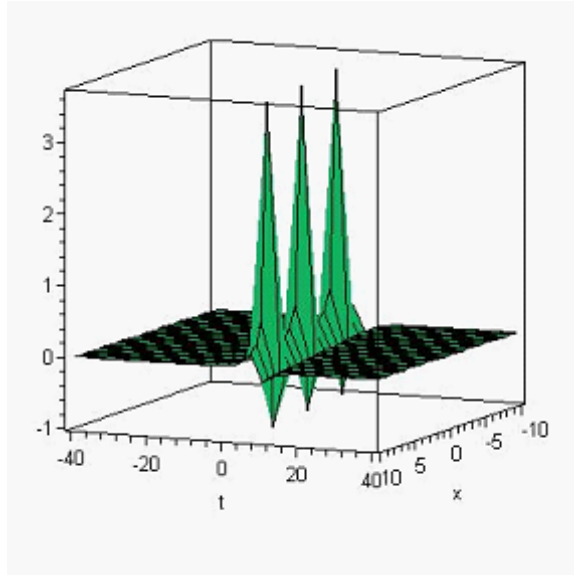


Figure a

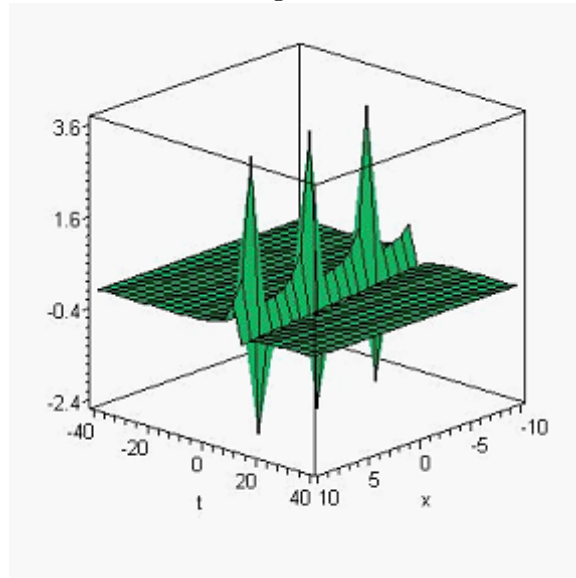


Figure b

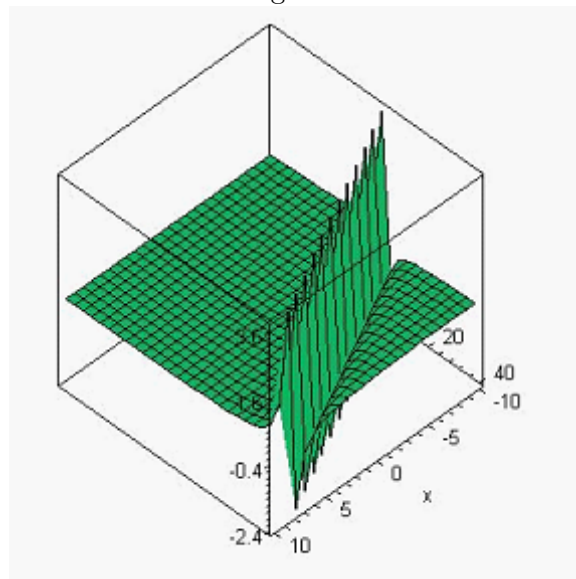


Figure c

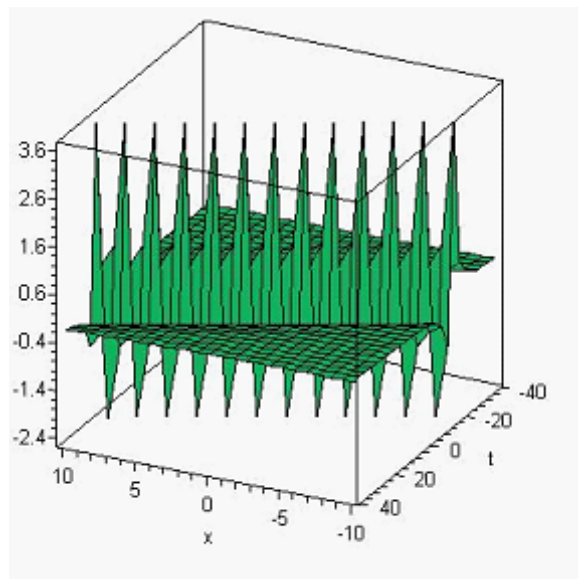


Figure d

## 5. CONCLUSIONS

Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. The homogeneous balance method is applied successfully for solving the system of non-linear evolution equations. The performance of this method is reliable and effective and gives more solutions. This method has more advantages: it is direct and concise. Thus, we deduce that the proposed method can be extended to solve many systems of non-linear fractional partial differential equations.

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