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A FIXED POINT APPROACH TO ALMOST TERNARY HOMOMORPHISMS AND TERNARY DERIVATIONS ASSOCIATED WITH THE ADDITIVE FUNCTIONAL EQUATION OF N-APOLLONIUS TYPE IN FUZZY TERNARY BANACH ALGEBRAS

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Abstract. In this paper by using the fixed point method, we investigate the generalized Hyers-Ulam-Rassias stability of the ternary homomorphisms and ternary derivations associated with the additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i) = -\frac{1}{n} \sum_{1\le i<j\le n} f(x_i+x_j) + nf(z - \frac{1}{n^2}\sum_{i=1}^{n} x_i)$$

for a fixed positive n with $n \geq 2$ in fuzzy ternary Banach algebras.

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1. Introduction and preliminaries

A classical equation in the theory of functional equations is the following: "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [30] in 1940. In the next year Hyers [11] gave a first affirmative answer to the question of Ulam in context of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [29] for linear mappings by considering an unbounded Cauchy difference. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Găvruţa

[9] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem [1, 12, 13, 14].

The theory of fuzzy sets was introduced by Zadeh in 1965 [33]. Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [17] introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year Wu and Fang [31] introduced a notion fuzzy normed apace to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [8] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [15]. Some mathematics have define fuzzy normed on a vector form various point of view [21, 32, 27]. In particular, Bang and Samanta [3] following Cheng and Mordeson [7], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type [20]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

In the following, we will given some notations that are needed in this paper.

Definition 1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,*
$(N1)$ $N(x,t) = 0$ *for* $t \leq 0$*;*
$(N2)$ $N(x,t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$;
$(N3)$ $N(cx,t) = N(x, \frac{t}{|c|})$ *for each* $c \neq 0$;
$(N4)$ $N(x+y, s+t) \geq \min\{N(x,t), N(y,s)\}$;
$(N5)$ $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$;
(N6) $N(x,.)$ *is continuous on* $\mathbb{R}$ *for* $x \neq 0$.

The pair (X, N) is called a fuzzy normed linear space.

On may regard $N(x,t)$ as the truth value of the statement "the norm of x is less than or equal to the real number t".

Example 1. *Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty}N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty}x_n = x$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy normed is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then f is said to be continuous on X [4].

Ternary algebraic operations have propounded originally in nineteenth century by several mathematicians such as Cayley [6] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [16]. The application of ternary algebra in supersymmetry is presented in [18] and in Yang-Baxter equation in [24]. Cubic analogue of Laplace and d'alembert equations have been consider for first order by Himbert in [10, 19].

Let X be a linear space over a complex field equipped a mapping $[.,.,.] : X \times X \times X \longrightarrow X$ with $(x, y, z) \longrightarrow [x, y, z]$ that is linear in variables x, y, z and satisfy the associative identity, i.e. $[x, y, [z, u, v]] = [x, [y, z, u], v] = [[x, y, z], u, v]$ for all $x, y, z, u, v \in X$. The pair $(X, [.,.,.])$ is called a ternary algebra. The ternary algebra $(X, [.,.,.])$ is called unital if it has an identity element, i.e. an element $e \in X$ such that $[x, e, e] = [e, e, x] = x$ for every $x \in X$.

X is called normed ternary algebra if X is a ternary algebra and there exists a norm $\|.\|$ on X which satisfies $\|[x, y, z]\| \leq \|x\|\|y\|\|z\|$ for all $x, y, z \in X$. Whenever the ternary algebra X is unital with unit element e, we repute $\|e\| = 1$. A normed ternary algebra X is called a Banach ternary algebra, if $(X, \|.\|)$ is a Banach space.

Definition 2. *Let X be a ternary algebra and (X, N) be a fuzzy normed space.*
(1) *The fuzzy normed space (X, N) is called a fuzzy ternary normed algebra if*

$$N([x, y, z], tsr) \geq N(x, t)N(y, s)N(z, r)$$

for all $x, y \in X$ and all positive real numbers t, s, r.
(2) *A complete fuzzy ternary normed algebra is called a fuzzy ternary Banach algebra.*

Example 2. *Let $(X, \|.\|)$ be a ternary normed (Banach) algebra. Let*

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X. \end{cases}$$

Then (x,t) is a fuzzy norm on X and (X,N) is a fuzzy ternary normed (Banach) algebra.

Definition 3. *Let (X,N) and (Y,N') be two fuzzy ternary Banach algebras.*
(1) The $\mathbb{C}$-linear mapping $H:(X,N)\to(Y,N')$ is called a fuzzy ternary homomorphism if

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z \in X$.
(2) The $\mathbb{C}$-linear mapping $D:(X,N)\to(X,N)$ is called a fuzzy ternary derivation if

$$D([xyz]) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$$

for all $x,y,z \in X$.

We recall the fundamental result in fixed point theory.

Theorem 1. *(see.[22, 28]) Let (X,d) be a complete generalized metric space and $J:X\to X$ be a strictly contractive mapping with Lipshitz constant $L<1$. Then, for each given $x\in X$, either*

$$d(J^n x, J^{n+1}x) = \infty \quad \textit{for all} \quad n \geq 0,$$

or there exists a natural number n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}, y) < \infty\}$;
(4) $d(y,y^*) \leq \frac{1}{1-L} d(y,Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 26]).

In this paper we consider a mapping $f:X\to Y$ satisfying the following of additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i) = -\frac{1}{n}\sum_{1\leq i<j\leq n} f(x_i+x_j) + nf(z-\frac{1}{n^2}\sum_{i=1}^{n} x_i) \tag{1}$$

for all $z, x_1, x_2, ..., x_n \in X$, which n is fixed positive integer with $n \geq 2$ and establish the ternary homomorphisms and ternary derivations of functional equation (1) in fuzzy ternary Banach algebras.

Throughout this article, assume that (X,N) and (Y,N') be two fuzzy ternary Banach algebra.

2. Approximate fuzzy ternary homomorphisms in fuzzy ternary Banach algebras

In this section, we prove the Hyers- Ulam-Rassias stability of fuzzy ternary homomorphisms in fuzzy ternary Banach algebras related to additive functional equation of n-Apollonius type.

Theorem 2. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < (\frac{n^2-1}{n^2})^2$ with*

$$\varphi(\frac{n^2-1}{n^2}z, \frac{n^2-1}{n^2}x_1, \frac{n^2-1}{n^2}x_2, ..., \frac{n^2-1}{n^2}x_n) \leq \frac{n^2-1}{n^2}L\varphi(z, x_1, x_2, ..., x_n) \tag{2}$$

for all $z, x_1, x_2, ..., x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ with

$$N(\sum_{i=1}^{n} \mu f(z-x_i)+\frac{1}{n}\sum_{1\leq i<j\leq n} f(\mu x_i+\mu x_j)-nf(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i), t) \geq \frac{t}{t+\varphi(z, x_1, x_2, ..., x_n)} \tag{3}$$

$$N(f([x, y, z]) - [f(x), f(y), f(z)], t) \geq \frac{t}{t + \varphi(x, y, z, 0, ..., 0)} \tag{4}$$

for all $x, y, z, x_1, x_2, ..., x_n \in X$, all $\mu \in \mathbb{T}^1 := \{u \in \mathbb{C} : |u| = 1\}$ and all $t > 0$. Then $H(x) = N - \lim_{k\to\infty}(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)$ exists for each $x \in X$, and defines a unique fuzzy ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \tag{5}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set $S := \{g : X \to Y, g(0) = 0\}$ and introduce the generalized metric

$$d(g, h) = \inf\{\eta \in \mathbb{R}^+ : N(g(x) - h(x), \eta t) \geq \frac{t}{t + \varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)}$$

where $\inf \emptyset = +\infty$. The proof of the fact (S, d) is a complete generalized metric space can be found in [5]. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := \frac{n^2}{n^2-1}g(\frac{n^2-1}{n^2}x)$$

for all $g \in S$ and $x \in X$. Let $\varepsilon > 0$ and $f, g \in S$ be given such that $d(g,h) < \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N(\frac{n^2}{n^2-1} g(\frac{n^2-1}{n^2} x) - \frac{n^2}{n^2-1} h(\frac{n^2-1}{n^2} x), L\varepsilon t) \\ &= N(g(\frac{n^2-1}{n^2} x) - h(\frac{n^2-1}{n^2} x), \frac{n^2-1}{n^2} L\varepsilon t) \\ &\geq \frac{\frac{n^2-1}{n^2} Lt}{\frac{n^2-1}{n^2} Lt + \varphi(\frac{n^2-1}{n^2} x, 0, 0, .., \underbrace{\frac{n^2-1}{n^2} x}_{jth}, 0, 0, ..., 0)} \\ &\geq \frac{\frac{n^2-1}{n^2} Lt}{\frac{n^2-1}{n^2} Lt + \frac{n^2-1}{n^2} L\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \\ &= \frac{t}{t + \varphi(x, 0, ..., \underbrace{x}_{jth}, 0, ...0)}, \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) < \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$, for all $g, h \in S$. Letting $\mu = 1$ and $z = x_j = x$ for each $1 \leq k \leq n$ with $k \neq j$, $x_k = 0$ in (3) , we have

$$N(\frac{n^2-1}{n} f(x) - nf(\frac{n^2-1}{n^2} x), t) \geq \frac{t}{t + \varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \tag{6}$$

for all $x \in X$ and all $t > 0$. It follows from (6) that $d(f, Jf) \leq \frac{n}{n^2-1}$. By Theorem 1.7 there exists a mapping $H : X \to Y$ such that the following holds:
(1) H is a fixed point of J, that is,

$$H(\frac{n^2-1}{n^2} x) = \frac{n^2-1}{n^2} H(x) \tag{7}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Delta = \{h \in S : d(g,h) < \infty\}$. This implies that H is a unique mapping satisfying (7) such that there exists $\eta \in (0,\infty)$ satisfying

$$N(f(x)-H(x),\eta t) \geq \frac{t}{t+\varphi(x,0,0,..,\underbrace{x}_{jth},0,0,...,0)}$$

for all $x \in X$ and all $t > 0$.
(2) $d(J^k f, H) \to 0$ as $k \to \infty$. This implies the equality

$$N-\lim_{k\to\infty}(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x) = H(x)$$

exists for each $x \in X$,
(3)$d(f,H) \leq \frac{1}{1-L} d(f,Jf)$, which implies inequality

$$d(f,H) \leq \frac{1}{\frac{n^2-1}{n} - \frac{n^2-1}{n}L}$$

and so

$$N(f(x)-H(x),t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x,0,0,..,\underbrace{x}_{jth},0,0,...,0)}.$$

Thus (5) holds.
It follows from (2) and (3) that

$$\begin{aligned}
&N(\sum_{i=1}^{n}\mu H(z-x_i)+\frac{1}{n}\sum_{1\leq i<j\leq n}H(\mu x_i+\mu x_j)-nH(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i),t)\\
&= N-\lim_{k\to\infty}((\frac{n^2}{n^2-1})^k\sum_{i=1}^{n}\mu f((\frac{n^2-1}{n^2})^k(z-x_i))\\
&+\frac{1}{n}(\frac{n^2}{n^2-1})^k\sum_{1\leq i<j\leq n}f((\frac{n^2-1}{n^2})^k(\mu x_i+\mu x_j))-n(\frac{n^2}{n^2-1})^k f(((\frac{n^2-1}{n^2})^k)\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i),t)\\
&\geq \lim_{k\to\infty}\frac{t}{t+(\frac{n^2}{n^2-1})^k\varphi((\frac{n^2-1}{n^2})^k z,(\frac{n^2-1}{n^2})^k x_1,...,(\frac{n^2-1}{n^2})^k x_n)}\\
&\geq \lim_{k\to\infty}\frac{t}{t+L^k\varphi(z,x_1,...,x_n)} \longrightarrow 1
\end{aligned}$$

for all $z, x_1, x_2, ..., x_n \in X$, $t > 0$ and $\mu \in \mathbb{T}^1$. Thus

$$\sum_{i=1}^{n}\mu H(z-x_i) = -\frac{1}{n}\sum_{1\leq i<j\leq n}H(\mu x_i+\mu x_j)+nH(\mu z-\frac{1}{n^2}\sum_{i=1}^{n}\mu x_i) \tag{8}$$

for all $z, x_1, x_2, ..., x_n \in X$. By [23] $H : X \to Y$ is Cauchy additive, that is, $H(x+y) = H(x) + H(y)$ for all $x, y \in X$.

By a Similar method to the proof of [25], one can show that the mapping is $\mathbb{C}$-linear.

By (4), we have

$$\begin{aligned}
&N((\frac{n^2}{n^2-1})^{3k} f([(\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, (\frac{n^2-1}{n^2})^k z]) \\
&\quad - (\frac{n^2}{n^2-1})^{3k} [f((\frac{n^2-1}{n^2})^k x), (f((\frac{n^2-1}{n^2})^k y), f((\frac{n^2-1}{n^2})^k y)], t) \\
&\quad \geq \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + \varphi((\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, (\frac{n^2-1}{n^2})^k z, ..., 0)} \\
&\quad \geq \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, z, ..., 0)}
\end{aligned}$$

for all $x, y, z \in X$ and $t > 0$. Since

$$\lim_{k \to \infty} \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, 0, ..., 0)} = 1$$

for all $x, y, z \in X$ and $t > 0$, hence

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in X$. This means that H is a fuzzy ternary homomorphism. This complete the proof.

Corollary 3. *Let X be a ternary Banach algebra with norm $\|.\|$, $\delta \geq 0$ and p be a real number with $p > 2$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}
N(\sum_{i=1}^{n} \mu f(z - x_i) + \frac{1}{n} \sum_{1 \leq i < j \leq n} f(\mu x_i + \mu x_j) - n f(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i), t) \\
\geq \frac{t}{t + \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)} \qquad (9)
\end{aligned}$$

$$N(f([x, y, z]) - [f(x), f(y), f(z)], t) \geq \frac{t}{t + \delta(\|x\|^p + \|y\|^p + \|z\|^p)} \qquad (10)$$

for all $x, y, z, x_1, x_2, ..., x_n \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then there exists a unique fuzzy ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{((n^2-1)^{1-p} - n^{2(1-p)})t}{((n^2-1)^{1-p} - n^{2(1-p)})t + 2n\delta(n^2-1)^{-p}\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(z, x_1, x_2, ..., x_n) := \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)$$

for all $z, x_1, x_2, ..., x_n \in X$. It follows from (9) that $f(0) = 0$, we can choose $L = (\frac{n^2}{n^2-1})^{1-p}$ to get the desired result.

Theorem 4. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ such that*

$$\varphi(\frac{n^2}{n^2-1}z, \frac{n^2}{n^2-1}x_1, \frac{n^2}{n^2-1}x_2, ..., \frac{n^2}{n^2-1}x_n) \leq \frac{n^2}{n^2-1}L\varphi(z, x_1, x_2, ..., x_n) \quad (11)$$

for all $z, x_1, x_2, ..., x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3) and (4).
Then the limit $H(x) = N - \lim_{k\to\infty}(\frac{n^2-1}{n^2})^k f((\frac{n^2}{n^2-1})^k x)$ exists for each $x \in X$, and defines a unique ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + nL\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \quad (12)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ defined by

$$Jg(x) := \frac{n^2-1}{n^2}g(\frac{n^2}{n^2-1}x)$$

for all $g \in S$ and $x \in X$. We can conclude that J is a strictly contractive self mapping of S with the Lipschitz constant L.

It follows from (6) that

$$N(nf(\frac{n^2-1}{n^2}x) - \frac{n^2-1}{n}f(x), t) \geq \frac{t}{t + \varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \quad (13)$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{n^2}{n^2-1}x$ in (13), we obtain

$$N(\frac{n^2-1}{n^2}f(\frac{n^2}{n^2-1}x) - f(x), t) \geq \frac{nt}{nt + \varphi(\frac{n^2}{n^2-1}x, 0, 0, .., \underbrace{\frac{n^2}{n^2-1}x}_{jth}, 0, 0, ..., 0)}$$

$$\geq \frac{nt}{nt + \frac{n^2}{n^2-1}L\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)}.$$

It follows that $d(f, Jf) \leq \frac{nL}{n^2-1}$.

By Theorem 1.7, there exists a mapping $H : X \to Y$ satisfying
(1) H is a fixed point of J, that is,

$$H(\frac{n^2}{n^2-1}x) = \frac{n^2}{n^2-1}H(x) \tag{14}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Delta = \{h \in S : d(g, h) < \infty\}$. This implies that H is a unique mapping satisfying (14) such that there exists $\eta \in (0, \infty)$ satisfying

$$N(H(x) - f(x), \eta t) \geq \frac{t}{t + \varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)}$$

for all $x \in X$ and all $t > 0$.
(2) $d(J^k f, H) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty}(\frac{n^2-1}{n^2})^k f((\frac{n^2}{n^2-1})^k x) = H(x)$$

exists for each $x \in X$,
(3)$d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies inequality

$$d(f, H) \leq \frac{nL}{(n^2-1) - (n^2-1)L}.$$

The rest the proof is similar to the proof of Theorem 2.1.

Corollary 5. *Let X be a ternary Banach algebra with norm $\|.\|$, $p < 1$ and $\delta \geq 0$. Let $f : X \to Y$ be a mapping satisfying (9) and (10) . Then there exists a unique ternary homomorphism $H : X \to Y$ such that*

$$N(f(x) - H(x), t) \geq \frac{((n^2-1)^{p-1} - n^{2(p-1)})t}{((n^2-1)^{p-1} - n^{2(p-1)})t + 2\delta n^{2p-1}(n^2-1)^{-1}\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(z, x_1, x_2, ..., x_n) := \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)$$

for all $z, x_1, x_2, ..., x_n \in X$. It follows from (9) that $f(0) = 0$, we can choose $L = (\frac{n^2}{n^2-1})^{p-1}$ to get the desired result.

3. Approximate fuzzy ternary derivations in ternary fuzzy Banach algebras

In this section, we prove the Hyers-Ulam-Rassias stability of fuzzy ternary derivations in fuzzy ternary Banach algebras related to additive functional equation of n-Apollonius type.

Theorem 6. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < (\frac{n^2-1}{n^2})^2$ with*

$$\varphi(\frac{n^2-1}{n^2}z, \frac{n^2-1}{n^2}x_1, \frac{n^2-1}{n^2}x_2, ..., \frac{n^2-1}{n^2}x_n) \le \frac{n^2-1}{n^2}L\varphi(z, x_1, x_2, ..., x_n) \quad (15)$$

for all $z, x_1, x_2, ..., x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$ with

$$N(\sum_{i=1}^{n} \mu f(z-x_i) + \frac{1}{n} \sum_{1\le i<j\le n} f(\mu x_i + \mu x_j) - nf(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i), t) \ge \frac{t}{t + \varphi(z, x_1, x_2, ..., x_n)} \quad (16)$$

$$N(f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]), t) \ge \frac{t}{t + \varphi(x, y, z, ..., 0)} \quad (17)$$

for all $x, y, z, x_1, x_2, ..., x_n \in X$, all $\mu \in \mathbb{T}^1 := \{u \in \mathbb{C} : |u| = 1\}$ and all $t > 0$. Then $D(x) = N - \lim_{k\to\infty}(\frac{n^2}{n^2-1})^k f((\frac{n^2-1}{n^2})^k x)$ exists for each $x \in X$, and defines a unique fuzzy ternary derivation $D : X \to X$ such that

$$N(f(x) - D(x), t) \ge \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \quad (18)$$

for all $x \in X$ and all $t > 0$.

Proof. By the same reasoning as that in the proof of Theorem 2.1, the mapping $D : X \to X$ is a unique $\mathbb{C}$-linear mapping which satisfies (18).

By (17), we have

$$\begin{aligned}
&N((\frac{n^2}{n^2-1})^{3k} f([(\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, (\frac{n^2-1}{n^2})^k z]) \\
&\quad - (\frac{n^2}{n^2-1})^{3k}([f((\frac{n^2-1}{n^2})^k x), y, z] - [x, f((\frac{n^2-1}{n^2})^k y), z] - [x, y, f((\frac{n^2-1}{n^2})^k y)]), t) \\
&\quad \geq \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + \varphi((\frac{n^2-1}{n^2})^k x, (\frac{n^2-1}{n^2})^k y, (\frac{n^2-1}{n^2})^k z, ..., 0)} \\
&\quad \geq \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, z, ..., 0)}
\end{aligned}$$

for all $x, y, z \in X$ and $t > 0$. Since

$$\lim_{k\to\infty} \frac{(\frac{n^2-1}{n^2})^{3k} t}{(\frac{n^2-1}{n^2})^{3k} t + (\frac{n^2-1}{n^2})^k L^k \varphi(x, y, z, ..., 0)} = 1$$

for all $x, y, z \in X$ and $t > 0$, hence

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)].$$

for all $x, y, z \in X$. This means that D is a fuzzy ternary derivation.

Theorem 7. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(\frac{n^2}{n^2-1} z, \frac{n^2}{n^2-1} x_1, \frac{n^2}{n^2-1} x_2, ..., \frac{n^2}{n^2-1} x_n) \leq \frac{n^2}{n^2-1} L \varphi(z, x_1, x_2, ..., x_n) \tag{19}$$

for all $z, x_1, x_2, ..., x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$, (16) and (17) . Then the limit $D(x) = N - \lim_{k\to\infty} (\frac{n^2-1}{n^2})^k f((\frac{(n^2-1)^k}{n^{2k}})^k x)$ exists for each $x \in X$, and defines a unique fuzzy ternary derivation $D : X \to X$ such that

$$N(f(x) - D(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + nL\varphi(x, 0, 0, .., \underbrace{x}_{jth}, 0, 0, ..., 0)} \tag{20}$$

for all $x \in X$ and all $t > 0$.

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