SOME CHARACTERIZATIONS FOR CURVES BY THE HELP OF SPHERICAL REPRESENTATIONS IN THE GALILEAN AND PSEUDO-GALILEAN SPACE

A.O. Öğrenmiş, M.Y. Yilmaz, M. Külahci

Abstract. In this paper, we focus on some characterizations for curves in the Galilean and Pseudo-Galilean.

1. Introduction

Discovering Galilean space-time is probably one of the major achievements of non-relativistic physics. One may consider Galilean space as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. Nowadays Galilean space is becoming increasingly popular as evidenced from the connection of the fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles as indicated in [8].

As a branch of science, geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group shows that, on some spacetimes of maximum symmetries there should be a principle of relativity, which requires the invariance of physical laws without gravity under transformations among inertial systems.

Galilean space G_3 has been investigated in [3], [4], [5], [6], [7] from the differential geometrical point of view.The mathematical model of Galilean space is a three dimensional complex projective space P_3 in which the absolute figure { w, f, I_1, I_2 } consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points I_1, $I_2 \in f$ (the absolute points).

We may also take, as a real model of the space G_3, a real projective space P_3 with the absolute { w, f } consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined.

Using homogeneous coordinates one may write

$$w...x_0 = 0, \quad f...x_0 = x_1 = 0$$
$$\varepsilon : (0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2)$$

On the contrary, for nonhomogeneous coordinates of the similarity group H_8 has the following form,

$$
\begin{aligned}
x' &= a_{11} + a_{12}x, \\
y' &= a_{21} + a_{22}x + a_{23}\cos\varphi y + a_{23}\sin\varphi z, \\
z' &= a_{31} + a_{32}x - a_{23}\sin\varphi y + a_{23}\cos\varphi z,
\end{aligned}
\tag{1}
$$

here we denote a_{ij} and φ as real numbers.

Taking $a_{12} = a_{23} = 1$ we obtain the subgroup B_6- the group of Galilean motions as follows;

$$
\begin{aligned}
& x' = a + x \\
B_6...& y' = b + cx + y\cos\varphi + z\sin\varphi \\
& z' = d + ex - y\sin\varphi + z\cos\varphi.
\end{aligned}
$$

We may observe four classes for lines in G_3 as indicated below:

a) (proper) nonisotropic lines-they don't meet the absolute line f.

b) (proper) isotropic lines -lines that don't belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines-all lines of w but f.

d) the absolute line f.

Planes $x = const.$ are Euclidean and so is the plane w. Other planes are isotropic.

Here the coefficients a_{12} and a_{23} play the special role.

In particular, for $a_{12} = a_{23} = 1$ (1) shows the group $B_6 \subset H_8$ of isometries of the Galilean space G_3 [6].

2. BASIC NOTIONS AND PROPERTIES

Let $\alpha : I \to G_3$, $I \subset IR$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t)),$$

where $x(t)$, $y(t)$, $z(t) \in C^3$ (the set of three times continuously differentiable functions) and t run through a real interval [6].

Let α be a curve in G_3, parameterized by arc length $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s)). \tag{2}$$

Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{y''^2(s) + z''^2(s)} \tag{3}$$

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2}$$

and associated moving trihedron is given by

$$T(s) = \alpha'(s) = (1, y'(s), z'(s)) \tag{4}$$

$$N(s) = \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s))$$

$$B(s) = \frac{1}{\kappa(s)}(0, -z''(s), y''(s)).$$

The vectors T, N, B are called the vectors of the tangent, principal normal and binormal line of α, respectively. For their derivatives the following Frenet formulas hold

$$T' = \kappa N \tag{5}$$

$$N' = \tau B$$

$$B' = -\tau N$$

Scalar product in the Galilean space G_3 is defined by

$$g(X, Y) = \{ \begin{array}{ll} x_1 y_1, & if \quad x_1 \neq 0 \ \vee \ y_1 \neq 0 \\ x_2 y_2 + x_3 y_3, & if \quad x_1 = 0 \ \wedge \ y_1 = 0 \end{array}, \tag{6}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$

Definition 1. *Let α be a curve in 3-dimensional Galilean space G_3, and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α . If κ and τ are positive constants along α , then α is called a circular helix with respect to the Frenet frame [4].*

Definition 2. *Let α be a curve in 3-dimensional Galilean space G_3, and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α . A curve α such that*

$$\frac{\kappa}{\tau} = const.$$

is called a general helix with respect to Frenet frame [4].

Definition 3. *Let's give curve of $\alpha \subset G_3$ with coordinate neighbourhood (I, α). $\kappa(s)$ and $\tau(s)$ be a curvature of α on point of $\alpha(s) \in \alpha$ corresponding to $\forall s \in I$. The function of H can be defined as*

$$H : I \to R$$
$$s \to H(s) = \frac{\kappa}{\tau}$$

where the function of H is called 1st harmonic curvature of α on point of $\alpha(s)$.

Remark 1. *Similar definitions can be given in the pseudo-Galilean space.*

Remark 2. *In [1], for the pseudo-Galilean Frenet trihedron of an admissible curve α, the following derivative Frenet formulas are true.*

$$T'(s) = \kappa(s)N(s)$$
$$N'(s) = \tau(s)B(s)$$
$$B'(s) = \tau(s)N(s)$$

where $T(s)$ is a spacelike, $N(s)$ is a spacelike and $B(s)$ is a timelike vector, $\kappa(s)$ is the pseudo-Galilean curvature given by above equations and $\tau(s)$ is the pseudo-Galilean torsion of α defined by

$$\tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}.$$

3. The Arc Length of Spherical Representations of The Curve

$\alpha \subset G_3$ In this section, using method in [2]., some characterizations related to spherical representations are obtained in Galilean and Pseudo-Galilean 3-space.

Theorem 1. *$\alpha \subset G_3$ is an ordinary helix if and only if*

$$s_T = \tau H s + c.$$

Proof. Let $T = T(s)$ be the tangent vector field of the curve

$$\alpha : I \subset R \to G_3$$
$$s \to \alpha(s)$$

The spherical curve $\alpha_T = T$ on S^2 is called first spherical representation of the tangent of α.

Let s be the arc length parameter of α. If we denote the arc length of the curve α_T by s_T, then we may write

$$\alpha_T(s_T) = T(s).$$

Letting $\frac{d\alpha_T}{ds_T} = T_T$ we have $T_T = \kappa\overrightarrow{N}\frac{ds}{ds_T}$. Hence we obtain $\frac{ds_T}{ds} = \kappa$. Thus we give the following result.

If κ is the first curvature of the curve $\alpha : I \to G_3$, then the arc length s_T of the tangentian representation α_T of α is

$$s_T = \int \kappa ds + c.$$

If the harmonic curvature of α is $H = \frac{\kappa}{\tau}$, we get

$$ds_T = \int \tau H ds + c$$

where c is an integral constant.

Theorem 2. $\alpha \subset G_3$ *is an ordinary helix if and only if*

$$s_N = \frac{\kappa}{H}s + c.$$

Proof. Let $\overrightarrow{N} = \overrightarrow{N}(s)$ be the principal normal vector field of the curve

$$\alpha : I \subset R \to G_3$$
$$s \to \alpha(s)$$

The spherical curve $\alpha_N = \overrightarrow{N}$ on S^2 is called second spherical representation for α or is called the spherical representation of the principal normals of α. Let $s \in I$ be the arc length parameter of α. If we denote the arc length of the curve α_N by s_N, we may write

$$\alpha_N(s_N) = \overrightarrow{N}(s).$$

Moreover letting $\frac{d\alpha_N}{ds_N} = T_N$ we obtain

$$T_N = \tau\overrightarrow{B}\frac{ds}{ds_N}.$$

Hence we have

$$\frac{ds_N}{ds} = \tau.$$

Thus we give the following result.

If τ is the second curvature of the curve $\alpha : I \to G_3$, then the arc length s_N of the principal normal representation α_N of α is

$$s_N = \int \tau ds + c.$$

If the harmonic curvature of α is $H = \frac{\kappa}{\tau}$, we get

$$s_N = \int \tau ds + c.$$

If the harmonic curvature of α is $H = \frac{\kappa}{\tau}$, we get

$$s_N = \int \frac{\kappa}{H} ds + c$$

where c is an integral constant.

Theorem 3. $\alpha \subset G_3$ *is an ordinary helix if and only if*

$$s_B = \frac{\kappa}{H} s + c.$$

Proof. Let $\vec{B} = \vec{B}(s)$ be the binormal vector field of the curve

$$\alpha : I \subset R \to G_3$$
$$s \to \alpha(s)$$

The spherical curve $\alpha_B = \vec{B}$ on S^2 is called third spherical representation for α and the spherical representation of the binormal of α.

Let $s \in I$ be the arc length parameter of α. If we denote the arc length parameter of the curve α_B by s_B, we may write

$$\alpha_B(s_B) = \vec{B}(s).$$

Moreover letting $\frac{d\alpha_B}{ds_B} = T_B$, we obtain

$$T_B = -\tau \vec{N} \frac{ds}{ds_B}.$$

Hence we have $\frac{ds_B}{ds} = \tau$ and $s_B = \int \tau ds + c$ or in terms of the harmonic curvature of α we obtain

$$s_B = \int \frac{\kappa}{H} ds + c.$$

Note. Same theorems can be given in the Pseudo-Galilean 3-space.

References

[1] B. Divjak, *Curves in Pseudo-Galilean Geometry*, Annales Univ. Sci. Budapest, 41 (1998), 117-128.

[2] H. H. Hacısalihoğlu, *A New Characterization For Inclined Curves By The Help Of Spherical Representations*, International Electronic Journal Of Geometry, 2, 2 (2009), 71-75.

[3] I. Kamenarovic, *Existence Theorems for Ruled Surfaces In The Galilean Space G_3* , Rad HAZU Math. 456, 10 (1991), 183-196.

[4] A. O. Öğrenmiş, M. Bektaş,and M. Ergüt , *On The helices in the Galilean space G_3*, Iranian Journal of science & Technology, Transaction A , 31, A2 (2007), 177-181.

[5] A. O. Öğrenmiş, H. Öztekin and M. Ergüt, *Bertrand Curves In Galilean Space and Their Characterizations*, Kragujevac J.Math. 32 (2009), 139-147.

[6] B. J. Pavkovic, I. Kamenarovic, *The Equiform Differential Geometry of Curves in the Galilean Space*, Glasnik Matematicki, 22, 42 (1987), 449-457.

[7] O. Röschel, *Die Geometrie Des Galileischen Raumes*, Berichte der Math.-Stat. Sektion im Forschungszentrum Graz, Ber. 256 (1986), 1-20.

[8] I. M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer-Verlag, New York Inc. 306 (1979), 201-214.

Alper Osman Öğrenmiş
Department of Mathematics, Faculty of Science,
University of Fırat,
Elazığ, Turkey.
email: *ogrenmisalper@gmail.com*

Münevver Yıldırım Yılmaz
Department of Mathematics, Faculty of Science,
University of Fırat,
Elazığ, Turkey.
email: *myildirim@firat.edu.tr*

Mihriban Külahcı
Department of Mathematics, Faculty of Science,
University of Fırat,
Elazığ, Turkey.
email: *mihribankulahci@gmail.com*