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ON SEMI-INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD ADMITTING A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, semi-invariant submanifolds of a generalized Kenmotsu manifold endowed with a semi-symmetric metric connection are studied. Necessary and sufficient conditions are given on a submanifold of a generalized Kenmotsu manifold to be semi-invarinat submanifold with the semi-symmetric metric connection. Moreover, the integrability conditions of the distribution on semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are studied.

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1. Introduction

In [3] , K. Kenmotsu has introduced a Kenmotsu manifold. A. Turgut Vanlı and R. Sarı [6], introduced the notion of a generalized Kenmotsu manifold. Semi-invariant submanifolds are studied by some authors (for examples, M. Kobayashi [4], B.B. Sinha, A.K. Srivastava [5]). In [9], K. Yano have introduced a semi-symmetric metric connection on a Riemannian manifold. He studied some properties of the curvature tensor with respect to the semi-symmetric metric connection. In this paper, semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection are studied.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. A lineer connection ∇ is said to be a semi-symmetric connection if it torsion tensor T is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$ where η is a $1-$form. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

The paper is organized as follows : In section 2, a brief introduction of a generalized Kenmotsu manifolds is given. A semi-symmetric metric connection on a generalized Kenmotsu manifold is defined . In section 3, some basic results for semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are given. In last section, some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are obtained.

2. Semi-invariant submanifolds of generalized Kenmotsu manifold

In [8], K.Yano has introduced the notion of a f-structure on a differentianable manifold M, i.e., a tensor fields φ of type $(1,1)$ and *rank* $2n$ satisfying $\varphi^3 + \varphi = 0$. The existence of which is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times O(s)$ [1]. Let M be $(2n+s)$ dimensional and a differentiable manifold with a f-structure of rank $2n$. If there exists on M vector fields ξ_i, $i \in \{1,2,...,s\}$ and η^i are dual $1-$forms such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \qquad \eta^i \circ \xi_j = \delta_{ij} \tag{1}$$

then M is called a f-manifold. Moreover, we have $\varphi \circ \xi_i = 0$, $\eta^i \circ \varphi = 0$, $i \in \{1,2,...,s\}$ [2].

Let M be a $(2n+s)$ dimensional f-manifold. M is called a metric f-manifold if there exists on M a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y). \tag{2}$$

In addition, we have

$$\eta^i(X) = g(X, \xi_i), \qquad g(X, \varphi Y) = -g(\varphi X, Y). \tag{3}$$

Then, a 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the *fundamental 2-form*. Moreover, a metric f-manifold is *normal* if

$$[\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$$

where $[\varphi,\varphi]$ is denoting the Nijenhuis tensor field associated to φ.

In [7], let M, $(2n+s)-$dimensional a metric $f-$manifold. If there exists $2-$form Φ such that

$$\eta^1 \wedge ... \wedge \eta^s \wedge \Phi^n \neq 0$$

on M then M is called an almost $s-$contact metric structure.

Definition 1. *Let M be an almost $s-$contact metric manifold of dimension $(2n+s)$, $s \geq 1$, with $\left(\varphi,\xi_i,\eta^i,g\right)$. M is said to be a generalized almost Kenmotsu manifold if for all $1 \leq i \leq s$, $1-$forms η^i are closed and $d\Phi = 2\sum\limits_{i=1}^{s}\eta^i \wedge \Phi$. A normal generalized almost Kenmotsu manifold M is called a generalized Kenmotsu manifold [6].*

In [6], A $(2n+s), s \geqslant 1$, dimensional almost $s-$contact metric manifold $\tilde{M}$ is a generalized Kenmotsu manifold if it satisfies the condition

$$(\widetilde{\nabla}_X\varphi)Y = \sum_{i=1}^{s}\{g(\varphi X,Y)\xi_i - \eta^i(Y)\varphi X\} \tag{4}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.In [6], from the formula (4) we have

$$\widetilde{\nabla}_X\xi_j = -\varphi^2 X. \tag{5}$$

Definition 2. *Let M be a submanifold of the $(2n+s)$-dimensional a generalized Kenmotsu manifold $\tilde{M}$. M is called a semi-invariant submanifold if vector fields ξ_i , $i \in \{1,2,...,s\}$ are tangent to M and there exists on M a pair of orthogonal distribution $\{D,D^{\perp}\}$ such that*

(i) $TM = D \oplus D^{\perp} \oplus sp\{\xi_1,...,\xi_s\}$

(ii) The distribution D is invariant under φ, that is $\varphi D_x = D_x$, for all $x \in M$

(iii) The distribution $D^{\perp}$ is anti-invariant under φ, that is $\varphi D_x^{\perp} \subset T_x^{\perp}M$, for all $x \in M$,

where T_xM is the tangent space of M at x.

A semi-invariant submanifold M is said to be an *invariant* (resp. a*nti-invariant*) submanifold if we have $D_x^{\perp} = \{0\}$ (resp. $D_x = \{0\}$) for each $x \in M$. We say that M is a proper semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

Let $\widetilde{\nabla}$ be the Riemannian connection of $\tilde{M}$ with respect to the induced metric g. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X^* Y + h(X,Y) \tag{6}$$

$$\widetilde{\nabla}_X N = \nabla^{*\perp}_X N - A_N X \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$. ∇^* is the induced connection on M, $\nabla^{*\perp}$ is the connection in the normal bundle, h is the second fundamental from of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{8}$$

Now, a linear connection $\overline{\nabla}$ is defined as

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}.$$

Theorem 1. *Let $\widetilde{\nabla}$ be the Riemannian connection on a generalized Kenmotsu manifold $\tilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} \qquad X, Y \in \Gamma(TM)$$

is a semi-symmetric metric connection on $\tilde{M}$.

Proof. Let $\bar{T}$ be the torsion tensor of $\overline{\nabla}$. Then,

$$\begin{aligned}
\bar{T}(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\
&= \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} \\
&\quad -\widetilde{\nabla}_Y X - \sum_{i=1}^{s}\{\eta^i(X)Y - g(Y,X)\xi_i\} - [X,Y] \\
&= \sum_{i=1}^{s}\{\eta^i(Y)X - \eta^i(X)Y\}.
\end{aligned}$$

Moreover we get,

$$\begin{aligned}
(\overline{\nabla}_X g)(Y,Z) &= X[g(Y,Z)] - g(\overline{\nabla}_X Y, Z) - g(Y, \overline{\nabla}_X Z) \\
&= X[g(Y,Z)] - g(\widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}, Z) \\
&\quad -g(Y, \widetilde{\nabla}_X Z + \sum_{i=1}^{s}\{\eta^i(Z)X - g(X,Z)\xi_i\}) \\
&= 0.
\end{aligned}$$

Corollary 2. *Let $\widetilde{\nabla}$ be the Riemannian connection on a generalized Kenmotsu manifold $\tilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} \qquad X, Y \in \Gamma(TM) \tag{9}$$

is a semi-symmetric metric connection on $\tilde{M}$.

Theorem 3. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$.Then we have*

$$(\overline{\nabla}_X \varphi)Y = 2\sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} \tag{10}$$

for all $X, Y \in \Gamma(TM)$.

Proof. From (4) and (9), we have

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\}$$

$$\overline{\nabla}_X \varphi Y - \sum_{i=1}^{s}\{\eta^i(\varphi Y)X - g(X,\varphi Y)\xi_i\} - \varphi\{\overline{\nabla}_X Y - \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}\}$$

$$= \sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\}$$

$$(\overline{\nabla}_X \varphi)Y = \sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X - g(X,\varphi Y)\xi_i - \eta^i(Y)\varphi X\}.$$

Theorem 4. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection $\overline{\nabla}$. Then*

$$\overline{\nabla}_X \xi_j = 2X - \sum_{i=1}^{s}\{\eta^i(X) + \eta^j(X)\}\xi_i \tag{11}$$

for all $X, Y \in \Gamma(TM)$.

Proof. Using (9) then we have

$$\overline{\nabla}_X \xi_j = \widetilde{\nabla}_X \xi_j + \sum_{i=1}^{s} \{\eta^i(\xi_j)X - g(X, \xi_j)\xi_i\}$$

from (5),

$$\overline{\nabla}_X \xi_j = X - \sum_{i=1}^{s} \eta^i(X)\xi_i + \sum_{i=1}^{s} \{X - \eta^j(X)\xi_i\}.$$

Example 1. *Now , we construct an example of generalized Kenmotsu manifold for 4-dimensional.*

Let, $n = 1$ *and* $s = 2$*. The vector fields*

$$e_1 = f_1(z_1, z_2)\frac{\partial}{\partial x} + f_2(z_1, z_2)\frac{\partial}{\partial y},$$

$$e_2 = -f_2(z_1, z_2)\frac{\partial}{\partial x} + f_1(z_1, z_2)\frac{\partial}{\partial y},$$

$$e_3 = \frac{\partial}{\partial z_1},$$

$$e_4 = \frac{\partial}{\partial z_2}$$

where f_1 *and* f_2 *are given by*

$$\begin{aligned} f_1(z_1, z_2) &= c_2 e^{-(z_1+z_2)} Cos(z_1 + z_2) - c_1 e^{-(z_1+z_2)} Sin(z_1 + z_2), \\ f_2(z_1, z_2) &= c_1 e^{-(z_1+z_2)} Cos(z_1 + z_2) + c_2 e^{-(z_1+z_2)} Sin(z_1 + z_2) \end{aligned}$$

for nonzero constant c_1, c_2*. It is obvious that* $\{e_1, e_2, e_3, e_4\}$ *are linearly independent at each point of* M*. Let* g *be the Riemannian metric defined by*

$$g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

for all $i, j \in \{1, 2, 3, 4\}$ *and given by the tensor product*

$$g = \frac{1}{f_1^2 + f_2^2}(dx \otimes dx + dy \otimes dy) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2,$$

where $\{x, y, z_1, z_2\}$ *are standart coordinates in* $\mathbb{R}^4$*. Let* η^1 *and* η^2 *be the 1-form defined by*

$$\eta^1(X) = g(X, e_3) \text{ and } \eta^2(X) = g(X, e_4),$$

respectively, for any vector field X on M and φ be the $(1,1)$ tensor field defined by

$$\varphi(e_1) = e_2, \qquad \varphi(e_2) = -e_1, \qquad \varphi(e_3 = \xi_1) = 0, \qquad \varphi(e_4 = \xi_2) = 0.$$

We have $\Phi(e_{1,}, e_2) = -1$ and otherwise $\Phi(e_{i,}, e_j) = 0$ for $i < j$. Therefore, the essential non-zero component of Φ is

$$\Phi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{1}{f_1^2 + f_2^2} = \frac{e^{2(z_1+z_2)}}{c_1^2 + c_2^2}$$

and hence

$$\Phi = \frac{e^{2(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy.$$

Consequently, the exterior derivative $d\Phi$ is given by

$$d\Phi = \frac{2e^{2(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy \wedge (dz_1 + dz_2).$$

Since $\eta^1 = dz_1$ and $\eta^2 = dz_2$, we find

$$d\Phi = 2(\eta^1 + \eta^2) \wedge \Phi.$$

So, we have 4-dimensional a generalized Kenmotsu manifold [6]. Let ∇ be the Riemannian connection (the Levi-Civita connection) of g. Then, we have

$$[e_1, e_4] = [e_1, e_3] = e_1 + e_2, \qquad [e_2, e_4] = [e_2, e_3] = e_1 + e_2,$$

$$[e_1, e_2] = 0, \qquad [e_3, e_4] = 0.$$

By Koszul's formula, we get

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = -e_3 - e_4,$$

$$\nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = e_1 + e_2$$

and anothers are zero.

$$\overline{\nabla}_X Y = \nabla_X Y + \eta^1(Y)X - g(X,Y)\xi_1 + \eta^2(Y)X - g(X,Y)\xi_2$$

is a semi-symmetric metric connection . Therefore, we have

$$\bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{e_2} e_2 = -2(e_3 + e_4), \qquad \bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{e_1} e_2 = -e_3 - e_4$$

$$\bar{\nabla}_{e_1} e_3 = \bar{\nabla}_{e_1} e_4 = 2e_1 + e_2, \qquad \bar{\nabla}_{e_2} e_3 = \bar{\nabla}_{e_2} e_4 = e_1 + 2e_2,$$

$$-\bar{\nabla}_{e_3} e_3 = \bar{\nabla}_{e_4} e_3 = e_4, \qquad \bar{\nabla}_{e_3} e_4 = -\bar{\nabla}_{e_4} e_4 = e_3$$

and anothers are zero.

We denote by same symbol g both metrices on $\tilde{M}$ and M. Let $\overline{\nabla}$ be the semi-symmetric metric connection on $\tilde{M}$ and ∇ be the induced connection on M. Then,

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y) \tag{12}$$

where m is a tensor field of type $(0,2)$ on a semi-invariant submanifold M. Using (6) and (9) we have,

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}.$$

So equation tangential and normal components from both the sides, we get

$$m(X,Y) = h(X,Y)$$

$$\nabla_X Y = \nabla^*_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}. \tag{13}$$

From (13) and (7)

$$\begin{aligned} \nabla_X N &= \nabla^*_X N + \sum_{i=1}^{s}\{\eta^i(N)X - g(X,N)\xi_i\} \\ &= -A_N X + \sum_{i=1}^{s}\eta^i(N)X \\ &= (-A_N + a)X \end{aligned}$$

where $a = \sum\limits_{i=1}^{s}\eta^i(N)$ is a function on M and $N \in \Gamma(TM)^{\perp}$.

Now, the Gauss and Weingarten formulas for semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{14}$$

$$\overline{\nabla}_X N = (-A_N + a)X + \nabla^{\perp}_X N \tag{15}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$, h second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X,Y),N) = g((-A_N + a)X, Y). \tag{16}$$

Theorem 5. *The connection induced on a semi-invariant submanifold of a generalized Kenmotsu manifold with the semi-symmetric metric connection is also a semi-symmetric metric connection.*

Proof. From (14) we have

$$\bar{T}(X,Y) = T(X,Y) \text{ and } (\overline{\nabla}_X g)(Y,Z) = (\nabla_X g)(Y,Z)$$

for any $X, Y \in \Gamma(TM)$, where T is the torsion tensor of ∇.

The projection morphisms of TM to D and $D^\perp$ are denoted by P and Q respectively. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$, we have

$$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i \tag{17}$$

$$\varphi N = BN + CN \tag{18}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN.

3. Basic Results

Lemma 6. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection, then we have*

$$\begin{aligned}(\overline{\nabla}_X \varphi)Y &= (\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y) \\ &\quad + (\nabla_X Q)Y + h(X,PY) - Ch(X,Y)\end{aligned} \tag{19}$$

$$\begin{aligned}(\overline{\nabla}_X \varphi)N &= (\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X \\ &\quad + (\nabla_X C)N + h(X,BN) + Q(-A_N + a)X\end{aligned} \tag{20}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^\perp$ where $a = \sum_{i=1}^{s} \eta^i(CN) = 0$.

Proof. Using (17) , (18), the Gauss and Weingarten formulas, necessary arrangements are made to obtain the desired.

Lemma 7. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection, we have*

$$(\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y) = -2\sum_{i=1}^{s} \eta^i(Y)PX \tag{21}$$

$$(\nabla_X Q)Y + h(X, PY) - Ch(X, Y) = -2\sum_{i=1}^{s}\eta^i(Y)QX \tag{22}$$

$$(\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X = 0 \tag{23}$$

$$(\nabla_X C)N + h(X, BN) + Q(-A_N + a)X = 0 \tag{24}$$

$$g(PX, Y) = 0 \tag{25}$$

$$g(QX, Y) = 0 \tag{26}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$.

Proof. Using (10) in (19) and (20) we get (21)-,(26).

Corollary 8. *Let* M *be a semi-invariant submanifold of a generalized Kenmotsu manifold* $\tilde{M}$ *with semi-symmetric metric connection such that* $\xi_i \in \Gamma(TM)$, *we have*

$$(\nabla_X P)\xi_j = -2PX \tag{27}$$

$$(\nabla_X Q)\xi_j = -2QX \tag{28}$$

$$(\nabla_{\xi_j} B)N = 0, \quad \nabla_{\xi_j} B = 0 \tag{29}$$

$$(\nabla_{\xi_j} C)N = 0, \quad \nabla_{\xi_j} C = 0. \tag{30}$$

Lemma 9. *Let* M *be a semi-invariant submanifold of a generalized Kenmotsu manifold* $\tilde{M}$ *with the semi-symmetric metric connection such that* $\xi_i \in \Gamma(TM)$, *we have*

$$\nabla_X \xi_j = 2X - \sum_{i=1}^{s}\{\eta^i(X) + \eta^j(X)\}\xi_i, \qquad h(X, \xi_j) = 0 \tag{31}$$

$$\nabla_{\xi_i}\xi_j = 0, \qquad h(\xi_i, \xi_j) = 0, \qquad A_N\xi_j = 0. \tag{32}$$

Proof. Using (9)and (11) we have (31).In addition, we get

$$0 = g(h(X, \xi_j), N) = g(h(\xi_j, X), N) = g(A_N\xi_j, X).$$

4. Integrability of distribution on a semi-invariant submanifold generalized Kenmotsu manifold

Theorem 10. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection. Then the distribution D is integrable.*

Proof. We have

$$\begin{aligned} g([X,Y],\xi_j) &= g(\widetilde{\nabla}_X Y,\xi_j) - g(\widetilde{\nabla}_Y X,\xi_j) \\ &= -g(Y,\widetilde{\nabla}_X \xi_j) + g(X,\widetilde{\nabla}_Y \xi_j) \end{aligned}$$

for all $X, Y \in \Gamma(D)$. Using (9) and (11), we get

$$\begin{aligned} g([X,Y],\xi_j) &= -g(Y,\overline{\nabla}_X \xi_j - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) + g(X,\overline{\nabla}_Y \xi_j - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= -g(Y, 2X - \sum_{i=1}^{s}\{\eta^i(X)+\eta^j(X)\}\xi_i - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) \\ &\quad + g(X, 2Y - \sum_{i=1}^{s}\{\eta^i(Y)+\eta^j(Y)\}\xi_i - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= 0. \end{aligned}$$

So $\eta^j([X,Y]) = 0$ for $j = 1, 2, ..., s$. Then, we have $[X,Y] \in \Gamma(D)$.

Theorem 11. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection. The distribution $D \oplus sp\{\xi_1, ..., \xi_s\}$ is integrable if and only if*

$$h(X,\varphi Y) = h(\varphi X, Y)$$

for all $X, Y \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ is satisfied.

Proof. Using (6) and (9), then

$$\begin{aligned} \varphi([X,Y]) &= \varphi(\nabla^*_X Y - \nabla^*_Y X) \\ &= \varphi(\widetilde{\nabla}_X Y - h(X,Y) - \widetilde{\nabla}_Y X + h(Y,X)) \\ &= \varphi(\overline{\nabla}_X Y - \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} - \overline{\nabla}_Y X + \sum_{i=1}^{s}\{\eta^i(X)Y - g(Y,X)\xi_i\}) \\ &= \overline{\nabla}_X \varphi Y - (\overline{\nabla}_X \varphi)Y - \sum_{i=1}^{s}\eta^i(Y)\varphi X - \overline{\nabla}_Y \varphi X + (\overline{\nabla}_Y \varphi)X + \sum_{i=1}^{s}\eta^i(X)\varphi Y. \end{aligned}$$

for all $X, Y \in \Gamma(D)$. For (10) and (14), we have

$$\varphi([X,Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \sum_{i=1}^{s}\{4g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\} + h(X,\varphi Y) - h(\varphi X, Y).$$

Then, we have $[X,Y] \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ if and only if $h(X,\varphi Y) = h(\varphi X, Y)$, where $\varphi([X,Y])$ shows the component of $\nabla_X Y$ from the ortohogonal complementary distribution of $D \oplus Sp\{\xi_{1,...,}\xi_s\}$ in M. Then, we have $[X,Y] \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ if and only if $h(X,\varphi Y) = h(Y, \varphi X)$.

Theorem 12. *Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection.The distribution $D^{\perp} \oplus sp\{\xi_1, ..., \xi_s\}$ is integrable if and only if*

$$A_{\varphi X} Y = A_{\varphi Y} X$$

for all $X, Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1, ..., \xi_s\})$ is satisfied.

Proof. We have for all $X, Y \in \Gamma(D^{\perp})$

$$\begin{aligned} g([X,Y],\xi_j) &= g(\widetilde{\overline{\nabla}}_X Y, \xi_j) - g(\widetilde{\overline{\nabla}}_Y X, \xi_j) \\ &= -g(Y, \widetilde{\overline{\nabla}}_X \xi_j) + g(X, \widetilde{\overline{\nabla}}_Y \xi_j). \end{aligned}$$

Using (9) and (11), we have

$$\begin{aligned} g([X,Y],\xi_i) &= -g(Y, 2X - \sum_{i=1}^{s}\{\eta^i(X) + \eta^j(X)\}\xi_i - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) \\ &\quad + g(X, 2Y - \sum_{i=1}^{s}\{\eta^i(Y) + \eta^j(Y)\}\xi_i - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= 0. \end{aligned}$$

Using (6) and (9), then

$$\begin{aligned} \varphi([X,Y]) &= \varphi(\nabla^*_X Y - \nabla^*_Y X) \\ &= \overline{\nabla}_X \varphi Y - (\overline{\nabla}_X \varphi)Y - \sum_{i=1}^{s}\eta^i(Y)\varphi X - \overline{\nabla}_Y \varphi X + (\overline{\nabla}_Y \varphi)X + \sum_{i=1}^{s}\eta^i(X)\varphi Y. \end{aligned}$$

For (10) and (15), we have

$$\begin{aligned} \varphi([X,Y]) &= (-A_{\varphi Y} + a)X + \nabla^{\perp}_X \varphi Y - 2\sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} - \sum_{i=1}^{s}\eta^i(Y)\varphi X \\ &\quad -(-A_{\varphi X} + a)Y - \nabla^{\perp}_Y \varphi X + 2\sum_{i=1}^{s}\{g(\varphi Y, X)\xi_i - \eta^i(X)\varphi Y\} + \sum_{i=1}^{s}\eta^i(X)\varphi Y \\ &= A_{\varphi X} Y - A_{\varphi Y} X + \nabla^{\perp}_X \varphi Y - \nabla^{\perp}_Y \varphi X + \sum_{i=1}^{s}\{4g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\}. \end{aligned}$$

Then we obtain,

$$[X,Y]\in\Gamma(D^{\perp}\oplus Sp\{\xi_1,...,\xi_s\})\Rightarrow A_{\varphi X}Y=A_{\varphi Y}X.$$

Conversely

$$\varphi^2([X,Y])=\varphi(A_{\varphi X}Y-A_{\varphi Y}X+\nabla^{\perp}_X\varphi Y-\nabla^{\perp}_Y\varphi X+\sum_{i=1}^{s}\{4g(X,\varphi Y)\xi_i+\eta^i(Y)\varphi X-\eta^i(X)\varphi Y\})$$

$$[X,Y]=\sum_{i=1}^{s}\{-\eta^i(Y)X+\eta^i(X)Y+\sum_{k=1}^{s}(\eta^i(Y)\eta^k(X)\xi_k-\eta^i(X)\eta^k(X)\xi_k)\}+\varphi(\nabla^{\perp}_X\varphi Y)-\varphi(\nabla^{\perp}_Y\varphi X)$$

then, we have $[X,Y]\in\Gamma(D^{\perp}\oplus Sp\{\xi_1,...,\xi_s\})$.

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