Acta Universitatis Apulensis No. 43/2015
ISSN: 1582-5329 pp. 187-198
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.43.15

FEKETE-SZEGO INEQUALITY FOR SUBCLASSES OF ANALYTIC FUNCTION OF COMPLEX ORDER

K. I. Noor, R. Fayyaz

Abstract. In this paper, we introduce certain new subclasses of analytic functions of complex order by using the convolution operator. For these classes several Fekete-Szego type coefficient inequalities are derived. Some special cases are also discussed.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Fekete-Szego inequality, subordination, convolution.

1. Introduction

Let A be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, z\in E, \tag{1}$$

which are analytic in the open unit disk $E=\{z\in\mathbb{C}\colon |z|<1\}$. Let S denote the subclass of A consisting of univalent functions in E. Let f, $g\in A$, with

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n, z\in E, \tag{2}$$

and $f(z)$ is given by (1). Then convolution (Hadamard product) of f and g is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_ng_nz^n, z\in E. \tag{3}$$

Also, if f and g are analytic in E, we say that f is subordinate to g written as $f(z)\prec g(z)$ if there exists a Schwarz function $w(z)$ with $w(0)=0$ and $|w(z)|<1$ such that

$$f(z) = g(w(z)), \quad z \in E.$$

It is well known that for $f \in S$, given by (1), the inequality $\left|a_3 - a_2^2\right| \leq 1$ holds [3]. Fekete and Szego [5] obtained the sharp upper bound for the functional $\left|a_3 - \mu a_2^2\right|$ as

$$\left|a_3 - \mu a_2^2\right| \leq 1 + \exp\left(\frac{-2\mu}{1-\mu}\right),$$

for $f \in S, 0 \leq \mu \leq 1$. The problemma of finding sharp upper bound for functional $\left|a_3 - \mu a_2^2\right|$ for different classes of functions in A is known as Fekete-Szego problemma. Many authors considered this problemma for different classes of univalent functions (see [10]-[6]). For brief history of this problemma for the classes of starlike, convex and close to convex functions see [16]. In [13], Fekete-Szego problemma for the classes $k - UCV$, $k - SP$ and some other related classes defined by using fractional calculus is settled. We discuss the Fekete-Szego type inequalities for the classes $S^*(b, g(z), \phi(z))$ and $C(b, g(z), \phi(z))$ defined as follows:

Definition 1. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 .., \tag{4}$$

be convex univalent function with $Re\,(\phi(z)) > 0$ *with real coefficients in* E *and let* $g(z)$ *given by (2) with real coefficients and let*

$$(f * g)(z) \neq 0, z \in E.$$

A function $f \in A$ *is in the class* $S^*(b, g(z), \phi(z))$ *if*

$$1 + \frac{1}{b}\left(\frac{z\,(f*g)'}{f*g} - 1\right) \prec \phi(z), z \in E.$$

Definition 2. *A function* $f \in A$ *is in the class* $C(b, g(z), \phi(z))$ *if*

$$1 + \frac{1}{b}\left(\frac{z\,(f*g)''}{(f*g)'}\right) \prec \phi(z), z \in E.$$

We have the following special cases.

1. $S^*(1, \frac{z}{1-z}, \frac{1+z}{1-z}) = S^*$, studied in [3].
2. $S^*(1, \frac{z}{1-z}, \frac{1+Az}{1+Bz}) = S^*[A, B]$, see [7].
3. $S^*(1, z\,{}_2F_1(a, b, c; z), p_k(z)) = k - SP_c^{a,b}$, introduced in [13].
4. $C(1, \frac{z}{1-z}, \frac{1+z}{1-z}) = C$, see [3].
5. $C(1, \frac{z}{1-z}, \frac{1+Az}{1+Bz}) = C[A, B]$ we refer to [7].

2. Preliminaries

Lemma 1. *[3] Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z + ..,$$

then

$$|c_n| \leq 2, n \geq 2.$$

Lemma 2. *[3] Let $p \in P$ with*

$$p(z) = 1 + c_1 z + c_2 z + ..,$$

then for any complex number ν

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\left\{1, |2\nu - 1|\right\},$$

and result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\ p(z) = \frac{1+z}{1-z}.$$

Lemma 3. *[3] Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z^2 + ..,$$

then

$$\left|c_2 - \frac{1}{2}\mu c_1\right| \leq 2 + \frac{1}{2}(|\mu - 1| - 1)\left|c_1\right|^2.$$

Lemma 4. *Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z + ..,$$

then

$$|c_2 - v c_1| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0 \\ 2 & if \quad 0 \leq \nu \leq 1 \\ 4\nu - 2 & if \quad \nu \geq 1 \end{cases}$$

3. Main Results

Theorem 5. *Let $\phi(z)$ be given by (4) and $g(z)$ by (2) where both $\phi(z)$ and $g(z)$ have real coefficients and $b \in \mathbb{C}\backslash\{0\}$.If $f \in S^*(b, g(z), \phi(z))$, then*

$$|a_2| \leq |b| \frac{B_1}{g_2},$$

$$|a_3| \leq |b| \frac{B_1}{2g_3} \max\left\{1, \left|\frac{B_2}{B_1} + bB_1\right|\right\}$$

and

$$\left|a_3 - \frac{1}{2}\left(\frac{g_2^2}{g_3}\right) a_2^2\right| \leq |b| \frac{B_2}{2g_3}.$$

Proof. Let $f \in S^*(b, g(z), \phi(z))$ then

$$1 + \frac{1}{b}\left(\frac{z\,(f*g)'}{f*g} - 1\right) \prec \phi(z), \quad z \in E,$$

so that

$$1 + \frac{1}{b}\left(\frac{z\,(f*g)'}{f*g} - 1\right) = \phi(w(z)), \quad z \in E,$$

where $w(z)$ is Schawarz function with $w(0) = 0$and $|w(z)| \leq 1$. Let us denote

$$(f*g)(z) = z + A_2z^2 + A_3z^3 + ..,$$

then by (3), we can write

$$A_2 = a_2g_2 \text{ and } A_3 = a_3g_3, \tag{5}$$

so that

$$\begin{aligned}\frac{z(1 + 2A_2z^2 + 3A_3z^2 + ...)}{z + A_2z^2 + A_3z^3 + ...} &= 1 - b + b(\phi(w(z))) \\ &= b(\phi(w(z)) - b + 1 \\ &= b(\phi(\frac{p(z) - 1}{p(z) + 1})) - b + 1\end{aligned}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with $Re(p(z)) > 0$. This implies that

$$\frac{z(1+2A_2z^2+3A_3z^2+...)}{z+A_2z^2+A_3z^3+...} = b\left(1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+...\right)-b+1$$

$$z\left(1+2A_2z^2+3A_3z^2+...\right) = \left(z+A_2z^2+A_3z^3+...\right)$$
$$\left(1+\frac{b}{2}B_1c_1z+\left(\frac{b}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{b}{4}B_2c_1^2\right)z^2+...\right)$$
$$= z+\left(A_2+\left(\frac{b}{2}B_1c_1\right)\right)z^2+$$
$$\left[\left(A_3+\left(\frac{b}{2}B_1c_1\right)A_2+\frac{b}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+b\left(\frac{1}{4}B_2c_1^2\right)\right)\right]z^3+...$$

Equating the coefficients on both sides, we have

$$A_2 = \frac{b}{2}B_1c_1 \text{ and } A_3 = \frac{bB_1}{4}\left(c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-bB_1\right)c_1^2\right) \quad (6)$$

Taking into account (5), (6) and lemmama 1, we have

$$|a_2| \leq |b|\frac{B_1}{g_2},$$

and lemmama 2 leads us to

$$|a_3| \leq |b|\frac{B_1}{2g_3}\max\left\{1,\left|\frac{B_2}{B_1}+bB_1\right|\right\}.$$

Moreover, by lemmama 1, we get

$$\left|a_3-\frac{1}{2}\left(\frac{g_2^2}{g_3}\right)a_2^2\right| \leq \frac{|b|\,B_2}{2g_3}.$$

which is our required result.

For $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B \leq A \leq 1)$ in theorem 5,we obtain the following corollaryollary.

Corollary 6. *If* $f \in S^*(b, g(z), \frac{1+Az}{1+Bz})$,$(-1 \leq B \leq A \leq 1)$ *then*

$$|a_2| \leq |b|\frac{(A-B)}{g_2},$$

$$|a_3| \leq |b|\frac{(A-B)}{2g_3}\max\left\{1,\left|-B+b\left(A-B\right)\right|\right\},$$

and

$$\left|a_3 - \frac{1}{2}\left(\frac{g_2^2}{g_3}\right)a_2^2\right| \leq |b|\,\frac{-B(A-B)}{2g_3}.$$

We take $g(z) = \frac{z}{1-z}$, $\phi(z) = \frac{1+z}{1-z}$, $z \in E$ and $b = 1$ in theorem 5, it follows the known result, given in [10]

Corollary 7. *If* $f \in S^*\left(1, \frac{z}{1-z}, \frac{1+z}{1-z}\right)$, *then*

$$\left|a_3 - \frac{1}{2}a_2^2\right| \leq 1.$$

For $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n$, $a > 0$, $\delta \geq 0$ and $\phi(z) = \frac{1+z}{1-z}$, $z \in E$, in theorem 5, we get the following known result, see [2].

Corollary 8. *If* $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n, \frac{1+z}{1-z}\right), (a > 0,\ \delta \geq 0,\ z \in E)$, *then*

$$\left|a_3 - \frac{1}{2}\left(\frac{a(a+2)}{(a+1)^2}\right)^{\delta} a_2^2\right| \leq |b|\left(\frac{a+2}{a}\right)^{\delta}.$$

Theorem 9. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{C}\backslash\{0\}$. *If* $f \in S^*(b, g(z), \phi(z))$ *then for any* $\mu \in \mathbb{C}$ *we have*

$$\left|a_3 - \mu a_2^2\right| \leq |b|\,\frac{B_1}{2g_3}\times \max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] bB_1\right|\right\}.$$

Proof. Taking into account (6), we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \left|\begin{array}{c}\frac{bB_1}{4g_3}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1\right)c_1^2\right) - \\ \mu\left(\frac{g_2^2}{g_3}\right)\left(\frac{b}{2g_2}B_1 c_1\right)^2\end{array}\right| \\ &= \left|\frac{bB_1}{4g_3}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1 + 2\mu\left(\frac{g_3}{g_2^2}\right)bB_1\right)c_1^2\right)\right|.\end{aligned}$$

From lemmama 2 we obtain,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,B_1}{2g_3}\max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] bB_1\right|\right\}$$

We put $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B \leq A \leq 1)$ in theorem 9 and obtain the following result.

Corollary 10. *If $f \in S^*\left(b, g(z), \frac{1+Az}{1+Bz}\right)$, $(-1 \leq B \leq A \leq 1)$, then for any $\mu \in \mathbb{C}$ we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,(A-B)}{2g_3} \max\left\{1, \left|-B + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] b\,(A-B)\right|\right\}.$$

We take $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n$, $(a > 0,\ \delta \geq 0)$ and $\phi(z) = \frac{1+z}{1-z}$, $z \in E$, in theorem 5, to get the following result.

Corollary 11. *If $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n, \frac{1+z}{1-z}\right)$, $(a > 0,\ \delta \geq 0,\ z \in E)$, then we for any $\mu \in \mathbb{C}$, we obtain the inequality*

$$\left|a_3 - \mu a_2^2\right| \leq |b|\left(\frac{a+2}{a}\right)^{\delta} \times \max\left\{1, \left|1 + 2b - 4\mu b\left(\frac{(a+1)^2}{a(a+2)}\right)^{\delta}\right|\right\}.$$

This result has been proved in [2]

We take

$g(z) = z + \sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n$, $(\alpha, \beta, \lambda, \delta \geq 0, \beta > \alpha, \lambda > \delta, k \in \mathbb{N}_0)$, in theorem 9, this implies the result proved in [1].

Corollary 12. *If $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n, \phi(z)\right)$, $(\alpha, \beta, \lambda, \delta \geq 0, \beta > \alpha, \lambda > \delta, k \in \mathbb{N}_0)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}}\right)\right] bB_1\right|\right\}.$$

Now we consider the case when both μ and b are real.

Theorem 13. *Let $\phi(z)$ be given by (4), $g(z)$ by (2), both with real coefficients and $b \in \mathbb{R}$ with $b > 0$. If $f \in S^*(b, g(z), \phi(z))$ then for any $\mu \in \mathbb{R}$, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{bB_1}{2g_3}\left[\frac{B_2}{B_1} + bB_1 - 2\mu\frac{g_3}{g_2^2} bB_1\right] & \text{if } \mu \leq \delta_1 \\ \frac{bB_1}{2g_3} & \text{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{bB_1}{2g_3}\left[-\frac{B_2}{B_1} - bB_1 + 2\mu\left(\frac{g_3}{g_2^2}\right) bB_1\right] & \text{if } \mu \geq \delta_2 \end{cases}$$

where $\delta_1 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1^2} - \frac{1}{bB_1}\right)\left(\frac{g_2^2}{g_3}\right)$, $\delta_2 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1} + \frac{1}{bB_1^2}\right)\left(\frac{g_2^2}{g_3}\right)$.

Proof. Using (6), we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \left|\frac{bB_1}{4g_3}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1\right)c_1^2\right) - \mu\left(\frac{g_2^2}{g_3}\right)\left(\frac{b}{2g_2}B_1c_1\right)^2\right| \\ &= \left|\frac{bB_1}{4g_3}\left\{c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1 + 2\mu\left(\frac{g_3}{g_2^2}\right)bB_1\right)c_1^2\right\}\right|,\end{aligned}$$

which can be written as

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{bB_1}{4g_3}\right|\left|\left(c_2 - \nu c_1^2\right)\right|,$$

where $\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \left(1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1\right)$. From lemmama 4, it follows that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{bB_1}{2g_3}[\frac{B_2}{B_1} + \left(1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1] & \text{if } \mu \leq \delta_1 \\ \frac{bB_1}{2g_3} & \text{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{bB_1}{2g_3}\left[-\frac{B_2}{B_1} - \left(1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1\right] & \text{if } \mu \geq \delta_2 \end{cases}$$

where $\delta_1 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1^2} - \frac{1}{bB_1}\right)\left(\frac{g_2^2}{g_3}\right)$ and $\delta_2 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1} + \frac{1}{bB_1^2}\right)\left(\frac{g_2^2}{g_3}\right)$.

We take $\phi(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1)$ in theorem 5, to obtain the following result.

Corollary 14. *If* $f \in S^*(b, g(z), \frac{1+Az}{1+Bz})$, $(-1 \leq B \leq A \leq 1)$ *then for any* $\mu \in \mathbb{R}$ *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{b(A-B)}{2g_3}\left[-B + \left(1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right)b\,(A-B)\right] & \textit{if } \mu \leq \delta_1 \\ \frac{b(A-B)}{2g_3} & \textit{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{b(A-B)}{2g_3}\left[B - \left(1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right)b\,(A-B)\right] & \textit{if } \mu \geq \delta_2 \end{cases}$$

where $\delta_1 = \frac{1}{2}\left(1 + \frac{-B}{b(A-B)} - \frac{1}{b(A-B)}\right)\left(\frac{g_2^2}{g_3}\right)$ *and* $\delta_1 = \frac{1}{2}\left(1 + \frac{-B}{b(A-B)} - \frac{1}{b(A-B)}\right)\left(\frac{g_2^2}{g_3}\right)$.

For $g(z) = z + \sum_{n=2}^{\infty}\left[(\lambda - \delta)(\beta - \alpha)(n-1) + 1\right]^k z^n$, $(\alpha, \beta, \lambda, \delta \geq 0,\ \beta > \alpha, \lambda > \delta, k \in \mathbb{N}_0)$, in theorem 3.3, we obtain the known result provided in [1].

Corollary 15. *If* $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n, \phi(z)\right)$, $(\alpha,\ \beta,\ \lambda,\ \delta \geq 0,\ \beta > \alpha,\ \lambda > \delta, k \in \mathbb{N}_0)$, *then for any* $\mu \in \mathbb{R}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \left\{ \begin{cases} c\frac{b}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left\{B_2 - \left[2\mu\left(\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^k}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}\right)-1\right]bB_1^2\right\} & \text{if } \mu \leq \delta_1 \\ \frac{bB_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k} & \text{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{b}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left\{-B_2 + \left[2\mu\left(\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^k}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}\right)-1\right]bB_1^2\right\} & \text{if } \mu \geq \delta_2 \end{cases} \right\},$$

where

$$\delta_1 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1^2} - \frac{1}{bB_1}\right)\left(\frac{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}}{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k}\right),$$

and

$$\delta_2 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1} + \frac{1}{bB_1^2}\right)\left(\frac{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}}{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k}\right).$$

Remark 1. *By setting* $g(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}$, *and* $\phi(z) = \frac{1+z}{1-z}$ *in theorem 5, theorem 9 and theorem 13, we get the known results proved by Kanas and Darwish in [8].*

Using similar techniques as given in proof of theorem **??**, we obtain the following result.

Theorem 16. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{C}/\{0\}$. *If* $f \in C(b, g(z), \phi(z))$, *then*

$$|a_2| \leq \frac{|b|}{2}\frac{B_1}{g_2},$$

$$|a_3| \leq |b|\frac{B_1}{6g_3}\max\left\{1, \left|\frac{B_2}{B_1} + bB_1\right|\right\}$$

and

$$\left|a_3 - \frac{2}{3}\left(\frac{g_3}{g_2^2}\right)a_2^2\right| \leq \frac{|b|}{6g_3}B_2.$$

When $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B \leq A \leq 1)$ in theorem 3.4, we obtain the following result.

Corollary 17. *If* $f \in C(b, g(z), \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1))$, *then*

$$|a_2| \leq \frac{|b|}{2} \frac{(A-B)}{g_2},$$

$$|a_3| \leq |b| \frac{(A-B)}{6g_3} \max\{1, |-B + b(A-B)|\},$$

and

$$\left| a_3 - \frac{2}{3}\left(\frac{g_3}{g_2^2}\right) a_2^2 \right| \leq \frac{|b|}{6g_3}\left[-B(A-B)\right].$$

Reasoning in the same lines as in the proof of theorem 3.2, we obtain the following theorm.

Theorem 18. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{C}/\{0\}$*. If* $f \in C(b, g(z), \phi(z))$, *then for any* $\mu \in \mathbb{C}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq |b| \frac{B_1}{2g_3} \max(1, \left| \frac{B_2}{B_1} + \left(1 - 2\mu\frac{g_3}{g_2^2}\right) bB_1 \right|.$$

We take $\phi(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1)$ in 18, to obtain the following corollary-ollary.

Corollary 19. *If* $f \in C(b, g(z), \frac{1+Az}{1+Bz}),\ (-1 \leq B \leq A \leq 1)$, *then for any* $\mu \in \mathbb{C}$ *we have*

$$\left|a_3 - \mu a_2^2\right| \leq |b| \frac{(A-B)}{2g_3} \max(1, \left| -B + \left(1 - 2\mu\frac{g_3}{g_2^2}\right) b(A-B) \right|.$$

When $g(z) = \frac{z}{1-z},\ \phi(z) = \frac{1+z}{1-z},\ z \in E$ and $b = 1$ in theorem 3.1, we obtain the known result, see [10].

Corollary 20. *If* $f \in C\left(1, \frac{z}{1-z}, \frac{1+z}{1-z}\right)$, *then for any* $\mu \in \mathbb{C}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \max\left\{\frac{1}{3}, |\mu - 1|\right\}.$$

If we take $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n,\ (a > 0,\ \delta \geq 0)$, and $\phi(z) = \frac{1+z}{1-z},\ z \in E$, in theorem 3.5, we obtain the following corollaryollary.

Corollary 21. *If* $f \in C\left(1, z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n, \frac{1+z}{1-z}\right),\quad (a > 0,\ \delta \geq 0, z \in E,)$ *,then for* $\mu \in \mathbb{C}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|}{3} \max\left\{1, \left|1 + 2b - 3\mu b\left(\frac{(a+1)^2}{a(a+2)}\right)^{\delta}\right|\right\},$$

which has been proved in [2].

Remark 2. *By setting* $g(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}$, *and* $\phi(z) = \frac{1+z}{1-z}$ *in theorem 3.4 and theorem 3.5, we get the results given in [8].*

Acknowledgements. The authors would like to thank Dr. S.M. Junaid Zaidi, Rector CIIT, for providing excellent research facilities.

References

[1] Aouf, M.K., El-Ashwah, R. M., Hassan, A. A. M. & Hasan, A. H. *Fekete-Szego problemma for a New Class of Analytic Functions Defined by Using a Generalized Differential Operator*, Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, Mathematica 52(1), (2013) 21-34.

[2] Bulut, S. *Fekete–Szegö roblemma for subclasses of analytic functions defined by Komatu integral operator*, Arabian Journal of Mathematics 2(2), (2013) 177-183.

[3] Duren, P.L. *Univalent functions* (Grundlehren der mathematischen Wissenschaften, New York, Berlin, Heidelberg (Tokyo), Springer-Verlag 259(1). (1983)

[4] Dziok, J. *A general solution of the Fekete-Szegö problemma*, Boundary Value Problemmas, 1, (2013).1-13.

[5] Fekete, M. & Szegö, G. Eine Bemerkung *Über ungerade schlichte Funktionen*, Journal of the London Mathematical Society, 1(2), (1933).85-89.

[6] Goyal, S.P. & Kumar, R. *Fekete-Szego problemma for a class of complex order related to Salagean operator*, Bulletin of Mathematical Analysis and Applications, 3(4), (2011) 240-246.

[7] Janowski, W. *Some extremarkal problemmas for certain families of analytic functions*, Bulletin de L academie polonaise des sciences-serie des sciences mathematiques astronomiques et physiques 21(1) (1972), 17-25.

[8] Kanas, S. & Darwish, H.E. *Fekete–Szegö problemma for starlike and convex functions of complex order*, Applied Mathematics Letters 23(7), (2010).777-782.

[9] Kanas, S. *An unified approach to the Fekete–Szegö problemma*, Applied Mathematics and Computation 218(17), (2012), 8453-8461.

[10] Keogh, F.R. & Merkes, E.P. (1969). *A coefficient inequality for certain classes of analytic functions*, Proceedings of the American Mathematical Society, 8-12.

[11] London, R.R. (1993) *Fekete-Szego inequalities for close-to-convex functions*, Proceeding of American Mathematical Society, 117, 947-950.

[12] Ma, W. Minda, D. (1994) *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis, 157-169.

[13] Mishra, A.K. & Gochhayat, P. (2008) *The Fekete–Szegö problemma for k-uniformly convex functions and for a class defined by the Owa–Srivastava operator*, Journal of Mathematical Analysis and Applications 347(2), 563-572.

[14] Mishra, A.K. & Gochhayat, P. (2010) *Fekete-Szegö problemma for a class defined by an integral operator*, Kodai Mathematical Journal 33(2), 310-328.

[15] Mishra, A.K. & Panigrahi, T. (2012) *The Fekete-Szego problemma for a class defined by the Hohlov operator*, Acta Universitatis Apulensis 29, 241-254.

[16] Srivastava, H.M., Mishra, A.K. & Das, M.K. (2001). *The Fekete-Szego problemma for a subclass of close-to-convex functions*, Complex variables Theory and Applications, 44, 145-163.

Khalida Inayat Noor, Department of Mathematics,
COMSATS Institute of Information Technology,
Islamabad, Pakistan
email: *khalidanoor@hotmail.com*

Rabia Fayyaz
Department of Mathematics,
COMSATS Institute of Information Technology,
Islamabad, Pakistan
email: *rabia_fayyaz@comsats.edu.pk*