Acta Universitatis Apulensis No. 44/2015
ISSN: 1582-5329 pp. 153-166
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.44.10

MULTIPLE SOLUTIONS FOR A CLASS OF KIRCHHOFF TYPE PROBLEMS IN ORLICZ-SOBOLEV SPACES

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Abstract. This article deals with the existence of at least three weak solutions for the following Kirchhoff type problems in Orlicz-Sobolev spaces. Our main tool is a variational principle due to G. Bonanno [4].

2010 *Mathematics Subject Classification*: 35J60, 35J92, 58E05, 76A02

Keywords: Kirchhoff type problems; Orlicz-Sobolev spaces; Three critical points theorem

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Assume that $a : (0, \infty) \to \mathbb{R}$ is a function such that the mapping, defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For the function φ above, let us define

$$\Phi(t) = \int_0^t \varphi(s)ds \quad \text{for all } t \in \mathbb{R},$$

on which will be imposed some suitable conditions later.

In this article, we are concerned with a class of Kirchhoff type problems in Orlicz-Sobolev spaces of the form

$$\begin{cases} -M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $M : [0, +\infty) \to \mathbb{R}$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions, and λ is a positive real parameter.

Firstly, it should be noticed that if $\varphi(t) = p|t|^{p-2}t$ for all $t \in \mathbb{R}$, $p > 1$ then problem (1) becomes the well-known p-Kirchhoff-type equation

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

which has been intensively studied in recent years, see the papers [3, 7, 16, 21, 22, 26]. In the case when $p(.)$ is a function, problem (2) has been also studied by many authors, see for examples [2, 15, 17, 18]. Since the first equation in (2) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [8]. Moreover, problem (2) is related to the stationary version of the Kirchhoff equation which was presented by Kirchhoff in 1883, see [20] for details.

We point out the fact that if $M(t) \equiv 1$ and the function $\varphi(t)$ is defined above, problem (1) becomes a nonlinear and non-homogeneous problem, namely,

$$\begin{cases} -\operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

which has been studied by some authors in Orlicz-Sobolev spaces, we refer to [5, 6, 12, 13, 14, 19, 23, 24].

In this article, motivated by the works mentioned above, we shall study the existence of solutions for problem (1). It is clear that this is a natural extension from the earlier studies on Kirchhoff type problems in classical Sobolev spaces and on nonlinear non-homogeneous problems in Orlicz-Sobolev spaces. More precisely, using the ideas firstly introduced in the paper [4] and developed in [17] we want to illustrate how to handle problem (1) in Orlicz-Sobolev spaces by using three critical points theorem. Our situation here is different from those presented in the previous papers [9, 10, 11] on the topic. Indeed, while in [9] we deal with problem (1) and the superlinear and subcritical growth conditions, the main tools in [10, 11] are the mountain pass theorem, the minimum principle and genus theory. To our best knowledge, the result of the present paper is new even in the case $M(t) \equiv 1$, see [6, 12, 19, 23].

In order to study problem (1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, useful for what follows, for more details we refer the readers to the books by Adams [1], Rao and Ren [25], the papers by Clément et al. [13, 14].

For $\varphi : \mathbb{R} \to \mathbb{R}$ and Φ introduced at the start of the paper, we can see that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$, the function Φ is then called an N-function. The function Φ^* defined by the formula

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s)ds \text{ for all } t \in \mathbb{R}$$

is called the complementary function of Φ and it satisfies the condition

$$\Phi^*(t) = \sup\{st - \Phi(s) : \ s \geq 0\} \quad \text{for all } t \geq 0.$$

We observe that the function Φ^* is also an N-function in the sense above and the following Young inequality holds

$$st \leq \Phi(s) + \Phi^*(t) \quad \text{for all } s, t \geq 0.$$

The Orlicz class defined by the N-function Φ is the set

$$K_\Phi(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega \Phi(|u(x)|)\, dx < \infty\right\}$$

and the Orlicz space $L_\Phi(\Omega)$ is then defined as the linear hull of the set $K_\Phi(\Omega)$. The space $L_\Phi(\Omega)$ is a Banach space under the following Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{u(x)}{k}\right) dx \leq 1\right\}$$

or the equivalent Orlicz norm

$$\|u\|_{L_\Phi} := \sup\left\{\left|\int_\Omega u(x)v(x)\, dx\right| : \ v \in K_{\Phi^*}(\Omega), \ \int_\Omega \Phi^*(|v(x)|)\, dx \leq 1\right\}.$$

For Orlicz spaces, the Hölder inequality reads as follows (see [25]):

$$\int_\Omega uv\, dx \leq 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L^*_\Phi(\Omega)} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^*}(\Omega).$$

The Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ building upon $L_\Phi(\Omega)$ is the space defined by

$$W^1L_\Phi(\Omega) := \left\{u \in L_\Phi(\Omega) : \ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, 2, ..., N\right\}.$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

Now, we introduce the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1 L_\Phi(\Omega)$. It turns out that the space $W_0^1 L_\Phi(\Omega)$ can be renormed by using as an equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

For an easier manipulation of the spaces defined above, we define the numbers

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}. \tag{4}$$

Throughout this paper, we assume that

$$1 < \varphi_0 \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 < \infty, \quad t \geq 0, \tag{5}$$

which assures that Φ satisfies the Δ_2-condition, i.e.,

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq 0, \tag{6}$$

where K is a positive constant, see [24, Proposition 2.3].

In this paper, we also need the following condition

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{7}$$

We notice that Orlicz-Sobolev spaces, unlike the Sobolev spaces they generalize, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the Δ_2-condition (6). Actually, condition (6) assures that both $L_\Phi(\Omega)$ and $W_0^1 L_\Phi(\Omega)$ are separable, see [1]. Conditions (6) and (7) assure that $L_\Phi(\Omega)$ is a uniformly convex space and thus, a reflexive Banach space (see [24]); consequently, the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ is also a reflexive Banach space.

Proposition 1 (see [6, 23, 24])**.** *Let $u \in W_0^1 L_\Phi(\Omega)$. Then we have*

(i) $\|u\|^{\varphi^0} \leq \int_\Omega \Phi(|\nabla u(x)|)\,dx \leq \|u\|^{\varphi_0}$ *if* $\|u\| < 1$.

(ii) $\|u\|^{\varphi_0} \leq \int_\Omega \Phi(|\nabla u(x)|)\,dx \leq \|u\|^{\varphi^0}$ *if* $\|u\| > 1$.

We also find that with the help of condition (5), the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$, as a result, $W_0^1 L_\Phi(\Omega)$ is continuously and compactly embedded in the classical Lebesgue space $L^q(\Omega)$ for all $1 \leq q < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$.

Example 1 (See [6, 12, 23])**.**

(1) *Let* $\varphi(t) = p|t|^{p-2}t$, $t \in \mathbb{R}$, $p > 1$. *A simple computation shows that* $\varphi_0 = \varphi^0 = p$. *In this case, the corresponding Orlicz space* $L_\Phi(\Omega)$ *is the classical Lebesgue space* $L^p(\Omega)$ *while the Orlicz-Sobolev space* $W_0^1 L_\Phi(\Omega)$ *is the classical Sobolev space* $W_0^{1,p}(\Omega)$. *Therefore, we obtain the p-Kirchhoff type problems as in [3, 7, 16, 21, 22, 26] and the references cited there.*

(2) *Let* $\varphi(t) = \log(1+|t|^s)|t|^{p-2}t$, $t \in \mathbb{R}$, $p, s > 1$. *Then we can deduce that* $\varphi_0 = p$ *and* $\varphi^0 = p + s$.

(3) *Let* $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ *if* $t \neq 0$, $\varphi(0) = 0$ *with* $p > 2$. *Then we can deduce that* $\varphi_0 = p - 1$ *and* $\varphi^0 = p$.

Before stating and proving the main result of this paper in the next section, in the rest of this section we recall a variational principle due to G. Bonanno [4] that plays an important role in our arguments.

Proposition 2 (See [4, Theorem 2.1]). *Let* $(X, \|.\|)$ *be a separable and reflexive real Banach space,* $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that there exists* $x_0 \in X$ *such that* $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$, $\mathcal{A}(x) \geq 0$ *for all* $x \in X$ *and there exist* $x_1 \in X$, $\rho > 0$ *such that*

(*i*) $\rho < \mathcal{A}(x_1)$,

(*ii*) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho \frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x)}, \text{ with } \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ *is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and*

(*iii*) $\lim_{\|x\|\to\infty}[\mathcal{A}(x) - \lambda\mathcal{F}(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ *and a positive real number* δ *such that each* $\lambda \in \Lambda$, *the equation*

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solutions in X *whose* $\|.\|$*-norms are less than* δ.

2. Multiple solutions

In this section, we shall state and prove the main result of the paper. We shall seek weak solutions of (1) in the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$. The norm in the space $L^p(\Omega)$ is defined by $|u|_p = \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}$. We denote by S_q the best constant in the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L^p(\Omega)$ while we use the letters C_i to denote general positive constants. This means that $S_p|u|_p \leq \|u\|$ for all $u \in W_0^1 L_\Phi(\Omega)$.

Definition 1. *A function $u \in W_0^1 L_\Phi(\Omega)$ is said to be a weak solution of problem (1) if it holds that*

$$M\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla v\,dx - \int_\Omega f(x,u)v\,dx = 0$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

Theorem 1. *Assume that M, f satisfy the following conditions*

(M_1) *There exist $m_0 > 0$ and $1 < \alpha < \frac{\varphi_0^*}{\varphi^0}$ such that*

$$M(t) \geq m_0 t^{\alpha-1}, \quad \forall t \in [0,+\infty);$$

(F_1) $\lim_{|t|\to+\infty} \frac{|f(x,t)|}{|t|^{\alpha\varphi_0 - 1}} = 0$ *uniformly for $x \in \overline{\Omega}$;*

(F_2) $\lim_{|t|\to+\infty} \frac{|f(x,t)|}{|t|^{\alpha\varphi^0 - 1}} = 0$ *uniformly for $x \in \overline{\Omega}$;*

(F_3) *There exist $x_0 \in \Omega$, $t_0 \in \mathbb{R}$ and $R_0 > 0$ so small that $B_N(x_0, R_0) = \{x \in \mathbb{R}^N : |x - x_0| \leq R_0\} \subset \Omega$ and we have $\operatorname{ess\,inf}_{x\in B_N(x_0,R_0)} F(x,t_0) = l_0 > 0$, $\operatorname{ess\,sup}_{x\in B_N(x_0,R_0)} \max_{|t|\leq|t_0|} |F(x,t)| = L_0 < \infty$, where $F(x,t) = \int_0^t f(x,s)\,ds$.*

Then there exist an open interval $\Lambda \subset (0,+\infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ problem (1) has at least three distinct weak solutions in $W_0^1 L_\Phi(\Omega)$, whose $W_0^1 L_\Phi(\Omega)$-norms are less than μ.

For each $\lambda \in \mathbb{R}$, we define the functional $\mathcal{J}_\lambda : W_0^1 L_\Phi(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}_\lambda(u) = \mathcal{A}(u) - \lambda\mathcal{F}(u), \quad u \in W_0^1 L_\Phi(\Omega), \tag{8}$$

where

$$\mathcal{A}(u) = \widehat{M}\left(\int_\Omega \Phi(|\nabla u|)\,dx\right), \quad \mathcal{F}(u) = \int_\Omega F(x,u)\,dx. \tag{9}$$

By (F_1), using the method as in [24], we can show that $\mathcal{J}_\lambda$ is of $C^1(W_0^1L_\Phi(\Omega),\mathbb{R})$ and its derivative is given by

$$\mathcal{J}_\lambda'(u)(v) = M\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla v\,dx - \int_\Omega f(x,u)v\,dx.$$

Hence, weak solutions of problem (1) are exactly the critical points of the functional $\mathcal{J}_\lambda$. Our idea is to prove Theorem 1 by verifying all the asumptions of Proposition 2.

Lemma 2. *The functional $\mathcal{J}_\lambda$ is weakly lower semi-continuous.*

Proof. Let $\{u_m\}$ be a sequence that converges weakly to u in X. Then, from the proof of [24, Lemma 4.3] we deduce that the functional $u\mapsto\int_\Omega\Phi(|\nabla u|)\,dx$ is weakly lower semi-continuous, i.e.,

$$\int_\Omega \Phi(|\nabla u|)dx \le \liminf_{m\to\infty}\int_\Omega \Phi(|\nabla u_m|)\,dx. \tag{10}$$

Combining (10) with the continuity and monotonicity of the function $\psi:\mathbb{R}^+\to\mathbb{R}$, $t\mapsto\psi(t)=\widehat{M}(t)$, we get

$$\begin{aligned}\liminf_{m\to\infty}\mathcal{M}(u_m) &= \liminf_{m\to\infty}\widehat{M}\Big(\int_\Omega\Phi(|\nabla u_m|)\,dx\Big)\\ &\ge \widehat{M}\Big(\liminf_{m\to\infty}\int_\Omega\Phi(|\nabla u_m|)\,dx\Big)\\ &\ge \widehat{M}\Big(\int_\Omega\Phi(|\nabla u|)\,dx\Big)\\ &= \mathcal{M}(u).\end{aligned} \tag{11}$$

Now, we shall show that

$$\lim_{m\to\infty}\mathcal{F}(u_m)=\mathcal{F}(u). \tag{12}$$

Indeed, by the condition (F_1), there exists a positive constant $C_1>0$ such that

$$|f(x,t)|\le C_1(1+|t|^{\alpha\varphi_0-1}),\quad \forall(x,t)\in\overline{\Omega}\times\mathbb{R}. \tag{13}$$

Hence, using the Hölder inequality, we get

$$\begin{aligned}
|\mathcal{F}(u_m) - \mathcal{F}(u)| &\leq \left| \int_\Omega F(x, u_n)\,dx - \int_\Omega F(x,u)\,dx \right| \\
&\leq \int_\Omega |F(x,u_n) - F(x,u)|\,dx \\
&\leq \int_\Omega |f(x, u + \theta_n(u_n - u))||u_n - u|\,dx \\
&\leq C_1 \int_\Omega (1 + |u + \theta_n(u_n - u))|^{\alpha\varphi_0 - 1})|u_n - u|\,dx \\
&\leq C_1 \left(|\Omega|^{\frac{\alpha\varphi_0 - 1}{\alpha\varphi_0}} + |u_n|^{\alpha\varphi_0}_{\alpha\varphi_0 - 1} + |u_n|^{\alpha\varphi_0 - 1}_{\alpha\varphi_0} \right) |u_n - u|_{\alpha\varphi_0}, \quad \theta_n \in (0,1),
\end{aligned} \tag{14}$$

which proves (8). From (11), (14) and the definition of $\mathcal{J}_\lambda$, the lemma is proved.

Lemma 3. *The functional $\mathcal{J}_\lambda$ is coercive.*

Proof. Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (F_1), there exists $\delta = \delta(\lambda) > 0$ such that

$$|f(x,t)| \leq \frac{m_0}{\alpha} S^{\alpha\varphi_0}_{\alpha\varphi_0} \alpha\varphi_0 (1 + |\lambda|)^{-1} |t|^{\alpha\varphi_0 - 1}, \quad \forall |t| \geq \delta \text{ and } x \in \overline{\Omega}. \tag{15}$$

Integrating the above inequality we have

$$|F(x,t)| \leq \frac{m_0}{\alpha} S^{\alpha\varphi_0}_{\alpha\varphi_0} (1 + |\lambda|)^{-1} |t|^{\alpha\varphi_0} + \max_{\overline{\Omega} \times \{|t| \leq \delta\}} |f(x,t)||t|, \quad \forall t \in \mathbb{R}. \tag{16}$$

Thus, for all $u \in W^1_0 L_\Phi(\Omega)$ with $\|u\| > 1$, we obtain

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \widehat{M}\left(\int_\Omega \Phi(|\nabla u|)\,dx \right) - \lambda \int_\Omega F(x,u)\,dx \\
&\geq \frac{m_0}{\alpha} \left(\int_\Omega \Phi(|\nabla u|)\,dx \right)^\alpha - |\lambda| \int_\Omega |F(x,u)|\,dx \\
&\geq \frac{m_0}{\alpha} \|u\|^{\alpha\varphi_0} - \frac{m_0}{\alpha} . \frac{|\lambda|}{1 + |\lambda|} S^{\alpha\varphi_0}_{\alpha\varphi_0} \int_\Omega |u|^{\alpha\varphi_0}\,dx - \max_{\overline{\Omega} \times \{|t| \leq \delta\}} |f(x,t)| \int_\Omega |u|\,dx \\
&\geq \frac{m_0}{\alpha(1 + |\lambda|)} \|u\|^{\alpha\varphi_0} - \frac{\max_{\overline{\Omega} \times \{|t| \leq \delta\}} |f(x,t)|}{S_1} \|u\|.
\end{aligned} \tag{17}$$

By (17) and the fact that $\alpha\varphi_0 > \varphi_0 > 1$, the functional $\mathcal{J}_\lambda$ is coercive.

Lemma 4. *The functional $\mathcal{J}_\lambda$ satisfies the (PS) condition.*

Proof. Let $\{u_m\} \subset W_0^1 L_\Phi(\Omega)$ be a sequence such that

$$\mathcal{J}_\lambda(u_m) \to C_2 > 0, \quad \mathcal{J}'_\lambda(u_m) \to 0 \text{ in } \left(W_0^1 L_\Phi(\Omega)\right)^*, \tag{18}$$

where $\left(W_0^1 L_\Phi(\Omega)\right)^*$ is the dual space of $W_0^1 L_\Phi(\Omega)$.

Since the functional $\mathcal{J}_\lambda$ is coercive, it follows from (18) that the sequence $\{u_m\}$ is bounded in $W_0^1 L_\Phi(\Omega)$. On the other hand, by conditions (5) and (6), the Banach space $W_0^1 L_\Phi(\Omega)$ is reflexive. Thus, there exists $u \in W_0^1 L_\Phi(\Omega)$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in $W_0^1 L_\Phi(\Omega)$. Therefore, $\{u_m\}$ converges strongly to u in $L^{\alpha\varphi_0}(\Omega)$. Using the Hölder inequality we deduce that

$$\begin{aligned} \left|\mathcal{F}'(u_m)(u_m - u)\right| &= \left|\int_\Omega f(x, u_m)(u_m - u)\, dx\right| \\ &\leq C_3 \int_\Omega (1 + |u_m|^{\alpha\varphi_0 - 1})|u_m - u|\, dx \\ &\leq C_3 \left(|\Omega|^{\frac{\alpha\varphi_0 - 1}{\alpha\varphi_0}} + |u_m|_{\alpha\varphi_0}^{\alpha\varphi_0 - 1}\right) |u_m - u|_{\alpha\varphi_0} \end{aligned} \tag{19}$$

which tends to 0 as $m \to \infty$.

On the other hand, by (18), we have

$$\lim_{m\to\infty} \mathcal{J}'_\lambda(u_m)(u_m - u) = 0. \tag{20}$$

From (18)-(20) and the definition of the functional $\mathcal{J}_\lambda$, we get

$$\lim_{m\to\infty} \mathcal{M}'(u_m)(u_m - u) = 0. \tag{21}$$

Using Proposition 1, since $\{u_m\}$ is bounded in $W_0^1 L_\Phi(\Omega)$, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \Phi(|\nabla u_m|)\, dx \to t_1 \geq 0 \text{ as } m \to \infty.$$

If $t_1 = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_1 > 0$ then since the function M is continuous, we get

$$M\left(\int_\Omega \Phi(|\nabla u_m|)\, dx\right) \to M(t_1) \text{ as } m \to \infty.$$

Thus, by (M_0), for sufficiently large m, we have

$$M\left(\int_\Omega \Phi(|\nabla u_m|)\,dx\right) \geq C_4 > 0. \tag{22}$$

From (21), (22), it follows that

$$\lim_{m\to\infty}\int_\Omega a(|\nabla u_m|)\nabla u_m\cdot(\nabla u_m - \nabla u)\,dx = 0.$$

Thus, using [23, Lemma 5], $\{u_m\}$ converges strongly to u in $W_0^1L_\Phi(\Omega)$ and the functional $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition.

Lemma 5. *The following property holds*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u):\ \mathcal{A}(u)<\rho\}}{\rho} = 0.$$

Proof. Due to (F_2), for an arbitrary small $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,t)| \leq \epsilon\alpha\varphi^0 S_{\alpha\varphi^0}^{\alpha\varphi^0}|t|^{\alpha\varphi^0-1}, \quad \forall |t| < \delta \text{ and } x\in\overline{\Omega}. \tag{23}$$

From (13) and (23) we have

$$|F(x,t)| \leq \epsilon S_{\alpha\varphi^0}^{\alpha\varphi^0}|t|^{\alpha\varphi^0} + K(\delta)|t|^q, \quad \forall t\in\mathbb{R} \text{ and } x\in\overline{\Omega}, \tag{24}$$

where $q\in(\alpha\varphi^0,\varphi_0^*)$ is fixed and $K(\delta)>0$ does not depend on t. For $\rho\in(0,+\infty)$, let us define the sets

$$B_\rho^1 = \{u\in W_0^1L_\Phi(\Omega):\ \mathcal{A}(u)<\rho\} \tag{25}$$

and

$$B_\rho^2 = \left\{u\in W_0^1L_\Phi(\Omega):\ \frac{m_0}{\alpha}\|u\|^{\alpha\varphi^0}<\rho\right\}. \tag{26}$$

For all $u\in W_0^1L_\Phi(\Omega)$ with $\|u\|<1$, by Proposition 1, we have

$$\mathcal{A}(u) \geq \frac{m_0}{\alpha}\|u\|^{\alpha\varphi^0},$$

which implies that $B_\rho^1\subset B_\rho^2$ for all $\rho\in\left(0,\frac{m_0}{\alpha}\right)$.

From (24) we obtain

$$\mathcal{F}(u) \leq \epsilon\|u\|^{\alpha\varphi^0} + K(\delta)S_q^{-q}\|u\|^q. \tag{27}$$

Since $0 \in B^1_\rho$ and $\mathcal{F}(0) = 0$ one has $0 \le \sup_{u \in B^1_\rho} \mathcal{F}(u)$. On the other hand, if $u \in B^2_\rho$, then

$$\|u\| \le \left(\frac{\alpha}{m_0}\right)^{\frac{1}{\alpha\varphi^0}} \rho^{\frac{1}{\alpha\varphi^0}}$$

and using (27) we get

$$\begin{aligned} 0 \le \frac{\sup_{u \in B^1_\rho} \mathcal{F}(u)}{\rho} &\le \frac{\sup_{u \in B^2_\rho} \mathcal{F}(u)}{\rho} \\ &\le \frac{\epsilon\alpha}{m_0} + \frac{\alpha K(\delta) S_q^{-q}}{m_0}\|u\|^{q-\alpha\varphi^0} \\ &\le \frac{\epsilon\alpha}{m_0} + S_q^{-q}\left(\frac{\alpha}{m_0}\right)^{\frac{q}{\alpha\varphi^0}} \rho^{\frac{q-\alpha\varphi^0}{\alpha\varphi^0}}. \end{aligned} \tag{28}$$

Because $\epsilon > 0$ is arbitrary and $\rho \to 0^+$, we get the desired result since $q > \alpha\varphi^0$.

Proof of Theorem 1. Let $x_0 \in \Omega$, $t_0 \in \mathbb{R}$ and $R_0 > 0$ be from the condition (F_3). Let us denote by $B_N(x_0, r)$ the $N-$dimensional closed euclidean ball with center $x_0 \in \mathbb{R}^N$ and radius $r > 0$.

For $\sigma \in (0, 1)$, we define the function u_σ by

$$u_\sigma(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \setminus B_N(0, R_0), \\ t_0, & \text{for } x \in B_N(0, \sigma R_0), \\ \frac{t_0}{R_0(1-\sigma)}(R_0 - |x|) & \text{for } x \in B_N(0, R_0) \setminus B_N(0, \sigma R_0). \end{cases}$$

It is clear that $u_\sigma \in W_0^{1,\varphi_0}(\Omega)$ and

$$|u_\sigma(x)| \le |t_0| \text{ for all } x \in \mathbb{R}^N.$$

Moreover, a simple computation implies that

$$\|u_\sigma\|^{\varphi_0}_{W_0^{1,\varphi_0}(\Omega)} = \int_\Omega |\nabla u_\sigma(x)|^{\varphi_0}\, dx = \frac{|t_0|^{\varphi_0}(1-\sigma^N)}{(1-\sigma)^{\varphi_0}} R_0^{N-\varphi_0} w_N > 0, \tag{29}$$

where w_N is the volume of $B_N(0, 1)$.

Since the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow W_0^{1,\varphi_0}(\Omega)$ is continuous, there exists $C_5 > 0$ such that

$$C_5 \|u\|_{W_0^{1,\varphi_0}(\Omega)} \le \|u\|, \quad \forall u \in W_0^1 L_\Phi(\Omega), \tag{30}$$

which helps us to get $\|u_\sigma\| > 0$ for all $\sigma \in (0,1)$. Using the definition of u_σ and the condition (F_3) we obtain

$$\begin{aligned}
\mathcal{F}(u_\sigma) &= \int_\Omega F(x,u_\sigma)\,dx \\
&= \int_{B_N(x_0,\sigma R_0)} F(x,u_\sigma)\,dx + \int_{B_N(x_0,R_0)\setminus B_N(x_0,\sigma R_0)} F(x,u_\sigma)\,dx \\
&\geq \int_{B_N(x_0,\sigma R_0)} F(x,t_0)\,dx - \int_{B_N(x_0,R_0)\setminus B_N(x_0,\sigma R_0)} |F(x,u_\sigma)|\,dx \\
&\geq \operatorname{ess\,inf}_{x\in B_N(x_0,R_0)} F(x,t_0)\sigma^N R_0^N w_N - \operatorname{ess\,sup}_{x\in B_N(x_0,R_0)} \max_{|t|\leq|t_0|} |F(x,t)|(1-\sigma^N)R_0^N w_N \\
&\geq \left[l_0\sigma^N - L_0(1-\sigma^N)\right] R_0^N w_N.
\end{aligned}$$

For σ close enough to 1, the right-hand side of the last inequality becomes strictly positive, let σ_0 be such a number. Then we have $\mathcal{F}(u_{\sigma_0}) > 0$.

Now, applying Lemma 5, we may choose $\rho_0 \in \left(0, \frac{m_0}{\alpha}\right)$ such that

$$\rho_0 < \frac{m_0}{\alpha}\|u_{\sigma_0}\|^{\alpha\varphi^0} \leq \mathcal{A}(u_{\sigma_0})$$

and

$$\begin{aligned}
\frac{\sup\{\mathcal{F}(u):\ \mathcal{A}(u) < \rho_0\}}{\rho_0} &< \frac{\left[l_0\sigma^N - L_0(1-\sigma^N)\right] R_0^N w_N}{2\mathcal{A}(u_{\sigma_0})} \\
&< \frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})}.
\end{aligned} \tag{31}$$

In Proposition 2 we choose $x_1 = u_{\sigma_0}$ and $x_0 = 0$ and observe that the hypotheses (i) and (ii) are satisfied. We define

$$\overline{a} := \frac{1+\rho_0}{\frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})} - \frac{\sup\{\mathcal{F}(u):\ \mathcal{A}(u)<\rho_0\}}{\rho_0}}. \tag{32}$$

Taking into account Lemmas 2 and 4, all the assumptions of Proposition 2 are verified. Thus, there exist an open interval $\Lambda \subset [0,\overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\mathcal{J}'_\lambda(u) = \mathcal{A}'(u) - \lambda\mathcal{F}'(u)$ admits at least three solutions in $W_0^1 L_\Phi(\Omega)$ having $W_0^1 L_\Phi(\Omega)$-norms less that μ. Theorem 1 is completely proved.

Acknowledgements. The author would like to thank the referees for their helpful comments and suggestions which improved the presentation of the original manuscript. This work is supported by Quang Binh University.

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