

## NEW SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL DIFFERENTIAL OPERATOR

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ABSTRACT. Using the Al-Oboudi differential operator  $\mathcal{D}_\lambda^{n,\alpha}$ , we define a new class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  in the open unit disk  $E = \{z \in \mathbb{C} : |z| < 1\}$ . The class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  generalizes the number of previously known classes. In this paper, we discuss some inclusion results, closure property. Results due to Al-Oboudi [1], Patel [21], Sharma [29] and Sokol [28] follows as special cases from our results.

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### 1. INTRODUCTION

Let  $\mathcal{H}$  denote the class of analytic functions in the open unit disk  $E = \{z : |z| < 1\}$ . Let  $\mathcal{A}$  denote the subclass of  $\mathcal{H}$  consisting of normalized analytic functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in E. \quad (1)$$

Let  $\mathcal{S}$ ,  $\mathcal{C}$  and  $\mathcal{S}^*$  denote the subclasses of  $\mathcal{A}$  consisting of functions that are univalent, convex and starlike in  $E$  respectively. Let  $\mathcal{P}(\beta)$  be the class of all functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots,$$

which are analytic in  $E$  with  $p(0) = 1$  and  $\Re(p(z)) > \beta, : 0 \leq \beta < 1$  and for all  $z \in E$ .

We say that  $f \in \mathcal{H}$  is subordinate to  $g \in \mathcal{H}$  in  $E$ , written as  $f \prec g$ , if there exists a Schwarz function  $\omega$  analytic in  $E$ , with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  for  $z \in E$ , such that  $f(z) = g(\omega(z)), z \in E$ . If  $g$  is univalent with  $f(0) = g(0)$ , then  $f(E) \subset g(E)$ , for details, see [9, 10, 16, 22].

For the functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \quad z \in E,$$

the convolution (Hadamard product) is defined as

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \quad z \in E. \quad (2)$$

The convolution has algebraic properties of ordinary multiplication.

The gamma function is denoted by  $\Gamma$  and is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx, \quad \Re(z) > 0. \quad (3)$$

For complex numbers  $a, c$  different from  $0, -1, -2, \dots$ , the incomplete beta function  $\phi(a, c; z)$  is defined as follows

$$\phi(a, c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(c)}{\Gamma(c+j)\Gamma(a)} z^j. \quad (4)$$

For complex numbers  $a, b, c$  different from  $0, -1, -2, \dots$ , the Gaussian Hypergeometric function  ${}_2F_1(a, b; c; z)$  is defined as follows

$${}_2F_1(a, b; c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)\Gamma(c)}{\Gamma(c+j)\Gamma(a)\Gamma(b)} \frac{z^j}{j!}, \quad z \in E.$$

Recently, the theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities and convolution structures for various subclasses of analytic functions.

The fractional derivative of order  $\alpha$ ,  $0 \leq \alpha < 1$  is defined in [20] as follows

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\alpha} dt, \quad 0 \leq \alpha < 1, \quad z \in E, \quad (5)$$

where the function  $f(z)$  is analytic in a simply connected domain of the complex plane containing the origin and the multiplicity of  $(z-t)^{-\alpha}$  is removed by requiring

$\log(z - t) \in \mathbb{R}$ , whenever  $(z - t) > 0$ .

Note that

$$D_z^0 f(z) = f(z).$$

In 1987, Owa and Srivastava [20] introduced the operator  $L_\alpha : \mathcal{A} \rightarrow \mathcal{A}$  as follows

$$\begin{aligned} L_\alpha f(z) &= \Gamma(2 - \alpha) z^\alpha D_z^\alpha f(z), \quad 0 \leq \alpha < 1, \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(1 + j) \Gamma(2 - \alpha)}{\Gamma(1 + j - \alpha)} a_j z^j, \quad z \in E, \\ &= \phi(2, 2 - \alpha; z) * f(z), \end{aligned} \tag{6}$$

which is known as the extension of the fractional derivative  $D_z^\alpha$  defined by (5).

It can be seen that

$$L_0 f(z) = f(z).$$

Mishra and Gochhayat [17] have studied the properties of the operator  $L_\alpha$  and introduced new class of  $k$ -uniformly convex functions. In a recent paper Ibrahim and Darus [12] introduced the fractional differential subordination based on the operator  $L_\alpha$ . Noor et al [18] used the operator  $L_\alpha$  to define the classes of  $k$ -uniformly star-like and  $k$ -uniformly close to convex functions and obtain some interesting results. Recently, Noor et al [19] used  $L_\alpha$  to define some new subclasses of analytic functions in the conic regions.

Al-Oboudi [2, 3] defined the linear fractional differential operator of order  $n$  as follows

$$\begin{aligned} (D_0^{0,0} f)(z) &= f(z) \\ (D_\lambda^{1,\alpha} f)(z) &= (1 - \lambda)L_\alpha f(z) + \lambda z (L_\alpha f(z))', \quad : 0 \leq \lambda \leq 1, \\ &: 0 \leq \alpha < 1, \\ &\dots \\ &\dots \\ (D_\lambda^{n,\alpha} f)(z) &= D \left( D_\lambda^{n-1,\alpha} f \right)(z), \end{aligned} \tag{7}$$

where  $L_\alpha$  is defined by (6).

Note that by using (1) and (3), we obtain

$$(D_\lambda^{n,\alpha} f)(z) = z + \sum_{j=2}^{\infty} \left[ \frac{\Gamma(j+1) \Gamma(2-\alpha)}{\Gamma(j+1-\alpha)} (1 + \lambda(j-1)) \right]^n a_j z^j. \tag{8}$$

If, we let

$$\sigma_j(\alpha, \lambda) = \frac{\Gamma(j+1) \Gamma(2-\alpha)}{\Gamma(j+1-\alpha)} (1 + \lambda(j-1)), \tag{9}$$

then

$$(D_\lambda^{n,\alpha} f)(z) = z + \sum_{j=2}^{\infty} [\sigma_j(\alpha, \lambda)]^n a_j z^j, : 0 \leq \lambda \leq 1, : 0 \leq \alpha < 1, z \in E. \quad (10)$$

The operator defined by (8) was extensively studied by Al-Oboudi [2, 3] and Bulut [5, 6, 7, 8].

For different choices of the parameters  $n, \alpha$  and  $\lambda$ , we obtain some well known operators.

If  $\alpha = 0$ , then

$$(D_\lambda^{n,0} f)(z) = z + \sum_{j=2}^{\infty} [(1 + \lambda(j-1))]^n a_j z^j,$$

which is the generalized Salagean operator defined by Al-Oboudi [1].

If  $\alpha = 0, \lambda = 1$ , then

$$(D_\lambda^{n,0} f)(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j,$$

which is the Salagean differential operator defined by Salagean [26]

For  $n = 1$  and  $\lambda = 0$ , we obtain the operator  $L_\alpha$  defined by (6).

Throughout this article we assume that  $h$  is convex univalent in  $E$  with  $h(0) = 1$  and  $\Re h(z) > 0, z \in E$ . In this paper, we obtain some inclusion results, closure property. Results due to Al-Oboudi [1], Patel [21], Sharma [29] and Sokol [28] follows as special cases from our results. Now using the operator  $D_\lambda^{n,\alpha}$ , we define the class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  as follows

**Definition 1.** A function  $f \in \mathcal{A}$  is said to be in the class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$ , if it satisfies the following condition

$$(D_\lambda^{n,\alpha} f)'(z) + \delta z (D_\lambda^{n,\alpha} f)''(z) \prec h(z),$$

for  $\delta \geq 0, : 0 \leq \lambda \leq 1, : 0 \leq \alpha < 1$  and for all  $z \in E$ .

We note that for different choices of convex univalent function  $h(z)$ , we obtain some well known subclasses of the class  $\mathcal{A}$ .

If  $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$  and  $\alpha = 0$ , we obtain

$$\mathcal{R}_\lambda^{n,\alpha} \left( \delta, \frac{1+Az}{1+Bz} \right) = \mathcal{R}_\lambda^{n,\alpha}(\delta, A, B),$$

which was introduced and study by Patel [21].

For  $\alpha = 0, : \delta = 0$  and  $h(z) = \frac{1-(2\beta-1)z}{1-z}, : 0 \leq \beta < 1$ , we have

$$\mathcal{R}_\lambda^{n,\alpha} \left( 0, \frac{1-(2\beta-1)z}{1-z} \right) = \mathcal{R}^n(\lambda, \beta),$$

which was introduced and study by Al-Oboudi [1].

Also

$$\mathcal{R}^1(\lambda, \beta) = \left\{ f \in \mathcal{A} : f'(z) + \lambda z f''(z) \prec \frac{1 - (2\beta - 1)z}{1 - z} \text{ in } E, \text{ see [23]} \right\}$$

and

$$\mathcal{R}_0^{n,0} \left( 0, \frac{1 - (2\beta - 1)z}{1 - z} \right) = \mathcal{R}^n(0, \beta) = \mathcal{R}^0(\lambda, \beta) = \mathcal{R}(\beta), \text{ see [31].}$$

The class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  generalizes the number of well known functions classes, see [1, 14, 15, 21, 23, 31].

## 2. MAIN RESULTS

To prove our main results we need the following Lemmas.

**Lemma 1.** [30]. *If  $p(z)$  is analytic in  $E$  with  $p(0) = 1$  and  $\Re(p(z)) > \frac{1}{2}$  ( $z \in E$ ) then for any function  $F$ , analytic in  $E$ , the function  $(p * F)(z)$  takes its values in the convex hull of  $F(E)$ . That is  $(p * F)(E) \subset F(E)$  in  $E$ .*

A sequence  $a_0, a_1, a_2, \dots, a_n, \dots$ , of non-negative numbers is called a convex null sequence of  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  and

$$a_0 - a_1 \geq a_1 - a_2 \geq \dots a_n - a_{n+1} \geq 0.$$

**Lemma 2.** [21]. *Let  $\{c_j\}_0^\infty$  be a convex null sequence. Then the function*

$$p(z) = \frac{c_0}{2} + \sum_{j=1}^{\infty} c_j z^j, \quad z \in E,$$

*is analytic in  $E$  and  $\Re(p(z)) > 0$  in  $E$ .*

**Lemma 3.** [16]. For complex numbers  $a, b, c$  different from  $0, -1, -2, \dots$ , we have

$$(i) \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z),$$

$$: \Re(c) > \Re(b) > 0,$$

$$(ii) \quad (a+1) {}_2F_1(1, a; a+1; z) = (a+1) + az {}_2F_1(1, a+1; a+2; z)$$

$$(iii) \quad {}_2F_1(a, b; c; z) = {}_2F_1\left(a, c-b; c; \frac{z}{1-z}\right)$$

$$(iv) \quad {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z).$$

**Lemma 4.** [11]. Let  $h$  be a convex univalent in  $E$  with  $h(0) = 1$ . Suppose that the function  $p$  given by

$$p(z) = 1 + c_1z + c_2z^2 + \dots, \tag{11}$$

is analytic in  $E$  with  $p(0) = 1$ .

If

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \Re(\gamma) \geq 0, \gamma \neq 0 \text{ and } z \in E,$$

then

$$p(z) \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z), z \in E,$$

and  $q(z)$  is the best dominant of  $h(z)$ .

**Lemma 5.** [21]. Let  $p$  defined by (11) be in the class  $P(\beta)$ .

Then

$$\Re(p(z)) \geq (2\beta - 1) + \frac{2(1-\beta)}{1+|z|}, \quad 0 \leq \beta < 1, z \in E.$$

The following Lemma is the generalization of a well known result due to Stankiewicz and Stankiewicz [31].

**Lemma 6.** [29]. If  $\beta_i \leq 1$  ( $i = 1, 2, 3, \dots, m$ ), then

$$P(\beta_1) * P(\beta_2) * \dots * P(\beta_m) = P(\eta),$$

where

$$\eta = 1 - 2^{m-1} (1 - \beta_1) (1 - \beta_2) \dots (1 - \beta_m). \tag{12}$$

Now we prove our main results.

**Theorem 7.** For  $h(z) = \frac{1+Az}{1+Bz}$ ,  $-1 \leq B < A \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 \leq \alpha < 1$ ,  $\delta = 0$  and for all  $z \in E$

$$\mathcal{R}_\lambda^{n+1,\alpha}(A, B) \subset \mathcal{R}_\lambda^{n,\alpha}(A, B).$$

*Proof.* Let  $f \in \mathcal{R}_\lambda^{n+1,\alpha}(A, B)$ . Then

$$\left( D_\lambda^{n+1,\alpha} f \right)'(z) \prec \frac{1+Az}{1+Bz} \text{ in } E.$$

That is

$$\Re \left[ \left( D_\lambda^{n+1,\alpha} f \right)'(z) \right] > \frac{1-A}{1-B}, \quad z \in E. \quad (13)$$

Note that by (10), we have

$$\left( D_\lambda^{n+1,\alpha} f \right)'(z) = 1 + \sum_{j=2}^{\infty} [\sigma_j(\alpha, \lambda)]^{n+1} j a_j z^{j-1}. \quad (14)$$

Now by using (13) and (14), we obtain

$$\Re \left[ 1 + \sum_{j=2}^{\infty} \{ \sigma_j(\alpha, \lambda) \}^{n+1} j a_j z^{j-1} \right] > \frac{1-A}{1-B}. \quad (15)$$

This implies that

$$\Re \left[ 1 + \frac{1-B}{2(A-B)} \sum_{j=2}^{\infty} j \{ \sigma_j(\alpha, \lambda) \}^{n+1} a_j z^{j-1} \right] > \frac{1}{2}, \quad z \in E. \quad (16)$$

Now

$$\begin{aligned} \left( D_\lambda^{n,\alpha} f \right)'(z) &= 1 + \sum_{j=2}^{\infty} [\sigma_j(\alpha, \lambda)]^n j a_j z^{j-1} \\ &= \left[ 1 + \frac{1-B}{2(A-B)} \sum_{j=2}^{\infty} j \{ \sigma_j(\alpha, \lambda) \}^{n+1} a_j z^{j-1} \right] \\ &\quad * \left[ 1 + \frac{2(A-B)}{1-B} \sum_{j=2}^{\infty} \frac{z^{j-1}}{(\sigma_j(\alpha, \lambda))} \right]. \end{aligned} \quad (17)$$

It is known [27] that the function

$$\sigma_j(\alpha) = \frac{\Gamma(j+1)\Gamma(2-\alpha)}{\Gamma(j+1-\alpha)},$$

is decreasing function of  $j$  and  $0 < \sigma_j(\alpha) \leq \sigma_2(\alpha) = \frac{1}{2-\alpha}$ . Also it is proved in [1] that the function

$$d_j = \frac{1}{1+\lambda_j}, \quad j = 1, 2, 3, \dots,$$

is a convex null sequence. This means that the sequence

$$\begin{aligned} c_{j-1} &= \left[ \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)\Gamma(2-\alpha)} \right] \left[ \frac{1}{(1+\lambda(j-1))} \right] \\ &= \frac{1}{\sigma_j(\alpha, \lambda)}, \quad j = 1, 2, 3, \dots, \end{aligned}$$

with  $c_0 = 1$ , is a convex null sequence. Therefore using Lemma 2.2, we have

$$\Re \left[ 1 + \frac{2(A-B)}{1-B} \sum_{j=2}^{\infty} c_{j-1} z^{j-1} \right] > \frac{1-A}{1-B}, \quad z \in E. \quad (18)$$

Now using Lemma 2.1 with (16) and (18), we obtain

$$\Re \left[ (D_{\lambda}^{n,\alpha} f)'(z) \right] > \frac{1-A}{1-B}, \quad z \in E.$$

This completes the proof.

**Remark 1.** For  $\alpha = 0$ ,  $\delta = 0$  and  $h(z) = \frac{1-(2\beta-1)z}{1-z}$ ,  $0 \leq \beta < 1$ , we obtain the following result due to [1].

**Corollary 8.** For  $0 \leq \lambda \leq 1$  and  $0 \leq \beta < 1$ ,

$$\mathcal{R}^{n+1}(\lambda, \beta) \subset \mathcal{R}^n(\lambda, \beta).$$

From Theorem 2.7, we have the following result.

**Theorem 9.** For  $h(z) = \frac{1+Az}{1+Bz}$ ,  $-1 \leq B < A \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 \leq \alpha < 1$ ,  $\delta = 0$  and for all  $z \in E$

$$\mathcal{R}_{\lambda}^{n+1,\alpha}(A, B) \subset \mathcal{R}_{\lambda}^{n,\alpha}(1-2\rho, -1),$$



where

$$\rho = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right) (1 - B)^{-1} {}_2F_1\left(1, \frac{B-A}{B}; 1 + \frac{1}{\sigma_j(\alpha, \lambda)}; \frac{B}{B-1}\right), & \text{if } B \neq 0, j = 2, 3, 4, \dots \\ 1 + \frac{A}{1 + \sigma_j(\alpha, \lambda)}, & \text{if } B = 0, j = 2, 3, 4, \dots \end{cases}$$

This result is sharp.

*Proof.* Let

$$p(z) = (D_\lambda^{n, \alpha} f)'(z), \tag{19}$$

where  $p(z)$  is analytic in  $E$  with  $p(0) = 1$ .

Note that by (10), we have

$$z (D_\lambda^{n, \alpha} f)'(z) = \frac{1}{\sigma_j(\alpha, \lambda)} (D_\lambda^{n+1, \alpha} f)(z) + \left(1 - \frac{1}{\sigma_j(\alpha, \lambda)}\right) (D_\lambda^{n, \alpha} f)(z). \tag{20}$$

Differentiating (19) and using (20), we obtain

$$p(z) + \frac{zp'(z)}{\frac{1}{\sigma_j(\alpha, \lambda)}} = (D_\lambda^{n+1, \alpha} f)'(z).$$

Since  $f \in \mathcal{R}_\lambda^{n+1, \alpha}(A, B)$ , this means that

$$p(z) + \frac{zp'(z)}{\frac{1}{\sigma_j(\alpha, \lambda)}} \prec \frac{1 + Az}{1 + Bz} \text{ in } E.$$

Now by using Lemma 2.4 with  $\gamma = \frac{1}{\sigma_j(\alpha, \lambda)} \neq 0$  and  $\Re(\gamma) \geq 0$ , we have

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{\sigma_j(\alpha, \lambda)} z^{-\frac{1}{\sigma_j(\alpha, \lambda)}} \int_0^z t^{\frac{1}{\sigma_j(\alpha, \lambda)}-1} \left(\frac{1 + At}{1 + Bt}\right) dt \\ &= \frac{1}{\sigma_j(\alpha, \lambda)} z^{-\frac{1}{\sigma_j(\alpha, \lambda)}} \int_0^z t^{\frac{1}{\sigma_j(\alpha, \lambda)}-1} (1 + At)(1 + Bt)^{-1} dt. \end{aligned} \tag{21}$$

If  $B = 0$ , then

$$\begin{aligned} q(z) &= \frac{1}{\sigma_j(\alpha, \lambda)} z^{-\frac{1}{\sigma_j(\alpha, \lambda)}} \int_0^z t^{\frac{1}{\sigma_j(\alpha, \lambda)}-1} (1 + At) dt \\ &= \frac{1}{\sigma_j(\alpha, \lambda)} z^{-\frac{1}{\sigma_j(\alpha, \lambda)}} \left[ \frac{z^{\frac{1}{\sigma_j(\alpha, \lambda)}}}{\frac{1}{\sigma_j(\alpha, \lambda)}} + A \frac{z^{\frac{1}{\sigma_j(\alpha, \lambda)}+1}}{\frac{1}{\sigma_j(\alpha, \lambda)}+1} \right]. \end{aligned}$$

Therefore

$$q(z) = 1 + \frac{A}{1 + \sigma_j(\alpha, \lambda)}, \text{ if } B = 0, j = 2, 3, 4, \dots$$

Now, if  $B \neq 0$ , then from (21), we have

$$\begin{aligned} q(z) &= \frac{1}{\sigma_j(\alpha, \lambda)} \int_0^1 s^{\frac{1}{\sigma_j(\alpha, \lambda)} - 1} \left( \frac{1 + As}{1 + Bs} \right) ds \\ &= \frac{1}{\sigma_j(\alpha, \lambda)} \int_0^1 s^{\frac{1}{\sigma_j(\alpha, \lambda)} - 1} (1 + Bs)^{-1} ds \\ &\quad + \frac{Az}{\sigma_j(\alpha, \lambda)} \int_0^1 s^{\frac{1}{\sigma_j(\alpha, \lambda)} - 1} (1 + Bs)^{-1} ds. \end{aligned}$$

Now using (i), (ii) and (iii) of Lemma 2.3 and after some simplification, we obtain

$$\begin{aligned} q(z) &= {}_2F_1 \left( 1, \frac{1}{\sigma_j(\alpha, \lambda)}; \frac{1}{\sigma_j(\alpha, \lambda)} + 1; -Bz \right) \\ &\quad + \frac{Az}{\sigma_j(\alpha, \lambda)} \frac{1}{\sigma_j(\alpha, \lambda) + 1} {}_2F_1 \left( 1, \frac{1}{\sigma_j(\alpha, \lambda)} + 1; \frac{1}{\sigma_j(\alpha, \lambda)} + 2; -Bz \right). \end{aligned}$$

This implies that

$$q(z) = \frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 - B)^{-1} {}_2F_1 \left( 1, 1; \frac{1}{\sigma_j(\alpha, \lambda)} + 1; \frac{Bz}{Bz + 1} \right).$$

To prove  $q(z)$  is best dominant, it is sufficient to show that

$$\inf \{ \Re(q(z)) \} = q(-1).$$

It is known [16] that for  $|z| \leq r < 1$ ,  $\Re \left( \frac{1+Az}{1+Bz} \right) \geq \frac{1-Ar}{1-Br}$ .

Let

$$G(t, z) = \frac{1 + Atz}{1 + Btz}, \quad d\mu(t) = \frac{1}{\sigma_j(\alpha, \lambda)} t^{\frac{1}{\sigma_j(\alpha, \lambda)} - 1} dt,$$

which is positive measure on  $[0, 1]$ .

Therefore

$$q(z) = \int_0^1 G(t, z) d\mu(t)$$

and

$$\Re(q(z)) \geq \int_0^1 \frac{1 - Atr}{1 - Btr} d\mu(t) = q(-r), \quad |z| \leq r < 1.$$

Letting  $r \rightarrow -1$ , we obtain  $\inf \{\Re(q(z))\} = q(-1)$ . This shows that the result is best possible because  $q$  is the best dominant.

**Remark 2.** For  $\alpha = 0$ , we obtain a known result in [21].

Our next result is the generalization of the result given in [29].

**Theorem 10.** Let for each  $i = 1, 2, 3, \dots, m$ ,  $f_i \in \mathcal{A}$  and  $\delta = 0$ . If  $f_i \in \mathcal{R}_\lambda^{n,\alpha}(A_i, B_i)$  for each  $i = 1, 2, 3, \dots, m$  and

$$\Phi(z) = [D_\lambda^{n,\alpha}(f_1 * f_2 * \dots * f_m)](z),$$

then

$$\Phi'(z) \prec h(z) \text{ in } E,$$

where

$$h(z) = 1 + \sum_{j=1}^{\infty} \left[ \prod_{i=1}^m \left\{ (-1)^j (B_i^j - A_i B_i^{j-1}) \right\} \right] \frac{z^j}{j+1}, \quad (22)$$

is convex univalent in  $E$ .

*Proof.* Let  $f_i \in \mathcal{R}_\lambda^{n,\alpha}(A_i, B_i)$  for each  $i = 1, 2, 3, \dots, m$ .

Then

$$(D_\lambda^{n,\alpha} f_i)'(z) \prec \frac{1 + A_i z}{1 + B_i z} = q_i(z) \text{ in } E. \quad (23)$$

Note that by using Lemma 2.6, we have

$$(D_\lambda^{n,\alpha} f_1)'(z) * (D_\lambda^{n,\alpha} f_2)'(z) * \dots * (D_\lambda^{n,\alpha} f_m)'(z) \prec q_1(z) * q_2(z) * \dots * q_m(z). \quad (24)$$

Now

$$q_1(z) * q_2(z) * \dots * q_m(z) = 1 + \sum_{j=1}^{\infty} \left[ \prod_{i=1}^m \left\{ (-1)^j (B_i^j - A_i B_i^{j-1}) \right\} \right] z^j.$$

We note that the function

$$h_1(z) = \frac{-2}{z} [z + \log(1-z)] = \sum_{j=1}^{\infty} \left( \frac{2}{j+1} \right) z^j,$$

belongs to the class  $\mathcal{C}$  and for  $f \in \mathcal{A}$

$$(f * h_1)(z) = \frac{2}{z} \int_0^z f(t) dt.$$

Therefore the function

$$h_2(z) = 1 + h_1(z), \quad z \in E,$$

being the translation of  $h_1$  is convex univalent in  $E$  and for  $p \in \mathcal{P}$

$$(p * h_2)(z) = -1 + \frac{2}{z} \int_0^z p(t) dt. \quad (25)$$

Now applying a result of Stankiewicz and Stankiewicz [31], to (24),  $(m-1)$  times with  $h_2 \prec h_2$  in  $E$ , we obtain

$$\begin{aligned} & (D_\lambda^{n,\alpha} f_1)' * (D_\lambda^{n,\alpha} f_2)' * \cdots * (D_\lambda^{n,\alpha} f_m)' * \underbrace{h_2 * h_2 * \cdots * h_2}_{(m-1) \text{ times}} \\ \prec & q_1 * q_2 * \cdots * q_m * \underbrace{h_2 * h_2 * \cdots * h_2}_{(m-1) \text{ times}}. \end{aligned}$$

This implies that

$$\begin{aligned} & \left[ (D_\lambda^{n,\alpha} f_1)' * h_2 \right] * \left[ (D_\lambda^{n,\alpha} f_2)' * h_2 \right] * \cdots * \left[ (D_\lambda^{n,\alpha} f_{m-1})' * h_2 \right] \\ & \quad * (D_\lambda^{n,\alpha} f_m)' \\ \prec & (q_1 * h_2) * (q_2 * h_2) * \cdots * (q_{m-1} * h_2) * q_m. \end{aligned} \quad (26)$$

Now using the series expansion of  $D_\lambda^{n,\alpha} f_i$ 's and  $q_i$ 's in view of (25), we can write (26) as follows

$$\begin{aligned} & \left( \frac{D_\lambda^{n,\alpha} f_1}{z} \right) * \left( \frac{D_\lambda^{n,\alpha} f_2}{z} \right) * \cdots * \left( \frac{D_\lambda^{n,\alpha} f_{m-1}}{z} \right) * (D_\lambda^{n,\alpha} f_m)' \\ \prec & \frac{1}{z} \int_0^z q_1(t) dt * \frac{1}{z} \int_0^z q_2(t) dt * \cdots * \frac{1}{z} \int_0^z q_{m-1}(t) dt * q_m(z) \\ = & 1 + \sum_{j=1}^{\infty} \left[ \prod_{i=1}^m \left\{ (-1)^j (B_i^j - A_i B_i^{j-1}) \right\} \right] \frac{z^j}{j+1} \\ = & h(z), \quad z \in E. \end{aligned} \quad (27)$$

The left hand side of (27) is

$$\begin{aligned} (D_\lambda^{n,\alpha} f_1 * D_\lambda^{n,\alpha} f_2 * \cdots * D_\lambda^{n,\alpha} f_m)'(z) &= [D_\lambda^{n,\alpha} (f_1 * f_2 * \cdots * f_m)]'(z) \\ &= \Phi'(z), \quad z \in E. \end{aligned} \quad (28)$$

Note that by using (27) and (28), we obtain  $\Phi'(z) \prec h(z)$  in  $E$ .

**Remark 3.** If we replace  $B_i = -1$  and  $A_i = 1 - 2\beta_i$  for each  $i = 1, 2, 3, \dots, m$ ,  $: 0 \leq \beta_i < 1$ , then  $h(z)$  defined by (22) can be written as

$$h(z) = 1 + 2^m (1 - \beta_1)(1 - \beta_2) \cdots (1 - \beta_{m-1}) \sum_{j=1}^{\infty} \frac{z^j}{(j+1)^m}. \quad (29)$$

For  $n = 0, : \alpha = 0, : \lambda = 0, : \delta = 0, B_i = -1, A_i = 1 - 2\beta_i, : 0 \leq \beta_i < 1$ , and  $h(z)$  defined by (29), we have the following known result in [29].

**Corollary 11.** Let for each  $i = 1, 2, 3, \dots, m, f_i \in \mathcal{A}$  and  $0 \leq \beta_i < 1$ . If  $f_i' \in \mathcal{P}(\beta_i)$  for each  $i = 1, 2, 3, \dots, m$  and

$$\Phi(z) = (f_1 * f_2 * \cdots * f_m)(z),$$

then

$$\Phi'(z) \prec h(z) \quad \text{in } E.$$

As  $h(z)$  defined by (22) is convex univalent function with real coefficient therefore we may easily get the following result from Theorem 2.12.

**Corollary 12.** Let for each  $i = 1, 2, 3, \dots, m, f_i \in \mathcal{A}$  and  $\delta = 0$ . If  $f_i \in \mathcal{R}_\lambda^{n,\alpha}(A_i, B_i)$  for each  $i = 1, 2, 3, \dots, m$  and

$$\Phi(z) = [D_\lambda^{n,\alpha} (f_1 * f_2 * \cdots * f_m)](z),$$

then

$$h(-1) \leq \Re(\Phi'(z)) \leq h(1),$$

where  $h(z)$  is defined by (22).

**Remark 4.** Note that from (22), we have

$$h(-1) = 1 + \sum_{j=1}^{\infty} \left[ \frac{\prod_{i=1}^m (B_i^j - A_i B_i^{j-1})}{(j+i)^m} \right]$$

and

$$h(1) = \sum_{j=1}^{\infty} \left[ \frac{\prod_{i=1}^m \left\{ (-1)^j (B_i^j - A_i B_i^{j-1}) \right\}}{(j+i)^m} \right].$$

**Corollary 13.** Let for each  $i = 1, 2, 3, \dots, m$ ,  $f_i \in \mathcal{A}$  and  $\delta = 0$ . If  $f_i \in \mathcal{R}_\lambda^{n,\alpha}(A_i, B_i)$  for each  $i = 1, 2, 3, \dots, m$  and

$$\Phi(z) = [D_\lambda^{n,\alpha}(f_1 * f_2 * \dots * f_m)](z),$$

then

$$\Re(\Phi'(z)) \geq 1 + \sum_{j=1}^{\infty} \left[ \prod_{i=1}^m (B_i^j - A_i B_i^{j-1}) \right] [(1 - 2^{2-m}) \zeta(m-1) - 1]$$

and

$$\Re(\Phi'(z)) \leq 1 + \sum_{j=1}^{\infty} \left[ \prod_{i=1}^m (B_i^j - A_i B_i^{j-1}) \right] [\zeta(m-1) - 1],$$

where  $\zeta$  is the well known zeta function, see [16].

**Remark 5.** Taking  $n = 0, \alpha = 0, \lambda = 0, \delta = 0, B_i = -1, A_i = 1 - 2\beta_i, 0 \leq \beta_i < 1$  in Corollary 2.15 and Corollary 2.17, we obtain some well known results due to [29].

For  $n = 0, m = 3, \alpha = 0, \lambda = 0, \delta = 0, B_i = -1, A_i = 1 - 2\beta_i, 0 \leq \beta_i < 1$  ( $i = 1, 2, 3$ ) and writing  $\zeta(2) = \frac{\pi^2}{6}$ , in Corollary 2.15 we obtain a result in [28].

**Theorem 14.** Let  $-1 \leq B_i < A_i \leq 1, i = 1, 2, 3, \dots, m$ . If  $f_i \in \mathcal{R}_\lambda^{n,\alpha}(\delta, A_i, B_i), \delta \geq 0$ , then the function  $g \in \mathcal{A}$  defined by

$$(D_\lambda^{n,\alpha} g)(z) = \int_0^z \left[ (D_\lambda^{n,\alpha} f_1)' * (D_\lambda^{n,\alpha} f_2)' * \dots * (D_\lambda^{n,\alpha} f_m)' \right] (t) dt, \quad (30)$$

belongs to the class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, 1 - 2\chi, -1)$ , where

$$\chi = \begin{cases} 1 - 2^{m-1} \left[ \frac{(A_1 - B_1)(A_2 - B_2) \dots (A_m - B_m)}{(1 - B_1)(1 - B_2) \dots (1 - B_m)} \right] \\ \quad \left[ 1 - \frac{1}{2} {}_2F_1 \left( 1, 1; 1 + \frac{1}{\delta}; \frac{1}{2} \right) \right], & \delta > 0 \\ 1 - 2^{m-1} \left[ \frac{(A_1 - B_1)(A_2 - B_2) \dots (A_m - B_m)}{(1 - B_1)(1 - B_2) \dots (1 - B_m)} \right], & \delta = 0. \end{cases}$$

*Proof.* Let  $\delta > 0$  and let

$$h_i(z) = (D_\lambda^{n,\alpha} f_i)'(z) + \delta z (D_\lambda^{n,\alpha} f_i)''(z) \prec \frac{1 + A_i z}{1 + B_i z}, \quad (31)$$

for all  $i = 1, 2, 3, \dots, m$  and  $z \in E$ .

That is

$$\Re(h_i(z)) > \frac{1 - A_i}{1 - B_i} = \gamma_i, \quad i = 1, 2, 3, \dots, m. \quad (32)$$

This implies that

$$h_i \in \mathcal{P}(\gamma_i) \text{ for all } i = 1, 2, 3, \dots, m.$$

Now using Lemma 2.6, we have

$$(h_1 * h_2 * \dots * h_m) \in \mathcal{P}(\xi),$$

where

$$\xi = 1 - 2^{m-1} (1 - \gamma_1) (1 - \gamma_2) \dots (1 - \gamma_m).$$

By using (32), we obtain

$$\xi = 1 - 2^{m-1} \left[ \frac{(A_1 - B_1)(A_2 - B_2) \dots (A_m - B_m)}{(1 - B_1)(1 - B_2) \dots (1 - B_m)} \right]. \quad (33)$$

Note that from (30), we can write

$$(D_\lambda^{n,\alpha} g)'(z) = \left[ (D_\lambda^{n,\alpha} f_1)' * (D_\lambda^{n,\alpha} f_2)' * \dots * (D_\lambda^{n,\alpha} f_m)' \right](z).$$

Also by using (31), we have

$$\begin{aligned} (D_\lambda^{n,\alpha} g)'(z) &= \left[ \frac{1}{\delta} z^{-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-1} h_1(t) dt \right] * \left[ \frac{1}{\delta} z^{-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-1} h_2(t) dt \right] \\ &\quad * \dots * \left[ \frac{1}{\delta} z^{-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-1} h_m(t) dt \right] \\ &= \frac{1}{\delta} z^{-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-1} (h_1 * h_2 * \dots * h_m)(t) dt \\ &= \frac{1}{\delta} z^{-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-1} h_0(t) dt. \end{aligned}$$

This means that

$$(D_\lambda^{n,\alpha} g)'(z) = \frac{1}{\delta} \int_0^1 u^{\frac{1}{\delta}-1} h_0(uz) du,$$

where

$$\begin{aligned} h_0(z) &= (D_\lambda^{n,\alpha} g)'(z) + \delta z (D_\lambda^{n,\alpha} g)''(z) \\ &= \frac{1}{\delta} \int_0^1 t^{\frac{1}{\delta}-1} (h_1 * h_2 * \dots * h_m)(t) dt. \end{aligned} \quad (34)$$

Now by using Lemma 2.5 with (34), we obtain

$$\begin{aligned} \Re(h_0(z)) &\geq \frac{1}{\delta} \int_0^1 t^{\frac{1}{\delta}-1} \left[ (2\xi - 1) + \frac{2(1-\xi)}{1+u|z|} \right] du \\ &> \frac{1}{\delta} \int_0^1 t^{\frac{1}{\delta}-1} \left[ (2\xi - 1) + \frac{2(1-\xi)}{1+u} \right] du. \end{aligned}$$

Using (33) and after some simplifications, we have

$$\begin{aligned} \Re(h_0(z)) &\geq 1 - 2^{m-1} \left[ \frac{(A_1 - B_1)(A_2 - B_2) \dots (A_m - B_m)}{(1 - B_1)(1 - B_2) \dots (1 - B_m)} \right] \\ &\quad \left[ 1 - \frac{1}{\delta} \int_0^1 \frac{t^{\frac{1}{\delta}-1}}{1+u} du \right] \\ &= 1 - 2^{m-1} \left[ \frac{(A_1 - B_1)(A_2 - B_2) \dots (A_m - B_m)}{(1 - B_1)(1 - B_2) \dots (1 - B_m)} \right] \\ &\quad \left[ 1 - \frac{1}{2} {}_2F_1 \left( 1, 1; 1 + \frac{1}{\delta}; z \right) \right] \\ &= \chi, \quad z \in E. \end{aligned}$$

This shows that  $g \in \mathcal{R}_\lambda^{n,\alpha}(\delta, 1 - 2\chi, -1)$ . For the case  $\delta = 0$ , the proof is simple so we omit the details.

Now we prove that the class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  is closed under convex convolution.

**Theorem 15.** *The class  $\mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  is closed under convex convolution. That is, if  $f \in \mathcal{R}_\lambda^{n,\alpha}(\delta, h)$  and  $\Psi \in \mathcal{C}$ , then  $(f * \Psi) \in \mathcal{R}_\lambda^{n,\alpha}(\delta, h)$ .*



*Proof.* Let  $\Psi \in \mathcal{C}$ . Then it is well known [16] that  $\Re \left[ \frac{\Psi(z)}{z} \right] > \frac{1}{2}$  in  $E$ .

Let

$$M(z) = (D_\lambda^{n,\alpha} f)'(z) + \delta z (D_\lambda^{n,\alpha} f)''(z)$$

and

$$N(z) = \frac{\Psi(z)}{z},$$

where both  $M$  and  $N$  are analytic in  $E$  with  $M(0) = N(0) = 1$ .

Now consider

$$\begin{aligned} & (1 - \delta) [D_\lambda^{n,\alpha} (f * \Psi)]'(z) + \delta [z (D_\lambda^{n,\alpha} (f * \Psi))']'(z) \\ &= (D_\lambda^{n,\alpha} (f * \Psi))'(z) + \delta z (D_\lambda^{n,\alpha} (f * \Psi))''(z) \\ &= \left[ (D_\lambda^{n,\alpha} f)'(z) * \frac{\Psi(z)}{z} \right] + \delta z \left[ (D_\lambda^{n,\alpha} f)'(z) * \frac{\Psi(z)}{z} \right]' \\ &= \left[ (D_\lambda^{n,\alpha} f)'(z) * N(z) \right] + \delta [z (D_\lambda^{n,\alpha} f)''(z) * N(z)] \\ &= \left[ (D_\lambda^{n,\alpha} f)'(z) + \delta z (D_\lambda^{n,\alpha} f)''(z) \right] * N(z). \end{aligned}$$

This implies that

$$(D_\lambda^{n,\alpha} (f * \Psi))'(z) + \delta z (D_\lambda^{n,\alpha} (f * \Psi))''(z) = (M * N)(z).$$

Since

$$M(z) = (D_\lambda^{n,\alpha} f)'(z) + \delta z (D_\lambda^{n,\alpha} f)''(z) \prec h(z) \text{ in } E$$

and

$$\Re(N(z)) = \Re \left[ \frac{\Psi(z)}{z} \right] > \frac{1}{2}, \quad z \in E.$$

Therefore by using Lemma 2.1, we obtain

$$(M * N)(E) \subset M(E).$$

Hence

$$(f * \Psi) \in \mathcal{R}_\lambda^{n,\alpha}(\delta, h), \quad z \in E.$$

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