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SUBORDINATION RESULTS FOR CLASSES OF MULTIVALENT NON-BAZILEVIC ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. By making use of the principle of subordination between analytic functions, we introduce non-Bazlevic classes of multivalent functions defined by linear operator. Various results as subordination properties, superordination properties, sandwich type results and distortion theorems are obtained.

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1. Introduction

Let $H[a,k]$ be the class of analytic functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ... \qquad (z \in \mathbb{U}),$$

and $\mathbb{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and $p-$valent in $\mathbb{U} = \{z : |z| < 1\}$.

Let M be the class of functions $\Phi(z)$ which are analytic and univalent in $\mathbb{U}$ and for which $\Phi(\mathbb{U})$ is convex with $\Phi(0) = 1$ and $\Re\{\ \Phi(z)\} > 0$.

For $m \in \mathbb{N}_0$, $\zeta \geq 0, \mu$ and $\eta \in \mathbb{R}$; $\mu < p+1$; $-\infty < \lambda < \eta + p + 1$; $\delta > -p$. Aouf et al. [2] defined the operator $N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta} : \mathbb{A}(p) \to \mathbb{A}(p)$ as follows:

$$\begin{aligned} \mathbb{N}^{1,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) &= \mathbb{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) \\ &= (1-\zeta)\, \mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z) + \zeta \frac{z}{p} \left[\mathbb{H}^{\lambda,\delta}_{p,\eta,\mu} f(z)\right]' \\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\zeta n}{p}\right) \frac{(\delta+p)_n (1+p-\mu)_n (1+p+\eta-\lambda)_n}{(1)_n (1+p)_n (1+p+\eta-\mu)_n} a_{p+n} z^{p+n}, \end{aligned}$$

$$\begin{aligned}\mathbb{N}^{2,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) &= (1-\zeta)\,\mathbb{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)+\zeta\frac{z}{p}\left[\mathbb{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right]' \\ &= z^p+\sum_{n=1}^{\infty}\left(\frac{p+\zeta n}{p}\right)^2\frac{(\delta+p)_n(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}a_{p+n}z^{p+n},\end{aligned}$$

and, in general

$$\begin{aligned}\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) &= \mathbb{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta}\left(N^{m-1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right) \\ &= z^p+\sum_{n=1}^{\infty}\left(\tfrac{p+\zeta n}{p}\right)^m\tfrac{(\delta+p)_n(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}a_{p+n}z^{p+n}. \qquad (2)\end{aligned}$$

From (1.2), we can easily obtain the following identities:

$$\zeta z(\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z))'=p\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)-p\,(1-\zeta)\,\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\quad(\zeta>0)\,, \qquad (3)$$

$$z(\mathbb{N}^{m,\delta,\zeta}_{p,\lambda+1,\mu,\eta}f(z))'=(p+\eta-\lambda)\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)-(\eta-\lambda)\mathbb{N}^{m,\delta,\zeta}_{p,\lambda+1,\mu,\eta}f(z) \qquad (4)$$

and

$$z(\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z))'=(p+\delta)\mathbb{N}^{m,\delta+1,\zeta}_{p,\lambda,\mu,\eta}f(z)-\delta\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z). \qquad (5)$$

Using the operator $\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)$ and for $\rho\in\mathbb{C}$, $-1\le B<A\le 1$, let:

(i) $R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)=$

$$\left\{f\in\mathbb{A}(p):\ \begin{array}{l}(1+\rho)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}-\rho\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}=\chi(z)\\ \prec\frac{1+Az}{1+Bz},\end{array}\right\}, \qquad (6)$$

(ii) $T^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)=$

$$\left\{f\in\mathbb{A}(p):\ \begin{array}{l}(1+\rho)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda+1,\mu,\eta}f(z)}\right)^{\alpha}-\rho\left(\frac{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda+1,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda+1,\mu,\eta}f(z)}\right)^{\alpha}\\ \prec\frac{1+Az}{1+Bz},\end{array}\right\}, \qquad (7)$$

(iii) $V^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)=$

$$\left\{f\in\mathbb{A}(p):\ \begin{array}{l}(1+\rho)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}-\rho\left(\frac{\mathbb{N}^{m,\delta+1,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}\\ \prec\frac{1+Az}{1+Bz}\end{array}\right\}, \qquad (8)$$

where $\prec$ denotes the subordination (see for details [1, 3, 6]; see also [9]).

Throughout this paper unless otherwise stated, the parameters $\eta, \mu, \lambda, \delta, \rho, \alpha, m, \zeta, A$ and B satisfy the constraints:

$$\begin{aligned} \eta, \mu &\in R, \ \mu < p+1, -\infty < \lambda < \eta + p + 1, \ \delta > -p; \\ 0 &< \alpha < 1, m \in \mathbb{N}_0, \zeta > 0, \text{ and } p \in \mathbb{N}, \end{aligned}$$

2. Preliminary results

In order to establish our main results, we need the following definition and Lemmas.

Definition 1. *[7]. Denote by* $\mathbf{L}$ *the set of all functions* f *that are analytic and injective on* $\bar{U} \backslash E(f)$*, where*

$$E(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \to \xi} f(z) = \infty \right\},$$

and such that $f'(\xi) \neq 0$ *for* $\xi \in \bar{U} \backslash E(f)$*.*

Lemma 1. *[6]. Let* $h(z)$ *be analytic and convex in* $\mathbb{U}$ *with* $h(0) = 1$*, and*

$$g(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + ... \tag{9}$$

be analytic in $\mathbb{U}$*. If*

$$g(z) + \frac{zg'(z)}{\gamma} \prec h(z) \qquad (\Re(\gamma) > 0 \ ), \tag{10}$$

then

$$g(z) \prec q(z) = \frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int h(t) t^{\frac{\gamma}{k}-1} dt \prec h(z),$$

and $q(z)$ *is the best dominant of* (2.2)*.*

Lemma 2. *[9]. Let* $q(z)$ *be a convex univalent in* $\mathbb{U}$ *and* $\sigma \in \mathbb{C}$*,* $\tau \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ *with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\sigma}{\tau}\right)\right\}.$$

If $g(z)$ *is analytic in* $\mathbb{U}$ *and*

$$\sigma g(z) + \tau z g'(z) \prec \sigma q(z) + \tau z q'(z),$$

then $g(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 3. *[7] Let $q(z)$ be convex univalent in $\mathbb{U}$ and $\tau \in \mathbb{C}$. Further assume that $\Re(\tau) > 0$. If $g(z) \in H[q(0), 1] \cap \mathbf{L}$, and $g(z) + \tau z g'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + \tau z q'(z) \prec g(z) + \tau z g'(z),$$

implies $q(z) \prec g(z)$ and $q(z)$ is the best subordinant.

Lemma 4. *[4]. let F be analytic and convex in $\mathbb{U}$. If f, $g \in \mathbb{A} = \mathbb{A}(1)$ and f, $g \prec F$ then*

$$\lambda f(z) + (1-\lambda) g(z) \prec F(z) \quad (0 \le \lambda \le 1).$$

Lemma 5. *[8]. Let $f(z) = 1 + \sum\limits_{k=1}^{\infty} a_k z^k$ be analytic in $\mathbb{U}$ and $g(z) = 1 + \sum\limits_{k=1}^{\infty} b_k z^k$ be analytic and convex in $\mathbb{U}$. If $f(z) \prec g(z)$, then*

$$|a_k| < |b_1| \quad (k \in \mathbb{N}).$$

3. Main results

In the remender of this paper, $\chi(z)$ is given by (6).

Theorem 6. . *Let $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$ with $\Re(\rho) > 0$. Then*

$$\begin{aligned} \left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z)}\right)^{\alpha} &\prec \quad q(z) = \frac{\alpha p}{\zeta\rho} \int\limits_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\alpha p}{\zeta\rho}-1} du \\ &\prec \quad \frac{1+Az}{1+Bz} \end{aligned} \tag{11}$$

and $q(z)$ is the best dominant.

Proof. let

$$g(z) = \left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z)}\right)^{\alpha}. \tag{12}$$

Then $g(z)$ is of the form (9) and is analytic in $\mathbb{U}$. Differentiating (12) and using (3), we get

$$\chi(z) = g(z) + \frac{\zeta\rho z g'(z)}{\alpha p}. \tag{13}$$

As $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$, we have

$$g(z) + \frac{\zeta\rho z g'(z)}{\alpha p} \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 1 with $\gamma = \frac{\alpha p}{\zeta\rho}$, leades to

$$\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \prec q(z) = \frac{\alpha p}{\zeta\rho} z^{-\frac{\alpha p}{\zeta\rho}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{\alpha p}{\zeta\rho}-1} dt$$

$$= \frac{\alpha p}{\zeta\rho} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\alpha(p+\delta)}{\rho}-1} du \prec \frac{1+Az}{1+Bz}, \tag{14}$$

and $q(z)$ is the best dominant, which ends the proof of Theorem 6. ∎

Theorem 7. *Let $\rho \in \mathbb{C}^*$ and $q(z)$ be univalent in $\mathbb{U}$ satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\alpha p}{\zeta\rho}\right)\right\}. \tag{15}$$

If $f(z) \in \mathbb{A}(p)$ satisfies

$$\chi(z) \prec q(z) + \frac{\zeta\rho z q'(z)}{\alpha p}, \tag{16}$$

then

$$\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let $g(z)$ be defined by (12). We know that (13) holds. Combining (13) and (16), we fined that

$$g(z) + \frac{\zeta\rho z g'(z)}{\alpha p} \prec q(z) + \frac{\zeta\rho z q'(z)}{\alpha p}. \tag{17}$$

By using Lemma 2 and (**??**), we easily get the assertion of Theorem 7. ∎

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 7, we get the following result.

Corollary 8. *Let $\rho \in \mathbb{C}^*$ and $-1 \le B < A \le 1$, suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{\alpha p}{\zeta\rho}\right)\right\}.$$

If $f(z) \in \mathbb{A}(p)$ satisfies

$$\chi(z) \prec \frac{1+Az}{1+Bz} + \frac{\zeta\rho\,(A-B)\,z}{\alpha p(1+Bz)^2},$$

then

$$\left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta}f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 9. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $\Re(\rho) > 0$. Also let*

$$\left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta}f(z)}\right)^{\alpha} \in H\left[q(0),1\right] \cap \mathbf{L}$$

and $\chi(z)$ be univalent in $\mathbb{U}$. If

$$q(z) + \frac{\zeta\rho z q'(z)}{\alpha p} \prec \chi(z),$$

then

$$q(z) \prec \left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta}f(z)}\right)^{\alpha},$$

and $q(z)$ is the best subdominant.

Proof. Let $g(z)$ be defined by (12) . Then

$$q(z) + \frac{\zeta\rho z q'(z)}{\alpha p} \prec \chi(z) = g(z) + \frac{\zeta\rho z g'(z)}{\alpha p}.$$

Applying Lemma 3 yields the assertion of Theorem 9. ∎

Taking $q(z) = \dfrac{1+Az}{1+Bz}\,(-1 \le B < A \le 1)$ in Theorem 9, we get the following result.

Corollary 10. *Let $q(z)$ be convex univalent in $\mathbb{U}$ and $-1 \le B < A \le 1$ with $\Re(\rho) > 0$. Also let*

$$\left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta}f(z)}\right)^{\alpha} \in H\left[q(0),1\right] \cap \mathbf{L},$$

and $\chi(z)$ be univalent in $\mathbb{U}$. If

$$\frac{1+Az}{1+Bz} + \frac{\zeta\rho(A-B)z}{\alpha p(1+Bz)^2} \prec \chi(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta} f(z)}\right)^{\alpha},$$

and $\frac{1+Az}{1+Bz}$ *is the best subdominant.*

Combining Theorem 7 and Theorem 9, we easily get the following "Sandwich type result".

Theorem 11. *Let* $q_1(z)$ *be convex univalent,* $q_2(z)$ *be univalent in* $\mathbb{U}$ *and satisfies (15) with* $\rho \in \mathbb{C}^*$*.If*

$$\left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta} f(z)}\right)^{\alpha} \in H\left[q_1(0), 1\right] \cap \mathbf{L},$$

and $\chi(z)$ *is univalent in* $\mathbb{U}$, *and if also*

$$q_1(z) + \frac{\zeta \rho z q_1'(z)}{\alpha p} \prec \chi(z) = q_2(z) + \frac{\zeta \rho z q_2'(z)}{\alpha p},$$

then

$$q_1(z) \prec \left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta} f(z)}\right)^{\alpha} \prec q_2(z),$$

and $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and dominant respectively.*

Theorem 12. *If* ρ, $\alpha > 0$ *and* $f(z) \in R_{p,\lambda,\eta,\mu}^{m,\delta,\zeta}(\alpha, 0; 1-2\psi, -1)$ $(0 \le \psi < 1)$, *then* $f(z) \in R_{p,\lambda,\eta,\mu}^{m,\delta,\zeta}(\alpha, \rho; 1-2\psi, -1)$ *for* $|z| < R$, *where*

$$R = \left(\sqrt{\left(\frac{\zeta\rho}{\alpha p}\right)^2 + 1} - \frac{\zeta\rho}{\alpha p}\right). \tag{18}$$

The bound R *is the best possible.*

Proof. We begin by writing

$$\left(\frac{z^p}{\mathbb{N}_{p,\lambda,\mu,\eta}^{m,\delta,\zeta} f(z)}\right)^{\alpha} = \psi + (1-\psi) g(z) \quad (0 \le \psi < 1). \tag{19}$$

Then, clearly, $g(z)$ is of the form (9), analytic and has positive real part in $\mathbb{U}$. Differentiating (19) and using (3), we obtain

$$\frac{1}{1-\psi}\left(\chi(z) - \psi\right) = g(z) + \frac{\zeta \rho z g'(z)}{\alpha p}. \tag{20}$$

By making use of the following well-known estimate (see [5]):

$$\frac{|zg'(z)|}{\Re\{g(z)\}} \leq \frac{2r}{1-r^2} \quad (|z| = r < 1)$$

(20) leads to

$$\Re\left(\frac{1}{1-\psi}\{\chi(z)-\psi\}\right) \geq \Re\{g(z)\}\left(1-\frac{2\zeta\rho r}{\alpha p(1-r^2)}\right). \tag{21}$$

It is seen that the right-hand side of (21) is positive, provided that $r < R$, where R is given by (18). In order to show that the bound R is the best possible, we consider the function $f(z) \in \mathbb{A}(p)$ defined by

$$\left(\frac{z^p}{N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} = \psi + (1-\psi)\left(\frac{1+z}{1-z}\right) \quad (0 \leq \psi < 1).$$

Noting that

$$\frac{1}{1-\psi}\{\chi(z)-\psi\} = \frac{1+z}{1-z} + \frac{2\zeta\rho z}{\alpha p(1-z)^2} = 0, \tag{22}$$

for $|z| = R$, we conclude that the bound is the best possible, which ends the proof. ∎

Theorem 13. *Let $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\Re(\rho) > 0$. Then*

$$f(z) = \left(z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}\right) * \left(z^p + \sum_{n=1}^{\infty}\left(\frac{p}{p+\zeta n}\right)^m \frac{(1)_n(1+p)_n(1+p+\eta-\mu)_n}{(\delta+p)_n(p+1-\mu)_n(1+p-\lambda+\eta)_n} z^{p+n}\right), \tag{23}$$

where $\omega(z)$ is analytic function with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\Re(\rho) > 0$. It follows from (11) that

$$\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{24}$$

where $\omega(z)$ is analytic with $\omega(0) = 0$ and $|\omega(z)| < 1$. By virtue of (24), we easily find that

$$\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) = z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}. \tag{25}$$

Combining (2) and (25), we have

$$\begin{aligned}&\left(z^p+\sum_{n=1}^{\infty}\left(\frac{p+\zeta n}{p}\right)^m \tfrac{(\delta+p)_n(p+1-\mu)_n(1+p-\lambda+\eta)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}z^{p+n}\right)*f(z)\\ =\ & z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\alpha}}. \qquad (26)\end{aligned}$$

The assertion (23) of Theorem 6 can now easily be derived from (26). ∎

Theorem 14. *Let* $f(z)\in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ *with* $\Re(\rho)>0$. *Then*

$$\begin{aligned}&\frac{1}{z^p}\left[\begin{array}{c}\left(1+Ae^{i\theta}\right)^{\frac{1}{\alpha}}\left(z^p+\sum\limits_{n=1}^{\infty}\left(\frac{p+\zeta n}{p}\right)^m\frac{(\delta+p)_n(p+1-\mu)_n(1+p-\lambda+\eta)_n}{(1)_n(1+p)_n(1+p+\eta-\mu)_n}z^{p+n}\right)\\ *f(z)-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\alpha}}\end{array}\right]\\ \neq\ & 0\ \ (0<\theta<2\pi). \qquad (27)\end{aligned}$$

Proof. Suppose that $f(z)\in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\Re(\rho)>0$. We know that (11) holds, implying that

$$\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\quad(\,0<\theta<2\pi\,). \qquad (28)$$

It is easy to see that the condition (28) can be written as follows:

$$\frac{1}{z^p}\left[\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\left(1+Ae^{i\theta}\right)^{\frac{1}{\alpha}}-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\alpha}}\right]\neq 0\quad(0<\theta<2\pi). \qquad (29)$$

Combining (2) and (29), we easily get the convolution property (27). ∎

Theorem 15. *Let* $\rho_2\geq\rho_1\geq 0$ *and* $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$. *Then*

$$R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho_2;A_2,B_2)\subset R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho_1;A_1,B_1). \qquad (30)$$

Proof. Suppose that $f(z)\in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho_2;A_2,B_2)$. We have

$$(1+\rho_2)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}-\rho_2\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}\prec\frac{1+A_2z}{1+B_2z}.$$

As $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$\begin{aligned}
&(1+\rho_2)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \rho_2\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \\
\prec\ & \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \qquad (31)
\end{aligned}$$

which means that $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho_2;A_1,B_1)$. Thus the assertion (30) holds for $\rho_2 = \rho_1 \geq 0$. If $\rho_2 > \rho_1 \geq 0$, by Theorem 6 and (31), we know that $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,0;A_1,B_1)$, that is,

$$\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \prec \frac{1+A_1z}{1+B_1z}. \qquad (32)$$

At the same time, we have

$$\begin{aligned}
&(1+\rho_1)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \rho_1\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \\
=\ & (1-\frac{\rho_1}{\rho_2})\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} + \frac{\rho_1}{\rho_2} \\
& \left[(1+\rho_2)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \rho_2\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha}\right]. \qquad (33)
\end{aligned}$$

Moreover

$$0 \leq \frac{\rho_1}{\rho_2} < 1,$$

and $\frac{1+A_1z}{1+B_1z}$ $(-1 \leq B_1 < A_1 \leq 1;\ z \in \mathbb{U})$ is analytic and convex in $\mathbb{U}$. Combining (31)-(33) and Lemma 4, we find that

$$(1+\rho_1)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \rho_1\left(\frac{\mathbb{N}^{m+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} \prec \frac{1+A_1z}{1+B_1z},$$

which means that $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho_1;A_1,B_1)$, which implies that the assertion (30) of Theorem 15 holds. ∎

Theorem 16. *Let* $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ *with* $\rho > 0$ *and* $-1 \le B < A \le 1$. *Then*

$$\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du < \Re\left(\frac{z^p}{N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} < \frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du. \quad (34)$$

The extremal function of (34), is given by

$$N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)F(z) = z^p\left(\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Az^n u}{1+Bz^n u}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{-1}{\alpha}}. \quad (35)$$

Proof. Let $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\rho > 0$. From Theorem 6, we know that (11) holds, which implies that

$$\begin{aligned}\Re\left(\frac{z^p}{N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} &< \sup_{z\in U}\Re\left\{\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right\} \\ &\le \frac{\alpha p}{\zeta\rho}\int_0^1 \sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\alpha p}{\zeta\rho}-1}du \\ &< \frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du, \quad (36)\end{aligned}$$

$$\begin{aligned}\Re\left(\frac{z^p}{N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} &> \inf_{z\in U}\Re\left\{\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right\} \\ &\ge \frac{\alpha p}{\zeta\rho}\int_0^1 \inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\alpha p}{\zeta\rho}-1}du \\ &> \frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du. \quad (37)\end{aligned}$$

Combining (36) and (37), we get (34). By noting that $\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)F(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$, we obtain that equality (34) is sharp. ∎

In a similar way, applying the method used in the proof of Theorem 16, we easily get the following result.

Corollary 17. *Let* $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$ *with* $\rho > 0$ *and* $-1 \le A < B \le 1$. *Then*

$$\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du < \Re\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\alpha} < \frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du. \tag{38}$$

The extremal function of (38), is given by (35).

In view of Theorem 16 and Corollary 17, we easily derive the following distortion theorems for the class $R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$.

Corollary 18. *Let* $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$ *with* $\rho > 0$ *and* $-1 \le B < A \le 1$. *Then for* $|z| = r < 1$, *we have*

$$r^p\left(\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{\alpha}} < \left|\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right| < r^p\left(\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{\alpha}}. \tag{39}$$

The extremal function of (39) is defined by (35).

Corollary 19. *Let* $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha, \rho; A, B)$ *with* $\rho > 0$ *and* $-1 \le A < B \le 1$. *Then for* $|z| = r < 1$, *we have*

$$r^p\left(\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{\alpha}} < \left|\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right| < r^p\left(\frac{\alpha p}{\zeta\rho}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{\alpha}}. \tag{40}$$

The extremal function of (40) is defined by (35).

By noting that

$$\left(\Re(v)\right)^{\frac{1}{2}} \leq \Re\left(v^{\frac{1}{2}}\right) \leq |v|^{\frac{1}{2}} \quad \left(v \in \mathbb{C};\ \Re(v) \geq 0\right). \tag{41}$$

we easily derive from Theorem 16 and Corollary 17 the following results.

Corollary 20. *Let $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\rho > 0$ and $-1 \leq B < A \leq 1$. Then*

$$\begin{aligned}\left(\frac{\alpha p}{\zeta\rho}\int\limits_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{2}} \quad &< \quad \Re\left(\frac{z^p}{\mathbb{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\frac{\alpha}{2}} \\ &< \quad \left(\frac{\alpha p}{\zeta\rho}\int\limits_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{2}}.\end{aligned}$$

The extremal function is defined by (35).

Corollary 21. *Let $f(z) \in R^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$ with $\rho > 0$ and $-1 \leq A < B \leq 1$. Then*

$$\begin{aligned}\left(\frac{\alpha p}{\zeta\rho}\int\limits_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{2}} \quad &< \quad \Re\left(\frac{z^p}{N^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right)^{\frac{\alpha}{2}} \\ &< \quad \left(\frac{\alpha p}{\zeta\rho}\int\limits_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha p}{\zeta\rho}-1}du\right)^{\frac{1}{2}}.\end{aligned}$$

The extremal function is defined by (35).

Remark 1. *(i) Using (4) instead of (3) in the above results, we get the corresponding results for the class $T^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$;*

(ii) Using (5) instead of (3) in the above results, we get the corresponding results for the class $V^{m,\delta,\zeta}_{p,\lambda,\eta,\mu}(\alpha,\rho;A,B)$;

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