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NEW CLASSES OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. We define and investigate new classes of harmonic univalent functions defined by Sălăgean integral operator, denoted by $H(m,n,\alpha,\beta)$and $H^{-}(m,n,\alpha,\beta)$. We obtain coefficient inequalities and distortion bounds for the functions in the class $H(m,n,\alpha,\beta)$. We determine the extreme points of closed convex hulls of $H^{-}(m,n,\alpha,\beta)$, denoted by clco$H^{-}(m,n,\alpha,\beta)$. We show that $H^{-}(m,n,\alpha,\beta)$ is closed under convex combination of its members.

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1. Introduction

A continuous complex valued function $f=u+iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f=h+\overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)|>|g'(z)|, z\in D$.(See Clunie and Sheil-Small[2]).
Denote by $\mathcal{H}$ the class of functions $f=h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U=\{z:|z|<1\}$ so that $f=h+\overline{g}$ is normalized by $f(0)=h(0)=f'_z(0)-1=0$.
Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. We let:

$$A_n=\{f\in\mathcal{H}(U), f(z)=z+a_{n+1}z^{n+1}+...,z\in U\}, with A_1=A.$$

We let $\mathcal{H}[a,n]$ denote the class of analytic functions in $U$of the form

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+...,z\in U.$$

The integral operator I^n is defined in [4] by:

$$(i) I^0f(z)=f(z);$$

$$(ii) I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt;$$

$$(iii) I^n f(z) = I(I^{n-1}f(z)), n \in \mathbb{N} - \{0\}, f \in A.$$

Ahuja and Jahangiri [1] defined the class $H(n), n \in \mathbb{N}$, consisting of all univalent harmonic functions $f = h + \overline{g}$ that are sense preserving in U and h and g are of the form:

$$h(z) = z + \lceil\int \sum_{k=2}^{\infty} a_k z^k, g(z) = \lceil\int \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1}$$

For $f = h + \overline{g}$ given by (1) the integral operator I^n is defined as:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, z \in U, \tag{2}$$

where

$$I^n h(z) = z + \lceil\int \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

and

$$I^n g(z) = \lceil\int \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

For fixed positive integers n and for $0 \leq \alpha < 1, \beta \geq 0, m \in \mathbb{N}, m \geq 1$, we let $H(m, n, \alpha, \beta)$ denote the class of univalent harmonic functions of the form (1) that satisfy the condition:

$$Re\Big\{\frac{I^n f(z)}{I^{n+m} f(z)}\Big\} > \beta\Big|\frac{I^n f(z)}{I^{n+m} f(z)} - 1\Big| + \alpha. \tag{3}$$

The subclass $H^{-}(m, n, \alpha, \beta)$ consists of functions $f_n = h + \overline{g_n}$ in $H(m, n, \alpha, \beta)$ so that h and g_n are of the form

$$h(z) = z - \lceil\int \sum_{k=2}^{\infty} a_k z^k, g_n(z) = (-1)^{n-1} \lceil\int \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{4}$$

2. The main results

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H(m, n, \alpha, \beta)$.

Theorem 1. *Let $f = h + \overline{g}$ be given by (1). If*

$$\lceil\int\sum_{k=1}^{\infty}\{\psi(m,n,\alpha,\beta)|a_k| + \theta(m,n,\alpha,\beta)|b_k|\} \leq 2, \tag{5}$$

where

$$\psi(m,n,\alpha,\beta) = \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}}{1-\alpha},$$

and

$$\theta(m,n,\alpha,\beta) = \frac{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha},$$

$a_1 = 1, 0 \leq \alpha < 1, \beta \geq 0, n \in \mathbb{N}, m \in \mathbb{N}, m \geq 1,$ *then* $f \in H(m,n,\alpha,\beta)$.

Proof. According to (2) and(3) we only need to show that

$$Re\Big(\frac{I^n f(z) - \alpha I^{n+m} f(z) - \beta e^{i\theta}|I^n f(z) - I^{n+m} f(z)|}{I^{n+m} f(z)}\Big) \geq 0.$$

The case $r = 0$ is obvious. For $0 < r < 1$ it follows that

$$Re\Big(\frac{I^n f(z) - \alpha I^{n+m} f(z) - \beta e^{i\theta}|I^n f(z) - I^{n+m} f(z)|}{I^{n+m} f(z)}\Big) =$$

$$= Re\Big\{\frac{(1-\alpha)z + \lceil\int\sum_{k=2}^{\infty} a_k z^k[\gamma^n - \alpha\gamma^{n+m}]}{z + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^k + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}} +$$

$$+\frac{(-1)^n\lceil\int\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}[\gamma^n + \alpha\gamma^{n+m}]}{z + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^k + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}} -$$

$$-\frac{\beta e^{i\theta}|\lceil\int\sum_{k=2}^{\infty} a_k z^k[\gamma^n - \gamma^{n+m}] + (-1)^n\lceil\int\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}[\gamma^n + \gamma^{n+m}]|}{z + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^k + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}}\Big\} =$$

$$= Re\Big\{\frac{1-\alpha + \lceil\int\sum_{k=2}^{\infty} a_k z^{k-1}[\gamma^n - \alpha\gamma^{n+m}]}{1 + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^{k-1} + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}z^{-1}} +$$

$$\frac{(-1)^n\lceil\int\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}z^{-1}[\gamma^n - (-1)^m\alpha\gamma^{n+m}]}{1 + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^{k-1} + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}z^{-1}} -$$

$$\frac{\beta e^{i\theta}z^{-1}|\lceil\int\sum_{k=2}^{\infty}[\gamma^n - \gamma^{n+m}]a_k z^k + (-1)^n\lceil\int\sum_{k=1}^{\infty}[\gamma^n - (-1)^m\gamma^{n+m}]\overline{b_k}\overline{z^k}|}{1 + \lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k z^{k-1} + (-1)^{n+m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}\overline{b_k}\overline{z^k}z^{-1}}\Big\} =$$

$$= Re\frac{(1-\alpha) + A(z)}{1 + B(z)}, where \gamma = \frac{1}{k},$$

$$A(z) = \lceil\int \sum_{k=2}^{\infty} a_k z^{k-1}[\gamma^n - \alpha\gamma^{n+m}] + (-1)^n \lceil\int \sum_{k=1}^{\infty} \overline{b_k z^k} z^{-1}[\gamma^n - (-1)^m \alpha\gamma^{n+m}] -$$

$$-\beta e^{i\theta} z^{-1} |\lceil\int \sum_{k=2}^{\infty} [\gamma^n - \gamma^{n+m}] a_k z^k + (-1)^n \lceil\int \sum_{k=1}^{\infty} (\gamma^n - (-1)^m \gamma^{n+m}) \overline{b_k z^k}|,$$

$$B(z) = \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k z^{k-1} + (-1)^{n+m} \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} \overline{b_k} \overline{z^k} z^{-1}.$$

For $z = re^{i\theta}$ we have

$$A(re^{i\theta}) = \lceil\int \sum_{k=2}^{\infty} (\gamma^n - \alpha\gamma^{n+m}) a_k r^{k-1} e^{(k-1)\theta i} +$$

$$+(-1)^n \lceil\int \sum_{k=1}^{\infty} (\gamma^n - (-1)^m \gamma^{n+m} \alpha) \overline{b_k} r^{k-1} e^{-(k+1)\theta i} - \beta \mathcal{D}(n+m, n, \alpha),$$

where

$$\mathcal{D}(n+m, n, \alpha) =$$

$$= |\lceil\int \sum_{k=2}^{\infty} (\gamma^n - \gamma^{n+m}) a_k r^{k-1} e^{-ki\theta} + (-1)^n \lceil\int \sum_{k=1}^{\infty} (\gamma^n - (-1)^m \gamma^{n+m}) \overline{b_k} r^{k-1} e^{-ki\theta}|,$$

and

$$B(re^{i\theta}) = \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k r^{k-1} e^{(k-1)\theta i} + (-1)^{n+m} \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} \overline{b_k} r^{k-1} e^{-(k+1)\theta i}.$$

Setting $\frac{1-\alpha+A(z)}{1+B(z)} = (1-\alpha)\frac{1+w(z)}{1-w(z)}$.

The proof will be complete if we can show that $|w(z)| \le r < 1$. This is the case since, by the condition (5), we can write:

$$|w(z)| = \left|\frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)}\right| \le$$

$$\frac{\lceil\int \sum_{k=1}^{\infty} [(1+\beta)(\gamma^n - \gamma^{n+m})|a_k| + (1+\beta)(\gamma^n - (-1)^m \gamma^{n+m})|b_k|] r^{k-1}}{4(1-\alpha) - \lceil\int \sum_{k=1}^{\infty} \{[\gamma^n(1+\beta) - \delta\gamma^{n+m}]|a_k| + [\gamma^n(1+\beta) - (-1)^m \delta\gamma^{n+m}]|b_k|\} r^{k-1}}$$

$$< \frac{\lceil\int \sum_{k=1}^{\infty} (1+\beta)(\gamma^n - \gamma^{n+m})|a_k| + (\gamma^n - (-1)^m \gamma^{n+m})(1+\beta)|b_k|}{4(1-\alpha) - \lceil\int \sum_{k=1}^{\infty} \{[\gamma^n(1+\beta) - \delta\gamma^{n+m}]|a_k| + [\gamma^n(1+\beta) - (-1)^m \delta\gamma^{n+m}]|b_k|\}}$$

$$\le 1,$$

where $\delta = \beta + 2\alpha - 1$.

The harmonic univalent functions

$$f(z) = z + \lceil\int \sum_{k=2}^{\infty} \frac{1}{\psi(m,n,\alpha,\beta)} x_k z^k + \lceil\int \sum_{k=1}^{\infty} \frac{1}{\theta(m,n,\alpha,\beta)} \overline{y_k z^k},$$

where $n \in \mathbb{N}, 0 \le \alpha < 1, \beta \ge 0, m \in \mathbb{N}, m \ge 1$ and $\lceil\int \sum_{k=2}^{\infty} |x_k| + \lceil\int \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (5) is sharp.

In the following theorem it is show that the condition (5)is also necessary for the function $f_n = h + \overline{g_n}$, where h and g_n are of the form (4).

Theorem 2. *Let $f_n = h + \overline{g_n}$ be given by (4). Then $f_n \in H^-(m,n,\alpha,\beta)$ if and only if*

$$\lceil\int \sum_{k=1}^{\infty} [\psi(m,n,\alpha,\beta) a_k + \theta(m,n,\alpha,\beta) b_k] \le 2, \tag{6}$$
$$a_1 = 1, 0 \le \alpha < 1, n \in \mathbb{N}, m \in \mathbb{N}, m \ge 1.$$

Proof. Since $H^-(m,n,\alpha,\beta) \subset H(m,n,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (4), we note that the condition

$$Re\Big\{\frac{I^n f(z)}{I^{n+m} f(z)}\Big\} > \beta \Big|\frac{I^n f(z)}{I^{n+m} f(z)} - 1\Big| + \alpha$$

is equivalent to

$$Re\Big\{\frac{(1-\alpha)z - \lceil\int \sum_{k=2}^{\infty} (\gamma^n - \alpha\gamma^{n+m}) a_k z^k}{z - \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k z^k + (-1)^{2n+m-1} \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} b_k \overline{z^k}} +$$
$$+ \frac{(-1)^{2n-1} \lceil\int \sum_{k=1}^{\infty} (\gamma^n - (-1)^m \gamma^{n+m} \alpha) b_k \overline{z^k}}{z - \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k z^k + (-1)^{2n+m-1} \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} b_k \overline{z^k}} -$$
$$- \frac{\beta e^{i\theta} | - \lceil\int \sum_{k=2}^{\infty} (\gamma^n + \gamma^{n+m}) a_k z^k + (-1)^{2n-1} \lceil\int \sum_{k=1}^{\infty} (\gamma^n - (-1)^{2m} \gamma^{n+m}) \overline{b_k} \overline{z^k}|}{z - \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k z^k + (-1)^{2n+2m-1} \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} b_k \overline{z^k}}\Big\} \ge 0, \tag{7}$$

where $\gamma = \frac{1}{k}$.

The above required condition (7) must hold for all values of $z \in U$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, and using $Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, we must have

$$\frac{(1-\alpha) - \lceil\int \sum_{k=2}^{\infty} [\gamma^n (1+\beta) - (\alpha+\beta)\gamma^{n+m}] a_k r^{k-1}}{1 - \lceil\int \sum_{k=2}^{\infty} \gamma^{n+m} a_k r^{k-1} - (-1)^m \lceil\int \sum_{k=1}^{\infty} \gamma^{n+m} b_k r^{k-1}} - \tag{8}$$

$$-\frac{\lceil\int\sum_{k=1}^{\infty}[\gamma^{n}(1+\beta)+\gamma^{n+m}(\beta+\alpha)]b_k r^{k-1}}{1-\lceil\int\sum_{k=2}^{\infty}\gamma^{n+m}a_k r^{k-1}-(-1)^{m}\lceil\int\sum_{k=1}^{\infty}\gamma^{n+m}b_k r^{k-1}}\geq 0.$$

If the condition (7) does not hold, then the expression in (8) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which this quotient in (8) is negative. This contradicts the required condition for $f_n \in H^{-}(m,n,\alpha,\beta)$ and so the proof is complete.

The following theorem gives the distortion bounds for functions in $H^{-}(m,n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 3. *Let $f_n \in H^{-}(m,n,\alpha,\beta)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1+b_1)r + [\theta(m,n,\alpha,\beta) - \omega(m,n,\alpha,\beta)b_1]r^{n+m+1}$$

and

$$|f_n(z)| \geq (1-b_1)r - \{\phi(m,n,\alpha,\beta) - \omega(m,n,\alpha,\beta)b_1\}r^{n+m+1},$$

where

$$\phi(m,n,\alpha,\beta) = \frac{1-\alpha}{(1/2)^{n}(1+\beta)-(1/2)^{n+m}(\alpha+\beta)},$$

$$\omega(m,n,\alpha,\beta) = \frac{(1+\beta)-(-1)^{m}(\alpha+\beta)}{(1/2)^{n}(1+\beta)-(1/2)^{n+m}(\alpha+\beta)}.$$

Proof. We prove the right side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n \in H^{-}(m,n,\alpha,\beta)$. Taking the absolute value of f_n then by Theorem 2, we can obtain :

$$|f_n(z)| = |z - \lceil\int\sum_{k=2}^{\infty}a_k z^k + (-1)^{n-1}\lceil\int\sum_{k=1}^{\infty}b_k\overline{z^k}| \leq$$

$$\leq r + \lceil\int\sum_{k=2}^{\infty}a_k r^k + \lceil\int\sum_{k=1}^{\infty}b_k r^k = r + b_1 r + \lceil\int\sum_{k=2}^{\infty}(a_k+b_k)r^k \leq$$

$$\leq r + b_1 r + \lceil\int\sum_{k=2}^{\infty}(a_k+b_k)r^2 =$$

$$= (1+b_1)r + \phi(m,n,\alpha,\beta)\lceil\int\sum_{k=2}^{\infty}\frac{1}{\phi(m,n,\alpha,\beta)}(a_k+b_k)r^2 \leq$$

$$\leq (1+b_1)r + \phi(m,n,\alpha,\beta)r^{n+m+1}\lceil\int\sum_{k=2}^{\infty}[\psi(m,n,\alpha,\beta)a_k + \theta(m,n,\alpha,\beta)b_k] \leq$$

$$\leq (1+b_1)r + [\phi(m,n,\alpha,\beta) - \omega(m,n,\alpha,\beta)b_1]r^{n+m+1}.$$

The following covering result follows from the left hand inequality in Theorem 3.

Corollary 4. *Let $f_n \in H^-(m,n,\alpha,\beta)$. Then for $|z| = r < 1$ we have $\{w : |w| < 1 - b_1 - [\phi(m,n,\alpha,\beta) - \omega(n,\alpha,\eta)b_1] \subset f_n(U)\}$.*

Next we determine the extreme points of closed convex hulls of $H^-(m,n,\alpha,\beta)$, denoted by clco$H^-(m,n,\alpha,\beta)$.

Theorem 5. *Let f_n be given by (4). Then $f_n \in H^-(m,n,\alpha,\beta)$ if and only if*

$$f_n(z) = \lceil\int \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_{n_k}(z)],$$

where $h(z) = z$,

$$h_k(z) = z - \frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}} z^k, k = 2,3,...$$

and

$$g_{n_k}(z) = z + (-1)^{n-1} \frac{1-\alpha}{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}} \overline{z}^k, k = 1,2,3,...$$

$x_k \geq 0, y_k \geq 0, \lceil\int \sum_{k=1}^{\infty}(x_k + y_k) = 1.$

In particular, the extreme points of $H^-(m,n,\alpha,\beta)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (5) we have:

$$f_n(z) = \lceil\int \sum_{k=2}^{\infty} [x_k h_k(z) + y_k g_{n_k}(z)] =$$

$$= \lceil\int \sum_{k=1}^{\infty} (x_k + y_k)z - \lceil\int \sum_{k=2}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}} x_k z^k +$$

$$+(-1)^{n-1} \lceil\int \sum_{k=1}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}} y_k \overline{z}^k.$$

Then

$$\lceil\int\sum_{k=2}^{\infty}x_k\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+m)}}{1-\alpha}\cdot\frac{(1-\alpha)}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+m)}}+$$

$$+\lceil\int\sum_{k=1}^{\infty}y_k\frac{k^{-n}(1+\beta)-(-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha}\cdot$$

$$\cdot\frac{1-\alpha}{k^{-n}(1+\beta)-(-1)^m(\beta+\alpha)k^{-(n+m)}}$$

$$=\lceil\int\sum_{k=2}^{\infty}x_k+\lceil\int\sum_{k=1}^{\infty}y_k=1-x_1\leq 1$$

and so $f_n(z)\in H^-(m,n,\alpha,\beta)$.
Conversely, suppose $f_n(z)\in H^-(m,n,\alpha,\beta)$. Letting

$$x_1=1-\lceil\int\sum_{k=2}^{\infty}x_k-\lceil\int\sum_{k=1}^{\infty}y_k$$

$$x_k=\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+m)}}{1-\alpha}\cdot a_k, k=2,3,...$$

and

$$y_k=\frac{k^{-n}(1+\beta)-(-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha}\cdot b_k, k=1,2,3,...$$

we obtain the required representation, since

$$f_n(z)=z-\lceil\int\sum_{k=2}^{\infty}a_kz^k+(-1)^{n-1}\lceil\int\sum_{k=1}^{\infty}b_k\overline{z}^k=$$

$$=z-\lceil\int\sum_{k=2}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+m)}}x_kz^k+$$

$$+(-1)^{n-1}\lceil\int\sum_{k=1}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta)-(-1)^m(\beta+\alpha)k^{-(n+m)}}y_k\overline{z}^k=$$

$$=z-\lceil\int\sum_{k=2}^{\infty}[z-h_k(z)]x_k-\lceil\int\sum_{k=1}^{\infty}[z-g_{n_k}(z)]y_k=$$

$$=[1-\lceil\int\sum_{k=2}^{\infty}x_k-\lceil\int\sum_{k=1}^{\infty}y_k]z+\lceil\int\sum_{k=2}^{\infty}x_kh_k(z)+\lceil\int\sum_{k=1}^{\infty}y_kg_{n_k}(z)=$$

$$= \lceil\int \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)].$$

Now we show that $H^{-}(m,n,\alpha,\beta)$ is closed under convex combination of its members.

Theorem 6. *The family $H^{-}(m,n,\alpha,\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, ...$ suppose that $f_n^i \in H^{-}(m,n,\alpha,\beta)$, where

$$f_n^i(z) = z + \lceil\int \sum_{k=2}^{\infty} a_k^i z^k + (-1)^{n-1} \lceil\int \sum_{k=1}^{\infty} b_k^i \overline{z}^k,$$

then by Theorem 2,

$$\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}}{1-\alpha} a_k^i +$$

$$+\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha} b_k^i \leq 2, \tag{9}$$

for $\lceil\int \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_n^i may be written as

$$\lceil\int \sum_{i=1}^{\infty} t_i f_n^i(z) = z - \lceil\int \sum_{k=2}^{\infty}(\lceil\int \sum_{i=1}^{\infty} t_i a_k^i) z^k + (-1)^{n-1} \lceil\int \sum_{k=1}^{\infty}(\lceil\int \sum_{i=1}^{\infty} t_i b_k^i) \overline{z}^k.$$

Then by (8)

$$\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}}{1-\alpha} (\lceil\int \sum_{i=1}^{\infty} t_i a_k^i) +$$

$$+\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha} (\lceil\int \sum_{i=1}^{\infty} t_i b_k^i) =$$

$$= \lceil\int \sum_{i=1}^{\infty} t_i [\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+m)}}{1-\alpha} a_k^i +$$

$$+\lceil\int \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (-1)^m(\beta+\alpha)k^{-(n+m)}}{1-\alpha} b_k^i]$$

$$\leq 2 \lceil\int \sum_{i=1}^{\infty} t_i = 2$$

and therefore $\lceil\int \sum_{i=1}^{\infty} t_i f_n^i(z) \in H^{-}(m,n,\alpha,\beta)$.

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