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# UNIVALENT HARMONIC FUNCTIONS GENERATED BY RUSCHEWEYH DERIVATIVES OF ANALYTIC FUNCTIONS

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Abstract. For $\lambda \geq 0$, $p > 0$ and a normalized univalent function $f$ defined on the unit disk $\mathbb{D}$, we consider the harmonic function defined by

$$T_{\lambda,p}[f](z) = \frac{\mathcal{D}^\lambda f(z) + pz(\mathcal{D}^\lambda f(z))'}{p+1} + \overline{\frac{\mathcal{D}^\lambda f(z) - pz(\mathcal{D}^\lambda f(z))'}{p+1}}, \quad z \in \mathbb{D},$$

where the operator $\mathcal{D}^\lambda$ is the familiar $\lambda$-Ruscheweyh derivative operator. We find some necessary and sufficient conditions for the univalence, starlikeness and convexity as well as the growth estimate of the function $T_{\lambda,p}[f]$. An extension of the above operator is also given.

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*Keywords:* harmonic mappings; Ruscheweyh derivative; univalence; starlikeness; convexity; convexity in one direction

## 1. Introduction

Let $\mathcal{H}$ denote the class of complex-valued harmonic functions $f = h + \bar{g}$ defined in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $h$ and $g$ are analytic functions given by

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m. \tag{1}$$

By Lewy's Theorem [9], the function $f = h + \bar{g} \in \mathcal{H}$ is sense-preserving if and only if the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive, or equivalently $|g'| < |h'|$ in $\mathbb{D}$. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving functions. A domain is said to be convex in the direction of real (or imaginary) axis if every line parallel to the real (or imaginary) axis has a connected intersection with the domain. The following theorem due to Clunie and Sheil-Small [5] and Sheil-Small [16] gives a technique of constructing univalent harmonic mappings in a given direction, known as "Shearing Method."

**Theorem 1.** *Let the function $f = h + \bar{g}$ be harmonic and locally univalent function in $\mathbb{D}$. Then*

*(1) the function $F = h - g \in \mathcal{S}$ and $F(\mathbb{D})$ is convex in the direction of real axis $\Longleftrightarrow$ the function $f = h + \bar{g}$ is univalent and convex in the direction of real axis.*

*(2) the function $F = h + g \in \mathcal{S}$ and $F(\mathbb{D})$ is convex in the direction of imaginary axis $\Longleftrightarrow$ the function $f = h + \bar{g}$ is univalent and convex in the direction of imaginary axis.*

Using Theorem 1, Clunie and Sheil-Small [5] proved that if the functions $H_0$ and $G_0$ are analytic in $\mathbb{D}$ with $H_0(z) + G_0(z) = z/(1-z)$ and $G_0'(z)/H_0'(z) = -z$, then the resulting harmonic function $T_0 := H_0 + \overline{G_0}$ is univalent and maps $\mathbb{D}$ onto the right half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > -1/2\}$. In fact,

$$T_0(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) + \frac{1}{2}\overline{\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right)}$$

which may be expressed as

$$T_0(z) = \frac{1}{2}(I(z) + zI'(z)) + \frac{1}{2}\overline{(I(z) - zI'(z))}$$

where

$$I(z) = \frac{z}{1-z}$$

is the analytic right-half plane mapping. The function $T_0$ is well-known in the theory of univalent harmonic functions and it acts extremal for many harmonic inequalities concerning the subclass of $\mathcal{S}_H$ consisting of convex functions.

Let $\mathcal{A}$ denote the class of all analytic functions $f$ defined in $\mathbb{D}$ normalized by $f(0) = 0 = f'(0) - 1$ and suppose that $\mathcal{S}$ is its subclass consisting of univalent functions. Motivated by the description of the right half-plane mapping $T_0$, we define a differential operator which is closely related to Ruscheweyh derivatives. If $f \in \mathcal{A}$, then for each $\lambda \geq 0$ and $p > 0$, we define

$$T_{\lambda,p}[f](z) = \frac{\mathcal{D}^\lambda f(z) + pz(\mathcal{D}^\lambda f(z))'}{p+1} + \frac{\overline{\mathcal{D}^\lambda f(z) - pz(\mathcal{D}^\lambda f(z))'}}{p+1}, \quad z \in \mathbb{D}, \tag{2}$$

where the operator $\mathcal{D}^\lambda : \mathcal{A} \to \mathcal{A}$ is $\lambda$-Ruscheweyh derivative of

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{3}$$

given by

$$\mathcal{D}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z + \sum_{m=2}^{\infty} \frac{(\lambda+1)_{m-1}}{(m-1)!} a_m z^m, \tag{4}$$

where

$$(\lambda+1)_{m-1} = (\lambda+1)(\lambda+2)\cdots(\lambda+m-1).$$

Here $*$ is the convolution (or Hadamard product) of two power series of given two functions. For properties of Ruscheweyh derivatives, one may refer to [1, 2, 13]. The following example justify the need of the operator defined by (2).

**Example 1.** *The operator $T_0(z) = T_{0,1}[I]$ was introduced and studied by Clunie and Sheil-Small[5]. In 2008, Muir [10] proved that $T_p[I] = T_{0,p}[I]$ is a harmonic right half-plane mapping of $\mathbb{D}$ onto the right half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > -1/(1+p)\}$ for each $p > 0$. The operator $T_p[f] = T_{0,p}[f]$ for $f \in \mathcal{S}$ and $p > 0$ was studied by Muir [11]. In [15], Ruscheweyh and Suffridge defined continuous extension of the de la Vallee Poussin means $\mathcal{V}_{\mu} : \mathbb{D} \longrightarrow \mathbb{C}$, by*

$$\mathcal{V}_{\mu}(z) = \frac{\mu z}{\mu+1} {}_2\mathcal{F}_1(1, 1-\mu, 2+\mu, -z), \quad \mu > 0$$

*where ${}_2\mathcal{F}_1$ is the Gaussian hypergeometric function. These authors also proved that the mapping*

$$T_p[\mathcal{V}_{\mu}] = \frac{\mathcal{V}_{\mu} + pz\mathcal{V}_{\mu}'}{p+1} + \overline{\frac{\mathcal{V}_{\mu} - pz\mathcal{V}_{\mu}'}{p+1}} \tag{5}$$

*is a harmonic mapping of $\mathbb{D}$ onto a convex domain for each $\mu \geq \frac{1}{2}$ and $p \geq 0$.*

## 2. PRELIMINARIES

Let $\mathcal{R}_{\lambda}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left(\frac{z(\mathcal{D}^{\lambda} f(z))'}{\mathcal{D}^{\lambda} f(z)}\right) > \alpha$$

for some $\lambda > -1$, $\alpha < 1$, and for all $z \in \mathbb{D}$, where $\mathcal{D}^{\lambda} : \mathcal{A} \to \mathcal{A}$ is the $\lambda-$Ruscheweyh derivative operator defined by (4). The class $\mathcal{R}_{\lambda}(\alpha)$ was introduced and studied by first author in [1, 2]. In particular, note that $\mathcal{R}_0(\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{R}_1(\alpha) = \mathcal{K}(\alpha)$ are well-known subclasses of $\mathcal{S}$ consisting of starlike functions of order $\alpha$ and convex functions of order $\alpha$ respectively. Let $\mathcal{T}$, $\mathcal{TS}^*(\alpha)$, $\mathcal{TK}(\alpha)$ and $\mathcal{TR}_{\lambda}(\alpha)$ be respectively subclasses of $\mathcal{A}$, $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{R}_{\lambda}(\alpha)$ whose elements can be expressed in the form

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m, \quad z \in \mathbb{D}. \tag{6}$$

In [1], the first author observed that the family $\mathcal{R}_\lambda(\alpha)$ includes several other subclasses of $\mathcal{T}$. For example, the classes $\mathcal{R}[\alpha] \equiv \mathcal{R}_{1-2\alpha}(\alpha)$ and $\mathcal{R}[\alpha,\beta] \equiv \mathcal{R}_{1-2\alpha}(\beta)$ for $\alpha, \beta < 1$ were respectively, studied in [17] and [4]. Recall that a function $f$ in $\mathcal{R}[\alpha]$ is called prestarlike of order $\alpha$ (see [3]).

**Lemma 2.** *Let $f$ be an analytic function of the form* (6), $\lambda \geq -1$ *and* $0 \leq \alpha < 1$. *Then the following statements are equivalent:*

*(1)* $f \in \mathcal{T}\mathcal{R}_\lambda(\alpha)$

*(2)* $\left|\frac{z\left(\mathcal{D}^{\lambda+1} f(z)\right)}{\mathcal{D}^\lambda f(z)} - 1\right| \leq 1-\alpha, \quad z \in \mathbb{D}$

*(3)* $\sum_{m=2}^\infty \frac{(m-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m| \leq 1.$

For $0 \leq \alpha < 1$, let $\mathcal{FS}^*_{\mathcal{H}}(\alpha)$ and $\mathcal{FK}_{\mathcal{H}}(\alpha)$ denote the subclasses of $\mathcal{H}$, respectively, consisting of fully starlike of order $\alpha$ and fully convex of order $\alpha$. These classes were studied in [12]. Recall that

$$\mathcal{FS}^*_H(\alpha) = \left\{ f \in \mathcal{S}_H : \frac{\partial}{\partial\theta}\left(\arg\left(f(re^{i\theta})\right)\right) \geq \alpha,\ 0 < r < 1,\ 0 \leq \theta < 2\pi \right\}$$

$$\mathcal{FK}_{\mathcal{H}}(\alpha) = \left\{ f \in \mathcal{S}_H : \frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) \geq \alpha,\ 0 < r < 1,\ 0 \leq \theta < 2\pi \right\}.$$

Let $\mathcal{T}_H$, $\mathcal{TFS}^*_H(\alpha)$ and $\mathcal{TFK}_H(\alpha)$ be subclasses, respectively of $\mathcal{H}$, $\mathcal{FS}^*_H(\alpha)$ and $\mathcal{FK}_H(\alpha)$ consisting of functions $f = h + \overline{g}$, where

$$f(z) = z - \sum_{m=2}^\infty |a_m| z^m, \quad g(z) = \sum_{m=1}^\infty |b_m| z^m, \quad z \in \mathbb{D}. \tag{7}$$

**Lemma 3.** *[7] Let $\alpha \in [0,1)$ and the function $f = h + \bar{g}$ be given by* (1). *If the inequality*

$$\sum_{m=1}^\infty \left( \frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m| \right) \leq 2, \quad a_1 = 1, \tag{8}$$

*holds, then $f \in \mathcal{FS}^*_H(\alpha)$. However, if the function $f = h + \bar{g}$ is given by* (7), *then the coefficient inequality* (8) *is necessary and sufficient for $f$ to be in $\mathcal{TFS}^*_H(\alpha)$.*

**Lemma 4.** *[8] Let $\alpha \in [0,1)$ and the function $f = h + \bar{g}$ be given by* (1). *If the inequality*

$$\sum_{m=1}^\infty \left( \frac{m(m-\alpha)}{1-\alpha}|a_m| + \frac{m(m+\alpha)}{1-\alpha}|b_m| \right) \leq 2, \quad a_1 = 1, \tag{9}$$

*holds, then $f \in \mathcal{FK}_H(\alpha)$. However, if $f = h + \bar{g}$ is given by* (7), *then the coefficient inequality* (9) *is necessary and sufficient for $f$ to be in $\mathcal{TFK}_H(\alpha)$.*

If co-analytic part $g$ of the function $f = h + \bar{g}$ is zero, then Lemma 3 and Lemma 4 yield the following results.

**Lemma 5.** *Let $\alpha \in [0, 1)$ and the function $f \in \mathcal{T}$ be given by* (6). *Then*

*(a)* $f \in \mathcal{TS}^*(\alpha) \iff \sum_{m=2}^{\infty}(m-\alpha)|a_m| \leq 1-\alpha$,

*(b)* $f \in \mathcal{TK}(\alpha) \iff \sum_{m=2}^{\infty} m(m-\alpha)|a_m| \leq 1-\alpha$.

## 3. Main results

The first result of this section determines the condition for the local univalence of the operator $T_{\lambda,p}[f]$ defined by (2).

**Lemma 6.** *Let $p > 0$, $\lambda \geq 0$ and the function $f \in \mathcal{A}$. Then the function $T_{\lambda,p}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $\mathcal{D}^{\lambda} f$ is convex in $\mathbb{D}$.*

*Proof.* Write $T_{\lambda,p}[f] = H + \bar{G}$, where

$$H = \frac{\mathcal{D}^{\lambda} f(z) + pz(\mathcal{D}^{\lambda} f(z))'}{p+1} \quad \text{and} \quad G = \frac{\mathcal{D}^{\lambda} f(z) - pz(\mathcal{D}^{\lambda} f(z))'}{p+1}. \tag{10}$$

In view of Lewy's Theorem, $T_{\lambda,p}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|G'| < |H'|$, or equivalently if and only if

$$|(1-p)(\mathcal{D}^{\lambda} f(z))' - pz(\mathcal{D}^{\lambda} f(z))''| < |(1+p)(\mathcal{D}^{\lambda} f(z))' + pz(\mathcal{D}^{\lambda} f(z))''|.$$

Clearly $(D^{\lambda} f)' \neq 0$ in $\mathbb{D}$, above inequality is equivalent to

$$\left| \frac{1}{p} - \left(1 + \frac{z(\mathcal{D}^{\lambda} f(z))''}{(\mathcal{D}^{\lambda} f(z))'}\right) \right| < \left| \frac{1}{p} + \left(1 + \frac{z(\mathcal{D}^{\lambda} f(z))''}{(D^{\lambda} f(z))'}\right) \right|$$

or

$$\operatorname{Re}\left(1 + \frac{z(\mathcal{D}^{\lambda} f(z))''}{(\mathcal{D}^{\lambda} f(z))'}\right) > 0.$$

This last condition is equivalent to convexity of $\mathcal{D}^{\lambda} f$.

For $\lambda = 0$ and $p > 0$, we have

**Corollary 7.** *[10] For $f \in \mathcal{S}$, the function $T_p[f]$ defined in Example 1 is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $f$ is convex analytic in $\mathbb{D}$.*

**Corollary 8.** *Let $p > 0$, $\lambda \geq 0$ and the function $f \in \mathcal{T}$ be given by* (6)*. Then the function $T_{\lambda,p}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if*

$$\sum_{m=2}^{\infty} \frac{m^2(\lambda+1)_{m-1}}{(m-1)!}|a_m| \leq 1. \tag{11}$$

*Proof.* In view of Lemma 6, $T_{\lambda,p}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if

$$(\mathcal{D}^\lambda f)(z) = z - \sum_{m=2}^{\infty} \frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m|z^m,$$

is convex. The result now follows from Lemma 5(b).

**Theorem 9.** *Let $p > 0$, $\lambda \geq 0$ and the function $f \in \mathcal{A}$. Then the function $T_{\lambda,p}[f]$ is convex in the direction of imaginary axis if and only if $\mathcal{D}^\lambda f$ is convex in $\mathbb{D}$.*

*Proof.* Suppose $T_{\lambda,p}[f] = H + \overline{G}$, where $H$ and $G$ are given by (10). The necessary part is obviously true by Lemma 6. For the sufficient part, note that the analytic function

$$M(z) = H(z) + G(z) = \frac{2\mathcal{D}^\lambda f(z)}{p+1}$$

satisfies

$$M' = \frac{2}{p+1}(\mathcal{D}^\lambda f(z))' \neq 0$$

$$\operatorname{Re}\left(1 + \frac{zM''(z)}{M'(z)}\right) = \operatorname{Re}\left(1 + \frac{z(\mathcal{D}^\lambda f(z))''}{(\mathcal{D}^\lambda f(z))'}\right) > 0.$$

This, in particular, shows that $H + G$ is univalent and convex in the direction of imaginary axis. Using Theorem 1, we obtain the desired result.

**Corollary 10.** *Let $p > 0$, $\lambda \geq 0$ and the function $f \in \mathcal{T}$ be given by* (6)*. Then the function $T_{\lambda,p}[f]$ is convex in the direction of imaginary axis if and only if the coefficient inequality*

$$\sum_{m=2}^{\infty} \frac{m^2(\lambda+1)_{m-1}}{(m-1)!}|a_m| \leq 1$$

*is satisfied.*

**Theorem 11.** *Suppose $0 \leq \alpha < 1$ and $p > 1$. Let the function $f \in \mathcal{A}$ is given by* (3)*. If the condition*

$$\sum_{m=1}^{\infty} \frac{(pm^2-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!(p+1)}|a_m| \leq 1, \quad a_m = 1 \tag{12}$$

*is satisfied, then the function* $T_{\lambda,p}[f] \in \mathcal{FS}^*_H(\alpha)$ *and the function* $\mathcal{D}^\lambda f \in \mathcal{K}$. *Moreover, if the function* $f \in \mathcal{T}$ *is given by* (6)*, then* (12) *is necessary for the function* $T_{\lambda,p}[f]$ *to be in* $\mathcal{TFS}^*_H(\alpha)$.

*Proof.* Using (2) and (3), we have

$$T_{\lambda,p}[f](z) = \sum_{m=1}^{\infty} \frac{(1+pm)}{p+1}\frac{(\lambda+1)_{m-1}}{(m-1)!} a_m z^m + \sum_{m=1}^{\infty} \overline{\frac{(1-pm)}{p+1}\frac{(\lambda+1)_{m-1}}{(m-1)!} a_m z^m}.$$

For $m \geq 1$, setting

$$\mathcal{A}_m = \frac{(pm+1)(\lambda+1)_{m-1}}{(p+1)(m-1)!} a_m, \quad \text{and} \quad \mathcal{B}_m = \frac{(1-pm)(\lambda+1)_{m-1}}{(p+1)(m-1)!} a_m \tag{13}$$

it can be seen that the inequality

$$\sum_{m=1}^{\infty} \left( \frac{m-\alpha}{1-\alpha} |\mathcal{A}_m| + \frac{m+\alpha}{1-\alpha} |\mathcal{B}_m| \right) \leq 2,$$

is equivalent to (12). By Lemma 3, $T_{\lambda,p}[f] \in \mathcal{FS}^*_H(\alpha)$ and hence $\mathcal{D}^\lambda f \in \mathcal{K}$ by Lemma 6. In order to prove necessary condition, we assume that $a_m \leq 0$ for $m \geq 2$. It follows from (13) that $\mathcal{A}_m \leq 0$ for all $m \geq 2$ and $\mathcal{B}_m \geq 0$ for all $m \geq 1$. Again, it follows by Lemma 3 that (12) is satisfied if and only if $T_{\lambda,p}[f] \in \mathcal{TFS}^*_H(\alpha)$.

In [6], Goodman proved that if $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is in $\mathcal{A}$ and if $\sum_{m=2}^{\infty} m^2 |a_m| \leq 1$, then $f \in \mathcal{K}$. However, for $\alpha = 0$, Theorem 11 provides the following stronger result.

**Corollary 12.** *Under the hypothesis of Theorem 11, if the condition*

$$\sum_{m=2}^{\infty} \frac{m^2 (\lambda+1)_{m-1}}{(m-1)!} |a_m| \leq \frac{1}{p}, \tag{14}$$

*is satisfied, then the function* $T_{\lambda,p}[f] \in \mathcal{FS}^*_H$ *and the function* $\mathcal{D}^\lambda f \in \mathcal{K}$.

**Corollary 13.** *If the function* $f(z) = z - \sum_{m=2}^{\infty} p|a_m| z^m$, $p \geq 1$ *is in* $\mathcal{T}$*, then* $\mathcal{D}^\lambda f \in \mathcal{TK}$ *if and only if* $T_{\lambda,p}[f] \in \mathcal{TFS}^*_H$.

*Proof.* If $T_{\lambda,p}[f] \in \mathcal{TFS}^*_H$, then $T_{\lambda,p}[f]$ is locally univalent. By Lemma 6, $\mathcal{D}^\lambda f \in \mathcal{TK}$. Conversely, suppose that $\mathcal{D}^\lambda f \in \mathcal{TK}$. Note that

$$\mathcal{D}^\lambda f(z) = z - \sum_{m=2}^{\infty} \frac{p(\lambda+1)_{m-1}}{(m-1)!} |a_m| z^m.$$

It follows from Lemma 5 that $D^\lambda f \in \mathcal{TK}$ if and only if (14) holds. By Corollary 12, $T_{\lambda,p}[f] \in \mathcal{TFS}^*_H$.

**Theorem 14.** *Under the hypothesis of Theorem 11, if the condition*

$$\sum_{m=1}^{\infty} \frac{m(pm^2-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(p+1)(m-1)!}|a_m| \leq 1, \quad a_1 = 1, \tag{15}$$

*is satisfied, then* $T_{\lambda,p}[f] \in \mathcal{FK}_H(\alpha)$. *Furthermore, if* $a_m \leq 0$ *for all* $m \geq 2$, *then the condition* (15) *is necessary for* $T_{\lambda,p}[f]$ *to be in* $\mathcal{TFK}_H(\alpha)$.

*Proof.* Following the proof of Theorem 11, substituting $\mathcal{A}_m$ and $\mathcal{B}_m$ from (13), the inequality

$$\sum_{m=1}^{\infty} \left( \frac{m(m-\alpha)}{1-\alpha}|\mathcal{A}_m| + \frac{m(m+\alpha)}{1-\alpha}|\mathcal{B}_m| \right) \leq 2,$$

is equivalent to (15). By using Lemma 4, it follows that $T_{\lambda,p}[f] \in \mathcal{FK}_H(\alpha)$. On the other hand, if $a_m \leq 0$ for all $m \geq 2$, it is straight forward to see that $\mathcal{A}_m \leq 0$ for $m \geq 2$ and $\mathcal{B}_m \geq 0$ for $m \geq 1$. Thus $T_{\lambda,p}[f] \in \mathcal{TFK}_H(\alpha)$ if and only if (15) holds.

**Theorem 15.** *Suppose* $0 \leq \alpha < 1$ *and* $p \geq 1$. *If a function* $f$ *of the form* (6) *is in* $\mathcal{TR}_\lambda(\alpha)$, *then*

*(a)* $\left|(T_{\lambda,p}[f])(z)\right| \leq \frac{2p}{p+1}r + \frac{4p(1-\alpha)}{(p+1)(2-\alpha)}r^2,$

*(b)* $\left|(T_{\lambda,p}[f])(z)\right| \geq \frac{2p}{p+1}r - \frac{4p(1-\alpha)}{(p+1)(2-\alpha)}r^2$

*where* $|z| |= r < 1$. *The results are sharp.*

*Proof.* Using (6) and (2), we obtain

$$\begin{aligned} \left|(T_{\lambda,p}[f])(z)\right| &= \left| z - \sum_{m=2}^{\infty} \frac{(pm+1)}{p+1} \frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m|z^m \right| \\ &\quad + \left| \frac{(1-p)}{p+1}z + \sum_{m=2}^{\infty} \frac{(1-pm)}{p+1} \frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m|z^m \right| \\ &\leq \frac{2p}{p+1}r + \frac{2p}{p+1}\left( \sum_{m=2}^{\infty} m\frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m| \right) r^2 \\ &\leq \frac{2p}{p+1}r + \frac{2p(1-\alpha)}{p+1}\left( \sum_{m=2}^{\infty} \frac{(m-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m| \right) r^2 \\ &\quad + \frac{2p(1-\alpha)\alpha}{(p+1)(2-\alpha)}\left( \sum_{m=2}^{\infty} \frac{(2-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m| \right) r^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2p}{p+1}r + \frac{2p(1-\alpha)}{p+1}r^2 \\
&\quad + \frac{2p(1-\alpha)\alpha}{(p+1)(2-\alpha)}\left(\sum_{m=2}^{\infty}\frac{(m-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m|\right)r^2 \\
&\leq \frac{2p}{p+1}r + \frac{4p(1-\alpha)}{(p+1)(2-\alpha)}r^2,
\end{aligned}$$

by using Lemma 2.

For the other inequality

$$\begin{aligned}
\left|\left(T_{\lambda,p}[f]\right)(z)\right| &\geq r - \sum_{m=2}^{\infty}\frac{(pm+1)}{p+1}\frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m|r^m \\
&\quad - \frac{(p-1)}{p+1}r - \sum_{m=2}^{\infty}\frac{(pm-1)(\lambda+1)_{m-1}}{(p+1)(m-1)!}|a_m|r^m \\
&\geq \frac{2}{p+1}r - \frac{2p}{p+1}\left(\sum_{m=2}^{\infty}\frac{m(\lambda+1)_{m-1}}{(m-1)!}|a_m|\right)r^2 \\
&= \frac{2}{p+1}r - \frac{2p(1-\alpha)}{p+1}\left(\sum_{m=2}^{\infty}\frac{(m-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m|\right)r^2 \\
&\quad - \frac{2p(1-\alpha)\alpha}{(p+1)(2-\alpha)}\left(\sum_{m=2}^{\infty}\frac{(2-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m|\right)r^2 \\
&\geq \frac{2}{p+1}r - \frac{2p(1-\alpha)}{p+1}r^2 \\
&\quad - \frac{2p(1-\alpha)\alpha}{(p+1)(2-\alpha)}\left(\sum_{m=2}^{\infty}\frac{(m-\alpha)(\lambda+1)_{m-1}}{(1-\alpha)(m-1)!}|a_m|\right)r^2 \\
&\geq \frac{2}{p+1}r - \frac{2p(1-\alpha)}{p+1}r^2 - \frac{2p(1-\alpha)\alpha}{(p+1)(2-\alpha)}r^2 \\
&\geq \frac{2}{p+1}r - \frac{4p(1-\alpha)}{(p+1)(2-\alpha)}r^2.
\end{aligned}$$

## 4. Concluding remarks

In this section, we introduce a new operator $T_{\lambda,p,\alpha}$ which is an extension of the operator $T_{\lambda,p}$. We will give some results as remarks which are nice extensions of some of the results in the previous section.

For $f \in \mathcal{A}$, $\lambda \geq 0$, $p > 0$ and $0 \leq \alpha < 1$, we define for $z \in \mathbb{D}$,

$$T_{\lambda,p,\alpha}[f](z) = \frac{\mathcal{D}^\lambda f(z) + p\left(z(\mathcal{D}^\lambda f(z))' - \alpha\mathcal{D}^\lambda f(z)\right)}{1+p(1-\alpha)}$$

$$+\overline{\frac{\mathcal{D}^\lambda f(z) - p\left(z(\mathcal{D}^\lambda f(z))' - \alpha\mathcal{D}^\lambda f(z)\right)}{1+p(1-\alpha)}}.$$

Clearly, we see that $T_{\lambda,p,0}[f] = T_{\lambda,p}[f]$.

**Remark 1.** *Going through the lines of the proof of Lemma 6, we see $T_{\lambda,p,\alpha}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $\mathcal{D}^\lambda f$ is convex of order $\alpha$ in $\mathbb{D}$ for all $p > 0$ and $\lambda \geq 0$.*

**Remark 2.** *Suppose $f(z) = z - \sum_{m=2}^\infty |a_m| z^m \in \mathcal{T}$. Then the operator $T_{\lambda,p,\alpha}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if*

$$\sum_{m=2}^\infty \frac{m(m-\alpha)(\lambda+1)_{m-1}}{(m-1)!}|a_m| \leq (1-\alpha), \quad p > 0, \quad \lambda \geq 0, \quad 0 \leq \alpha < 1. \tag{16}$$

*Proof.* In view of Remark 1, $T_{\lambda,p,\alpha}[f]$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if

$$(\mathcal{D}^\lambda f)(z) = z - \sum_{m=2}^\infty \frac{(\lambda+1)_{m-1}}{(m-1)!}|a_m| z^m,$$

is convex of order $\alpha$. The result now follows from Lemma 5(b).

The reader can also check the corresponding results regarding the operator $T_{\lambda,p,\alpha}$ which are done for $T_{\lambda,p}$ in the previous section.

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