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SOME SUFFICIENT CONDITIONS ON ANALYTIC FUNCTIONS ASSOCIATED WITH POISSON DISTRIBUTION SERIES

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Abstract. The main object of this paper is to obtain some sufficient conditions for the convolution operator $I(m)f(z)$ belonging to the classes $\alpha - UCV(\beta)$ and $\alpha - ST(\beta)$.

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1. Introduction

Let $\mathcal{A}$ be the class of consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1) which are also univalent in U. Further, we denote by T the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

A function $f \in \mathcal{S}$ of the form (1) is said to be starlike of order α, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U,$$

and is said to be convex of order α, if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $C(\alpha)$, respectively, studied by Robertson [12].

In 1997, Bharti *et al.* [2] introduced the following subclasses of analytic univalent functions in the following way
A function f of the form (1) is in $\alpha - ST(\beta)$, if it satisfies the following condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta, \quad \alpha \geq 0, 0 \leq \beta < 1, \tag{3}$$

and $f \in \alpha - UCV(\beta)$ if and only if $zf' \in \alpha - ST(\beta)$.

By specialzing the parameters in $\alpha - UCV(\beta)$ and $\alpha - ST(\beta)$ we obtain the following known subclasses of $\mathcal{S}$ studied earlier by various researchers.

1. $\alpha - UCV(0) \equiv \alpha - UCV$ studied by Kanas and Wisniowska [5]
2. $\alpha - ST(0) \equiv \alpha - ST$ studied by Kanas and Wisniowska [6].
3. $1 - UCV(0) \equiv UCV$ studied by Goodman [3]
4. $1 - ST(0) \equiv SP$ studied by Goodman [4].
5. $0 - UCV(\beta) \equiv C(\beta)$ and $0 - ST(\beta) \equiv S^*(\beta)$ studied by Robertson [12].

Ruscheweyh [13] introduced the operator $D^\mu : \mathcal{A} \to \mathcal{A}$ defined by the Hadamard product

$$D^\mu f(z) = f(z) * \frac{z}{(1-z)^{\mu+1}}, \quad (\mu \geq -1, z \in U), \tag{4}$$

which implies that

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad (n \in N_0 = \{0, 1, 2, \ldots\}).$$

We observe that the power series of $D^\mu f(z)$ for the function f of the form (1) in view of (4) is given by

$$D^\mu f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\mu)}{\Gamma(1+\mu)(n-1)!} a_n z^n, \quad (z \in U).$$

Using the Ruscheweyh derivative Kanas and Yuguchi [7] introduced the class $UR(\mu, \alpha)$ as

$$UR(\mu, \alpha) = \left\{f \in \Re\left\{\frac{z(D^\mu f(z))'}{D^\mu f(z)}\right\} \geq \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z)} - 1\right|, \quad \alpha \geq 0, z \in U\right\}.$$

It is easy to see that $UR(1, \alpha) \equiv \alpha - UCV$ and $UR(0, \alpha) = \alpha - ST$.

The confluent hypergeometric function is given by power series

$$F(a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n(1)_n}z^n,$$

where a, c are complex numbers such that $c\neq 0,-1,-2,\dots$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$\begin{aligned}(a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)}\\ &= \begin{cases}1, & \text{if } n=0\\ a(a+1)\dots(a+n-1), & \text{if } n\in N=\{1,2,3,\dots\}\end{cases}\end{aligned}$$

is convergent for all finite value of z.

Recently, Porwal [9] introduced Poisson distribution series as follows

$$K\left(m,z\right)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n.$$

The convolution (or Hadamard product) of two power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ is defined as the power series

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

Now, we consider the linear operator $I(m):\mathcal{A}\to\mathcal{A}$ defined by

$$\begin{aligned}I(m)f &= K(m,z)*f(z)\\ &= z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_nz^n.\end{aligned}$$

In the present paper, motivated by the results of [9] and on connections between various subclasses of analytic univalent functions by using generalized Bessel functions [10], hypergeometric distribution series [1], Poisson distribution series [8], Confluent hypergeometric series [11], we establish some sufficient conditions for the convolution operator $I(m)f(z)$ belonging to the classes $\alpha-UCV(\beta)$ and $\alpha-ST(\beta)$.

2. Main Results

To establish our main results, we shall require the following lemmas.

Lemma 1. *([7]) Let $0 \leq k < \infty$ and let $f \in \mathcal{A}$ be of the form* (1)*. If $f \in UR(\mu, k)$, then*

$$|a_n| \leq \frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}, \quad n \in N/\{1\}, \tag{5}$$

where $P_1 = P_1(k)$ is the coefficient of z in the function

$$P_k(z) = 1 + \sum_{n=1}^{\infty} P_n(\alpha) z^n, \tag{6}$$

which is the extremal function for the class $UR(\mu, k)$ by the range of the expression $1 + \frac{zf''(z)}{f'(z)}$, $(z \in U)$, where $P_1 = P_1(k)$ is given as above above by (6)*.*

Lemma 2. *([7]) Let $f \in \mathcal{A}$ be of the form* (1)*. If*

$$\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)]|a_n| \leq 1 - \beta, \tag{7}$$

then $f \in \alpha - ST(\beta)$.

Lemma 3. *([7]) Let $f \in \mathcal{A}$ be of the form* (1)*. If*

$$\sum_{n=2}^{\infty} n[n(1+\alpha) - (\alpha+\beta)]|a_n| \leq 1 - \beta, \tag{8}$$

then $f \in \alpha - UCV(\beta)$.

Theorem 4. *If $m > 0$, $f \in UR(\mu, k)$ and the inequality*

$$(1+\alpha)\frac{P_1 m}{1+\mu} F(P_1+1; 2+\mu; m) + (1-\beta)F(P_1; 1+\mu; m) \leq (1-\beta)(e^m + 1), \tag{9}$$

is satisfied, then $I(m)f \in \alpha - ST(\beta)$.

Proof. Let f be of the form (1) belong to the class $UR(\mu, k)$. To show that $I(m)f \in \alpha - ST(\beta)$, we have to prove that

$$\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 1 - \beta.$$

Since $f \in UR(\mu, k)$, then by Lemma 1, we have

$$|a_n| \leq \frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}, \quad n \in N.$$

Now

$$
\begin{aligned}
T_1 &= \sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n| \\
&\leq \sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)} \\
&= e^{-m}\sum_{n=2}^{\infty}[(1+\alpha)(n-1)+(1-\beta)]\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)} \\
&= e^{-m}\left[(1+\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}+(1-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right] \\
&= e^{-m}\left[(1+\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}+(1-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right] \\
& e^{-m}\left[(1+\alpha)\frac{P_1 m}{1+\mu}F(P_1+1;2+\mu;m)+(1-\beta)\left(F(P_1;1+\mu;m)-1\right)\right] \\
&\leq 1-\beta
\end{aligned}
$$

by the given hypothesis.

This completes the proof of Theorem 4.

Theorem 5. *If $m>0$, $f\in UR(\mu,k)$ and the inequality*

$$
(1+\alpha)\frac{P_1(P_1+1)m^2}{(1+\mu)(2+\mu)}F(P_1+2;3+\mu;m)+(3+2\alpha-\beta)\frac{P_1 m}{1+\mu}F(P_1+1;2+\mu;m)+(1-\beta)F(P_1;1+\mu;m)\leq(1-\beta)(e^m+1), \quad (10)
$$

is satisfied, then $I(m)f\in\alpha-UCV(\beta)$.

Proof. Let f be of the form (1) belong to the class $UR(\mu,k)$. To show that $I(m)f\in\alpha-UCV(\beta)$, we have to prove that

$$
\sum_{n=2}^{\infty}n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|\leq 1-\beta.
$$

Since $f\in UR(\mu,k)$, then by Lemma 1, we have

$$
|a_n|\leq\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}, \quad n\in N.
$$

Now

$$T_2 = \sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|$$
$$\leq \sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}$$
$$= e^{-m}\sum_{n=2}^{\infty}[(1+\alpha)(n-1)(n-2)+(3+2\alpha-\beta)(n-1)+(1-\beta)]\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}$$
$$= e^{-m}\left[(1+\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)} + (3+2\alpha-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right.$$
$$\left. +(1-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right]$$
$$= e^{-m}\left[(1+\alpha)\frac{P_1(P_1+1)m^2}{(1+\mu)(2+\mu)}F(P_1+2;3+\mu;m)+(3+2\alpha-\beta)\frac{P_1 m}{1+\mu}F(P_1+1;2+\mu;m)\right.$$
$$\left. +(1-\beta)\left(F(P_1;1+\mu;m)-1\right)\right]$$
$$\leq 1-\beta$$

by the given hypothesis.

Thus the proof of Theorem 5 is established.

Theorem 6. *If* $m>0$*,* $f\in UR(\mu,k)$ *then* $G(m,z)=\int_0^z \frac{I(m)f(t)}{t}dt$ *is in* $\alpha-UCV(\beta)$ *if* (9) *is satisfied.*

Proof. It is easy to see that

$$G(m,z) = z + \sum_{n=2}^{\infty}\frac{m^{n-1}}{n!}e^{-m}z^n.$$

To show that $G(m,z)\in\alpha-UCV(\beta)$, we have to prove that

$$\sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{n!}e^{-m}|a_n| \leq 1-\beta.$$

Since $f\in UR(\mu,k)$, then by Lemma 1, we have

$$|a_n| \leq \frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}, \quad n\in N.$$

Now

$$
\begin{aligned}
T_3 = &\sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{n!}e^{-m}|a_n| \\
&\le \sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]\frac{m^{n-1}}{n!}e^{-m}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)} \\
&= e^{-m}\sum_{n=2}^{\infty}[(1+\alpha)(n-1)+(1-\beta)]\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)} \\
&= e^{-m}\left[(1+\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}+(1-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right] \\
&= e^{-m}\left[(1+\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}+(1-\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\frac{(P_1)_{n-1}\Gamma(1+\mu)}{\Gamma(n+\mu)}\right] \\
&e^{-m}\left[(1+\alpha)\frac{P_1 m}{1+\mu}F(P_1+1;2+\mu;m)+(1-\beta)\left(F(P_1;1+\mu;m)-1\right)\right] \\
&\le 1-\beta
\end{aligned}
$$

by the given hypothesis.

This completes the proof of Theorem 6.

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