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COEFFICIENT ESTIMATES FOR BI-CONCAVE FUNCTIONS OF SAKAGUCHI TYPE

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Abstract. In this study, a new class $\mathcal{CS}_{\Sigma}^{p,q}(s,t,\alpha)$ of analytic and bi-concave functions with Sakaguchi type in the open unit disc were presented. The estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found for functions belonging to this class.

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1. Introduction, Preliminaries and Definition

Let $\mathbb{C}$,$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{R}$ denote the set of complex numbers, the extended complex plain and the set of real numbers respectively. Let $\mathbb{D}$ denote the open unit disk. Let $\mathcal{A}$ indicate the class of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

normalized by the condition $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the set of all normalized analytic functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

A univalent function $f : \mathbb{D} \to \overline{\mathbb{C}}$ is called concave when $f(\mathbb{D})$ is concave, i.e. $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex. Concave univalent functions have already been studied in detail by several authors (see [1, 2, 3, 4, 7]).

A function $f : \mathbb{D} \to \mathbb{C}$ is called a member of concave univalent functions with an opening angle $\pi\alpha$ at infinity for $\alpha \in (1, 2]$ if f satisfies the conditions given below:

1. f is analytic in $\mathbb{D}$ which has normalization condition $f(0) = 0 = f'(0) - 1$. Additionaly, $f(1) = \infty$.

2. f maps $\mathbb{D}$ conformally onto a set whose complement is convex with respect to $\mathbb{C}$.

3. The opening angle of $f(\mathbb{D})$ at infinity is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

Let us denote the class of concave univalent functions of order β by $C_\beta(\alpha)$.

The analytic characterization for functions in $C_\beta(\alpha)$ are as follows : For $\alpha \in (1, 2]$ and $\beta \in [0, 1)$, $f \in C_\beta(\alpha)$ if and only if

$$Re P_f(z) > \beta, \qquad \forall z \in \mathbb{D}, \tag{2}$$

for

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)}\right] \quad and \quad f(0) = 0 = f'(0) - 1.$$

Also each $f \in C_\beta(\alpha)$ has the Taylor expansion given by (1). Especially, for $\beta = 0$, we can obtain the class of concave univalent functions $C_0(\alpha)$ which was studied in [2]. The closed set $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex and unbounded for $f \in C_0(\alpha)$, $\alpha \in (1, 2]$.

Now we define the class of concave functions with Sakaguchi type and order β by $CS_\beta(s, t, \alpha)$ as follows:
For $\alpha \in (1, 2]$, $\beta \in [0, 1)$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$, $f \in CS_\beta(s, t, \alpha)$ if and only if

$$Re P_f(z) > \beta, \qquad \forall z \in \mathbb{D}, \tag{3}$$

for

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - \frac{(s - t)(zf'(z))'}{(f(sz) - f(tz))'}\right].$$

It is obvious that $CS_\beta(1, 0, \alpha) \equiv C_\beta(\alpha)$.

For all $f \in \mathcal{S}$, the Koebe 1/4 theorem [8] confirms that the image of $\mathbb{D}$ under each univalent function $f \in \mathcal{S}$ covers a disk of radius 1/4. Hence, each $f \in \mathcal{A}$ has an inverse f^{-1}, described by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

If f is univalent and $g = f^{-1}$ is univalent in $\mathbb{D}$, the function $f \in \mathcal{A}$ is known to be bi-univalent in $\mathbb{D}$. If f given by (1) is bi-univalent, then $g = f^{-1}$ can be arranged in the form of Taylor expansion given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots \ . \tag{4}$$

Also, a function f is bi-concave if both f and f^{-1} are concave.

Let us denote Σ the class of all bi-univalent functions in $\mathbb{D}$. Lewin [10] investigated the class Σ and showed that $|a_2| < 1.51$ for the function $f(z) \in \Sigma$. Also, several researchers obtained the coefficient boundaries for $|a_2|$ and $|a_3|$ of bi-univalent functions for some subclasses of the class Σ in [9, 12, 13]. In addition, certain subclasses of bi-univalent functions, and also univalent functions consisting of strongly starlike, starlike and convex functions were studied by Brannan and Taha [5]. Some properties of bi-convex, bi-univalent and bi-starlike function classes have already been investigated by Brannan and Taha [5]. Furthermore, estimations for $|a_2|$ and $|a_3|$ were found by Bulut [6] for bi-starlike functions. The class of bi-concave functions was studied by Sakar and Güney in [11].

Now, we define the definition of bi-concave functions of Sakaguchi type as follows:

Definition 1. The function f in (1) is called $\sum_{CS_\beta(s,t,\alpha)}$ if the conditions given below are satisfied: $f \in \Sigma$ and

$$Re\left\{\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - \frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right]\right\} > \beta \quad , z \in \mathbb{D} \ and \ 0 \leq \beta < 1 \tag{5}$$

and

$$Re\left\{\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - \frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right]\right\} > \beta \quad , w \in \mathbb{D} \, and \, 0 \leq \beta < 1. \tag{6}$$

where g is given by (4) and $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$. In other words, $\sum_{CS_\beta(s,t,\alpha)}$ is the class of bi-concave functions of Sakaguchi type and order β.

It is obvious that $\sum_{CS_\beta(1,0,\alpha)} \equiv \sum_{C_\beta(\alpha)}$ (see [11]).

We next define the following subclass of $\mathcal{A}$, analogous to the definition given by Xu et al. [14].

Definition 2. Let us define the functions $p, q : \mathbb{D} \to \mathbb{C}$ satisfying the following condition

$$\min\{Re(p(z)), Re(q(z))\} > 0 \quad (z \in \mathbb{D}) \ and \ p(0) = q(0) = 1.$$

Also let the function f, defined by (1.1), be in $\mathcal{A}$. Then $f \in \mathcal{CS}_{\Sigma}^{p,q}(s,t,\alpha)$ if the following conditions are satisfied: $f \in \Sigma$ and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - \frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right] \in p(\mathbb{D}),\ (z \in \mathbb{D}) \tag{7}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - \frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right] \in q(\mathbb{D}),\ (w \in \mathbb{D}) \tag{8}$$

where the g is given in (4) and $s,t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$.

Remark. If we let

$$p(z) = \frac{1+(1-2\beta)z}{1-z} \qquad \textit{and} \qquad q(z) = \frac{1-(1-2\beta)z}{1+z} \qquad (0 \leq \beta < 1, z \in \mathbb{D}) \tag{9}$$

in the class $\mathcal{CS}_{\Sigma}^{p,q}(s,t,\alpha)$ then we have $\sum_{CS_{\beta}(s,t,\alpha)}$.

The aim of this paper is to estimate the initial coefficients for the bi-concave functions of Sakaguchi type in $\mathbb{D}$.

2. Initial Coefficient Boundary for $|a_2|$ and $|a_3|$

The estimations of initial coefficients for the class $\mathcal{CS}_{\Sigma}^{p,q}(s,t,\alpha)$ of bi-concave functions of Sakaguchi type are presented in this section.

Theorem 1. If the function $f(z)$ given by (1) is in $\mathcal{CS}_{\Sigma}^{p,q}(s,t,\alpha)$ then

$$|a_2| \leq \min\left\{\sqrt{\frac{1}{|[4-2u_2]|^2}\left\{(\alpha+1)^2 + \frac{(\alpha^2-1)}{2}[|p'(0)|+|q'(0)|] + \frac{(\alpha-1)^2}{8}[|p'(0)|^2+|q'(0)|^2]\right\}}\right.$$

$$\left. ;\sqrt{\frac{1}{|4(9-3u_3)-8u_2(4-2u_2)|}\left\{\frac{(\alpha-1)}{2}[|p''(0)|+|q''(0)|]+4(\alpha+1)\right\}}\right\} \tag{10}$$

and

$$|a_3| \leq \min\left\{\frac{(\alpha+1)^2}{|4-2u_2|^2} + \frac{(\alpha-1)}{8|9-3u_3|}(|p''(0)|+|q''(0)|)\right.$$

$$+\frac{(\alpha^2-1)}{2|4-2u_2|^2}(|p'(0)|+|q'(0)|) + \frac{(\alpha-1)^2}{8|4-2u_2|^2}(|p'(0)|^2+|q'(0)|^2)$$

$$;\frac{4}{|4(9-3u_3)-8u_2(4-2u_2)|}\times$$

$$\left[(\alpha+1)+\frac{(\alpha-1)}{4|9-3u_3|}(|(9-3u_3)-u_2(4-2u_2)||p''(0)|+|u_2(4-2u_2)||q''(0)|)\right]\Bigg\} \tag{11}$$

where $u_n=\sum_{k=1}^{n}s^{n-k}t^{k-1}$, $s,t\in\mathbb{C}$ with $s\neq t$, $|t|\leq 1$.

Proof. Firstly, we can write the argument inequalities in 7 and 8 in their equivalent forms as follows:

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right]=p(z) \qquad (z\in\mathbb{D}), \tag{12}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-\frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right]=q(w) \qquad (w\in\mathbb{D}). \tag{13}$$

In addition, $p(z)$ and $q(w)$ can be expended to Taylor-Maclaurin series as given below respectively

$$p(z)=1+p_1z+p_2z^2+...$$

and

$$q(w)=1+q_1w+q_2w^2+\cdot\quad.$$

Now upon equating the coefficients of $\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right]$ with those of $p(z)$ and the coefficients of $\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-\frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right]$ with those of $q(w)$, we can write $p(z)$ and $q(w)$ as follows.

$$p(z)=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right]=1+p_1z+p_2z^2+p_3z^3+... \tag{14}$$

and

$$q(w)=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-\frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right]=1+q_1w+q_2w^2+q_3w^3+... \quad .(15)$$

Since

$$\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}=\frac{1+\sum_{n=2}^{\infty}n^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}na_nu_nz^{n-1}}$$
$$=1+[4-2u_2]a_2z+([9-3u_3]a_3-2u_2[4-2u_2]a_2^2)z^2+...$$

where $u_n=\sum_{k=1}^{n}s^{n-k}t^{k-1}$ and $\frac{1+z}{1-z}=1+2\sum_{n=1}^{\infty}z^n=1+2z+2z^2+2z^3+...$ we obtain that

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z}-\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right]$$

$$= \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2} - 1 + (\alpha+1)z + (\alpha+1)z^2 + \ldots - [4-2u_2]a_2 z - ([9-3u_3]a_3 - 2u_2[4-2u_2]a_2^2)z^2 + \ldots\right]$$

$$= \frac{2}{(\alpha-1)}\left[\frac{(\alpha-1)}{2} + ((\alpha+1) - [4-2u_2]a_2)z + ((\alpha+1) - ([9-3u_3]a_3 - 2u_2[4-2u_2]a_2^2)z^2 + \ldots\right]$$

$$= 1 + \frac{2[(\alpha+1) - [4-2u_2]a_2]}{(\alpha-1)}z + \frac{2[(\alpha+1) - [9-3u_3]a_3 + 2u_2[4-2u_2]a_2^2]}{(\alpha-1)}z^2 + \ldots \quad .$$

Then

$$p_1 = \frac{2[(\alpha+1) - [4-2u_2]a_2]}{(\alpha-1)} \tag{16}$$

and

$$p_2 = \frac{2[(\alpha+1) - [9-3u_3]a_3 + 2u_2[4-2u_2]a_2^2]}{(\alpha-1)} \quad . \tag{17}$$

From (4) and (6), we have

$$\frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'} = \frac{1 - 4a_2w + 9(2a_2^2 - a_3)w^2 + \ldots}{1 - 2u_2a_2w + 3u_3(2a_2^2 - a_3)w^2}$$

$$= 1 + [2u_2 - 4]a_2w + \left[(9-3u_3)(2a_2^2 - a_3) + 2u_2(2u_2-4)a_2^2\right]w^2 + \ldots$$

where $u_n = \sum_{k=1}^{n} s^{n-k}t^{k-1}$ and we know $\frac{1-w}{1+w} = 1 + 2\sum_{n=1}^{\infty}(-1)^n w^n = 1 - 2w + 2w^2 - 2w^3 + \ldots$. Then from $q(w)$ given by (15), we have

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1-w}{1+w} - \frac{(s-t)(wg'(w))'}{(g(sw)-g(tw))'}\right] = \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2} - (\alpha+1)w + (\alpha+1)w^2 - \ldots\right.$$

$$\left. -1 - [2u_2-4]a_2w - \left[(9-3u_3)(2a_2^2 - a_3) + 2u_2(2u_2-4)a_2^2\right]w^2 + \ldots\right]$$

$$= 1 - \frac{2[(\alpha+1) + [2u_2-4]a_2]}{(\alpha-1)}w + \frac{2[(\alpha+1) - \left[(9-3u_3)(2a_2^2 - a_3) + 2u_2(2u_2-4)a_2^2\right]]}{(\alpha-1)}w^2 + \ldots \quad .$$

So we can obtain q_1 and q_2 as follows

$$q_1 = -\frac{2[(\alpha+1) + [2u_2-4]a_2]}{(\alpha-1)} \tag{18}$$

$$q_2 = \frac{2[(\alpha+1) - \left[(9-3u_3)(2a_2^2 - a_3) + 2u_2(2u_2-4)a_2^2\right]]}{(\alpha-1)} \quad . \tag{19}$$

From (16) and(18) we obtain

$$p_1 = -q_1 \tag{20}$$

and

$$a_2^2 = \frac{(\alpha+1)^2}{[4-2u_2]^2} - \frac{(\alpha^2-1)}{2[4-2u_2]^2}[p_1 - q_1] + \frac{(\alpha-1)^2}{8[4-2u_2]^2}[p_1^2 + q_1^2]$$

or

$$a_2^2 = \frac{1}{[4-2u_2]^2}\left\{(\alpha+1)^2 - \frac{(\alpha^2-1)}{2}[p_1-q_1] + \frac{(\alpha-1)^2}{8}[p_1^2+q_1^2]\right\}. \tag{21}$$

Also, from (17) and (19) we obtain that

$$a_2^2 = \frac{(1-\alpha)}{[4(9-3u_3)-8u_2(4-2u_2)]}[p_2+q_2] + \frac{4(\alpha+1)}{[4(9-3u_3)-8u_2(4-2u_2)]}$$

or

$$a_2^2 = \frac{1}{[4(9-3u_3)-8u_2(4-2u_2)]}\{(1-\alpha)[p_2+q_2]+4(\alpha+1)\} \quad . \tag{22}$$

Therefore, we find from (21) and (22)

$$|a_2|^2 = \frac{1}{|[4-2u_2]|^2}\left\{(\alpha+1)^2 + \frac{(\alpha^2-1)}{2}[|p'(0)|+|q'(0)|] + \frac{(\alpha-1)^2}{8}[|p'(0)|^2+|q'(0)|^2]\right\}.$$

and

$$|a_2|^2 = \frac{1}{|4(9-3u_3)-8u_2(4-2u_2)|}\left\{\frac{(\alpha-1)}{2}[|p''(0)|+|q''(0)|]+4(\alpha+1)\right\} \quad .$$

So we obtain the upper bound of $|a_2|$ as stated in (10).

Now, to obtain the upper bound for the coefficient $|a_3|$ we use (17) and (19). So we obtain

$$(\alpha-1)(p_2-q_2) = 4[9-3u_3]a_2^2 - 4[9-3u_3]a_3.$$

From (21), we find

$$4[9-3u_3]a_3 = -(\alpha-1)(p_2-q_2) + \frac{4[9-3u_3]}{[4(9-3u_3)-8u_2(4-2u_2)]}\{(1-\alpha)[p_2+q_2]+4(\alpha+1)\}$$

or

$$a_3 = -\frac{(\alpha-1)}{4[9-3u_3]}(p_2-q_2) + \frac{1}{[4(9-3u_3)-8u_2(4-2u_2)]}\{(1-\alpha)[p_2+q_2]+4(\alpha+1)\}$$

$$\Rightarrow a_3 = \frac{4(\alpha+1)}{[4(9-3u_3)-8u_2(4-2u_2)]}$$

$$-\frac{8(\alpha-1)}{4[9-3u_3]\,[4(9-3u_3)-8u_2(4-2u_2)]}[[(9-3u_3)-u_2(4-2u_2)]p_2+[u_2(4-2u_2)]q_2]. \tag{23}$$

We thus find that

$$|a_3| \leq \frac{4}{|4(9-3u_3)-8u_2(4-2u_2)|}\times$$
$$\left[(\alpha+1)+\frac{(\alpha-1)}{4|9-3u_3|}(|(9-3u_3)-u_2(4-2u_2)||p''(0)|+|u_2(4-2u_2)||q''(0)|)\right].$$

Also, we obtain from (22)

$$4[9-3u_3]a_3 = -(\alpha-1)(p_2-q_2)+\frac{4[9-3u_3]}{[4-2u_2]^2}\left\{(\alpha+1)^2-\frac{(\alpha^2-1)}{2}[p_1-q_1]+\frac{(\alpha-1)^2}{8}[p_1^2+q_1^2]\right\}$$

$$\Rightarrow a_3 = \frac{(\alpha+1)^2}{[4-2u_2]^2}-\frac{(\alpha-1)}{4[9-3u_3]}(p_2-q_2)-\frac{(\alpha^2-1)}{2[4-2u_2]^2}(p_1-q_1)+\frac{(\alpha-1)^2}{8[4-2u_2]^2}(p_1^2+q_1^2) \quad . \tag{24}$$

We thus find that

$$|a_3| \leq \frac{(\alpha+1)^2}{|4-2u_2|^2}+\frac{(\alpha-1)}{8|9-3u_3|}(|p''(0)|+|q''(0)|)+\frac{(\alpha^2-1)}{2|4-2u_2|^2}(|p'(0)|+|q'(0)|)+\frac{(\alpha-1)^2}{8|4-2u_2|^2}(|p'(0)|^2+|q'(0)|^2) \quad .$$

So, the proof of Theorem 1 is completed.

If we set

$$p(z)=\frac{1+(1-2\beta)z}{1-z} \qquad \text{and} \qquad q(z)=\frac{1-(1-2\beta)z}{1+z} \qquad (0\leq\beta<1, z\in\mathbb{D})$$

in Theorem 1, we can obtain the following corollary.

Corollary 1. Let f given by (1) be in the class $\sum_{CS_\beta(s,t,\alpha)}$ $(0\leq\beta<1)$. Then

$$|a_2| \leq \sqrt{\frac{4\{(\alpha-1)(1-\beta)+(\alpha+1)\}}{|4(9-3u_3)-8u_2(4-2u_2)|}}$$

and

$$|a_3| \leq \frac{4}{|4(9-3u_3)-8u_2(4-2u_2)|}\times$$
$$\left[(\alpha+1)+\frac{(\alpha-1)}{|9-3u_3|}(|(9-3u_3)-u_2(4-2u_2)|+|u_2(4-2u_2)|)(1-\beta)\right]$$

where $u_n=\sum_{k=1}^{n} s^{n-k}t^{k-1}$, $s,t\in\mathbb{C}$ with $s\neq t$, $|t|\leq 1$.

Last of all, if we take $s=1$ and $t=0$ in Theorem 1 and Corollary 1, we can obtain Theorem 2.1 and Corollary 3.1 in [11] respectively.

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