Acta Universitatis Apulensis No. 60/2019
ISSN: 1582-5329 pp. 61-73
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.60.05

ON Q-STARLIKE FUNCTIONS WITH RESPECT TO K-SYMMETRIC POINTS

A. Alsoboh and M. Darus

Abstract. In this paper, we define new subclass of analytic functions, the so-called q-starlike functions of order α with respect to k-symmetric points. We explore some inclusion properties and find some sufficient condition for this class. Finally, we obtain the integral representation for functions belonging to this class.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Analytic function, Univalent function, q-Starlike function, k-Symmetric points, q-calculus.

1. Introduction

We begin by letting $\mathcal{H}$ the class of analytic functions in the open unit disc of the complex plane $\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ containing all functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \qquad z \in \mathbb{U}, \tag{1}$$

which satisfying the condition of normalization; $f^{'}(0) = f(0)+1 = 1$. Let $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ containing of all functions that are univalent in $\mathbb{U}$. For any two analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there exist a Schwarz function $\omega(z)$ with $\omega(0) = 0$, $|\omega(z)| \leq 1$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$ [14].

The convolution of $f(z)$ as in (1) and $\beta(z) = z + \sum_{m=2}^{\infty} \phi_m z^m$ is defined by

$$(f * \beta)(z) = (\beta * f)(z) = z + \sum_{m=2}^{\infty} a_m \phi_m z^m.$$

The geometric properties of analytic functions played an important role in geometric function theory, such as convexity and starlikeness, these subclasses denoted by $\mathcal{C}$

and $\mathcal{S}^*$, respectively.

More generally, for $0 \leq \alpha \leq 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of starlike of order α and convex of order α, respectively, defined analytically by

$$\begin{aligned} \mathcal{S}^*(\alpha) &= \left\{ f : f \in \mathcal{A}, and \;\; Re\{\frac{zf^{'}(z)}{f(z)}\} > \alpha, z \in \mathbb{U} \right\}, \\ \mathcal{C}(\alpha) &= \left\{ f : f \in \mathcal{A}, and \;\; Re\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\} > \alpha, z \in \mathbb{U} \right\}. \end{aligned}$$

The application of q-calculus is very important in the theory of analytic functions. Jackson was 1^{st} developed q-calculus in a systematic way (for more details, see [10, 11]). There are several application of q-calculus on subclasses of analytic functions, especially subclasses of univalent functions in $\mathbb{U}$ like stalike and convex (for more details, see [1, 2, 3, 4, 7, 5, 16, 17]) that depends on replacing the usual derivative by q-derivative. Ismail et al. [9] introduced a general q-starlike function with replacing the right half plane by appropriate domains, Agrawal and Sahoo in [1] extend this idea to introduce the class of q-starlike functions of order α. Later on, Aldweby and Darus [4] introduced two subclasses of bounded q-starlike and q-convex functions. Some other application of q-calculus are studied by Alsoboh and Darus [6, 7, 8] and Mohammed and Darus [15].

Now, we give some basic concepts and definitions of the applications of q-calculus assuming that $0 < q < 1$, by:

Definition 1. *[10] For $0 < q < 1$, the q-numbers $[m]_q$ is given by:*

$$[m]_q = \begin{cases} \frac{1-q^n}{1-q} & , n \in \mathbb{C} \\ 1 + q + q^2 + ... + q^{n-1} & , n \in \mathbb{N} \end{cases},$$

and $\lim_{q \to 1^-} [m]_q = m$.

Definition 2. *[10] The Jackson q-derivative of a function f is given by:*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z-qz} & (z \in \mathbb{C} \backslash \{0\}) \\ f^{'}(0) & (z = 0) \end{cases},$$

where $\lim_{q \to 1} \partial_q f(z) = f^{'}(z)$.

Definition 3. *[11] The Jackson's q-integral of a function f is given by:*

$$\int_0^z f(t)d_qt = (1-q)z\sum_{n=0}^{\infty} q^n f(q^n z).$$

In case of $f(z) = z^m, m \in \mathbb{N}$, we have

$$\partial_q(z^m) = [m]_q z^{m-1},$$

$$\int_0^z t^{c-1}d_qt = (1-q)z\sum_{n=0}^{\infty}(zq^n)^{c-1}q^n = \frac{z^c}{[c]_q}.$$

Ismail and et al. in [9] introduced the class of q-starlike functions and the definition of class $\mathcal{S}_q^*$ is given as follows:

Definition 4. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$, if*

$$\left|\frac{z(\partial_q f)(z)}{f(z)} - \frac{1}{1-q}\right| < \frac{1}{1-q} \qquad (z \in \mathbb{U}). \tag{2}$$

If $q \to 1^-$ then $\mathcal{S}_q^*$ reduced to $\mathcal{S}^*$.

Later, Agrawal and Sahoo in [1] defined and investigated the subclass of generalized q-starlike functions of order α. The definition is as follows:

Definition 5. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*(\alpha)$, $0 \leq \alpha < 1$, if*

$$\left|\frac{\frac{z(\partial_q f)(z)}{f(z)} - \alpha}{1-\alpha} - \frac{1}{1-q}\right| < \frac{1}{1-q}, \quad (z \in \mathbb{U}).$$

If $\alpha = 0$, then $\mathcal{S}_q^*(\alpha) := \mathcal{S}_q^*$.

The authors in [6] introduced a q-differential operator $\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z) : \mathcal{A} \to \mathcal{A}$ by

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z) = z + \sum_{m=2}^{\infty}\left(\mathbf{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\right)^n a_m z^m \tag{3}$$

where

$$\mathbf{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m) = (\kappa-\lambda)(\delta-\mu)([m]_q-1)+1, \quad (\delta,\kappa,\lambda,\mu \geq 0, \kappa > \lambda, \delta > \mu, n \in \mathbb{N}_0).$$

Next, we introduce new subclass of q-starlike of order α with respect to k-symmetric points using the differential operator $\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f$ given as follows:

Definition 6. *A function $f \in \mathcal{A}$ is said to in the class $\mathcal{S}_q^{*(k)}(n,\alpha)$, if it satisfies the following inequality*

$$\left| \frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q}, \quad (z \in \mathbb{U}), \tag{4}$$

where $0 \leq \alpha < 1$, $n \in \mathbb{N}_0$, k is a fixed positive integer and f_k is defined by the equality

$$f_k(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-v} f(\varepsilon^v z), \qquad (\varepsilon^k = 1). \tag{5}$$

We observe that the class $\mathcal{S}_q^{*(k)}(n,\alpha)$ satisfies the following relation:

$$\bigcap_{0<q<1} \mathcal{S}_q^{*(k)}(n,\alpha) \subset \bigcap_{0<q<1} \mathcal{S}_q^{*}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*.$$

Throughout this paper, we will assuming that $0 \leq \alpha < 1$, $0 < q < 1$ and $\theta \in [0, 2\pi)$.

2. The main results

First, we need the following lemma of Liu [13].

Lemma 1. *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then we have*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Next, we give some meaningful conclusion about the class $\mathcal{S}_q^{*(k)}(n,\alpha)$.

Theorem 2. *If $f \in \mathcal{A}$ as in (1). Then $f \in \mathcal{S}_q^{*(k)}(n,\alpha)$ if and only if it satisfies the following subordination condition*

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} \prec \frac{1+(1-\alpha(1+q))\,z}{1-qz}, \tag{6}$$

where f_k as in (5).

Proof. Suppose that $f \in \mathcal{S}_q^{(k)}(n,\alpha)$, then by Definition 6, we have

$$\left| \frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q}$$

Consider $I(z) = \frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)}$, then

$$\left|\frac{1-q}{1-\alpha}I(z) - \frac{1-\alpha q}{1-\alpha}\right| < 1.$$

We can introduce the function $\Phi(z)$ by

$$\Phi(z) = \frac{(1-q)I(z) + \alpha q - 1}{1-\alpha}, \qquad (z \in \mathbb{U}, \quad |\Phi(z)| < 1).$$

Now, define the function $\omega(z)$, by

$$\omega(z) = \frac{\Phi(z) - \Phi(0)}{1 - \Phi(z)\overline{\Phi(0)}} = \frac{I(z) - 1}{1 - \alpha(1+q) + qI(z)}. \tag{7}$$

We note that $\omega(0) = 0$, and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$.

From the last equation, we have

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} = \frac{1 + \Big(1 - \alpha(1+q)\Big)\omega(z)}{1 - q\omega(z)},$$

this implies that

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} \prec \frac{1 + \Big(1 - \alpha(1+q)\Big)z}{1 - qz}.$$

Conversely, by assuming the equation (6) holds, then there exist a Schwarz function $\omega(z)$, such that

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} = \frac{1 + (1 - \alpha(1+q))\omega(z)}{1 - q\omega(z)}.$$

It is equivalent to

$$\begin{aligned}
\left|\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q}\right| &= \left|\frac{1 + (1 - \alpha(1+q))\omega(z)}{1 - q\omega(z)} - \frac{1-\alpha q}{1-q}\right| \\
&= \frac{1-\alpha}{1-q}\left|\frac{\omega(z) - q}{1 - q\omega(z)}\right| \\
&\leq \frac{1-\alpha}{1-q},
\end{aligned}$$

hence $f \in \mathcal{S}^{*(k)}_q(n,\alpha)$ and the proof is complete.

Theorem 3. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1$, then we have $\mathcal{S}_q^{(k)}(n, \alpha_2) \subset \mathcal{S}_q^{(k)}(n, \alpha_1)$.*

Proof. Suppose that $f \in \mathcal{S}_q^{(k)}(n, \alpha_2)$, by Theorem 2, we have

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} = \frac{1+\Big(1-\alpha_2(1+q)\Big)\omega(z)}{1-q\omega(z)}.$$

Since $\alpha_1 \leq \alpha_2$, this leads to $1-\alpha_2(1+q) \leq 1-\alpha_2(1+q)$ and

$$\frac{1+\Big(1-\alpha_2(1+q)\Big)\omega(z)}{1-q\omega(z)} < \frac{1+\Big(1-\alpha_1(1+q)\Big)\omega(z)}{1-q\omega(z)}.$$

By Lemma 1, we have

$$\frac{1+\Big(1-\alpha_2(1+q)\Big)z}{1-q\omega(z)} \prec \frac{1+\Big(1-\alpha_1(1+q)\Big)\omega(z)}{1-q\omega(z)},$$

this means that $\mathcal{S}_q^{*(k)}(n, \alpha_2) \subset \mathcal{S}_q^{*(k)}(n, \alpha_1)$ and hence the proof is complete.

Theorem 4. *Let $f \in \mathcal{S}_q^{*(k)}(n, \alpha)$, then $\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k \in \mathcal{S}_q^{*}(\alpha)$.*

Proof. Since $f \in \mathcal{S}_q^{*(k)}(n, \alpha)$, then by Definition 6, we have

$$\Big|\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q}\Big| < \frac{1-\alpha}{1-q}.$$

Then substituting z by $\varepsilon^\gamma z$ where $\gamma = 0, 1, ..., k-1$, in the last inequality, we have

$$\Big|\frac{\varepsilon^\gamma z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(\varepsilon^\gamma z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(\varepsilon^\gamma z)} - \frac{1-\alpha q}{1-q}\Big| < \frac{1-\alpha}{1-q}, \quad (\gamma = 0, 1, 2, ..., k-1). \tag{8}$$

According to the definition of f_k and $\varepsilon^k = 1$, we have $f_k(\varepsilon^\gamma z) = \varepsilon^\gamma f_k(z)$ and summing the last equation for $\gamma = 0, 1, 2, ..., k-1$, we can get

$$\Big|\frac{1}{k}\sum_{\gamma=0}^{k-1}\frac{\varepsilon^\gamma z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(\varepsilon^\gamma z)}{\varepsilon^\gamma(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q}\Big| < \frac{1-\alpha}{1-q}.$$

Note that

$$\frac{1}{k}\sum_{\gamma=0}^{k-1}\frac{\varepsilon^\gamma z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(\varepsilon^\gamma z)}{\varepsilon^\gamma(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} = \frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(\varepsilon^\gamma z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)},$$

therefore,

$$\Big|\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} - \frac{1-\alpha q}{1-q}\Big| < \frac{1-\alpha}{1-q},$$

hence $f \in \mathcal{S}^*_q(\alpha)$.

Theorem 5. *Let f be defined as in (1). If for $0 \le \alpha < 1$, and*

$$\sum_{m=2}^{\infty}(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(mk+1))^n([mk+1]_q-\alpha)|a_{mk+1}| + \sum_{\substack{m=2\\ m\neq lk+1}}^{\infty}[m]_q(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m))^n|a_m| \le (1-\alpha) \tag{9}$$

then $f \in \mathcal{S}^{*(k)}_q(n,\alpha)$.

Proof. Suppose that f and $f_k(z)$ is defined by (1) and (5), respectively. For $z \in \mathbb{U}$, we have

$$M = \Big|(1-q)z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z) - (1-\alpha q)(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)\Big| - (1-\alpha)\Big|(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)\Big|$$

$$\begin{aligned}
M &= \Big|(1-q)z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z) - (1-\alpha q)(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)\Big| - (1-\alpha)\Big|(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)\Big| \\
&\le q(1-\alpha)r + \sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n a_m\Big\{(1-q)[m]_q - (1-\alpha q)c_m\Big\}r^m \\
&\quad - (1-\alpha)r + (1-\alpha)\sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n a_m c_m r^m \\
&< -(1-q)(1-\alpha)r + \sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n a_m\Big\{(1-q)[m]_q - (1-\alpha q)c_m\Big\}r^m \\
&\quad + (1-\alpha)\sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n a_m c_m r^m \\
&< \Big\{\sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n a_m(1-q)[m]_q - \alpha(1-q)c_m - (1-q)(1-\alpha)\Big\}r
\end{aligned}$$

therefore,

$$M < (1-q)\left[\sum_{m=2}^{\infty}\Big(\boldsymbol{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\Big)^n\Big([m]_q - \alpha c_m\Big)|a_m| - (1-\alpha)\right]. \tag{10}$$

From definition of c_m, we know that

$$c_m = \sum_{v=0}^{k-1} \varepsilon^{(m-1)v} = \begin{cases} 1 & , if m = lk+1 \\ o & , if m \neq lk+1 \end{cases} \qquad (k, l \geqslant 1, m \geqslant 2). \tag{11}$$

Substituting (11) into (10), we have

$$M < (1-q)\Big[\sum_{m=2}^{\infty} (\mathbf{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(mk+1))^n([mk+1]_q - \alpha)|a_{mk+1}| + \sum_{\substack{m=2 \\ m\neq lk+1}}^{\infty} [m]_q(\mathbf{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m))^n|a_m| - (1-\alpha)\Big].$$

From inequality (5), we know that $M < 0$, then the proof is complete.

Theorem 6. *The function f of the form (1) is in the class $\mathcal{S}_q^{*(k)}(n,\alpha)$ if and only if*

$$\frac{e^{i\theta}(e^{-i\theta} - q)}{z}\Big[\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f(z) * \Big(\frac{z - N(1-z)(1-qz)h(z)}{(1-z)(1-qz)}\Big)\Big] \neq 0, \tag{12}$$

for all $N = \frac{(e^{-i\theta} + [1-\alpha(1+q)])}{e^{-i\theta} - q}$, $0 \le \theta \le 2\pi$, $z \in \mathbb{U}$.

Proof. Let $f \in \mathcal{S}_q^{*(k)}(n,\alpha)$ of the form (1), then it satisfies the equality (6), i.e

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)} \prec \frac{1 + (1-\alpha(1+q))\,z}{1-qz}.$$

Since $\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)}$ is analytic in $\mathbb{U}$, this means $(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z) \neq 0, z \in \mathbb{U}^*$, i.e $\frac{1}{z}(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z) \neq 0, z \in \mathbb{U}$, according to (6), then by definition of subordination, there exist a Schwarz function $\omega(z)$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ such that

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)} = \frac{1 + \Big(1-\alpha(1+q)\Big)\omega(z)}{1 - q\omega(z)}, \quad z \in \mathbb{U}$$

which is equivalent to

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)} \neq \frac{1 + \Big(1-\alpha(1+q)\Big)e^{i\theta}}{1 - qe^{i\theta}}, \quad (z \in \mathbb{U}; 0 \le \theta \le 2\pi), \tag{13}$$

or

$$\frac{1}{z}\left[\left(1-qe^{i\theta}\right)z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)-\left(1+[1-\alpha(1+q)]e^{i\theta}\right)(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)\right]\neq 0,$$

$$(z\in\mathbb{U};0\leq\theta\leq 2\pi). \tag{14}$$

And from the definition of $f_k(z)$, we know

$$f_k(z)=z+\sum_{m=2}^{\infty}a_m c_m z^m=(f*h)(z),$$

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)=z+\sum_{m=2}^{\infty}\left(\mathbf{\Delta}_{[\kappa,\lambda,\delta,\mu;q]}(m)\right)^n a_m c_m z^m=(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f*h)(z) \tag{15}$$

where $h(z)=z+\sum_{m=2}^{\infty}c_m z^m$, for

$$c_m=\begin{cases}1 & if\ \ m=lk+1,\\ o & if\ \ m\neq lk+1.\end{cases}$$

And also

$$f(z)=f(z)*\frac{z}{1-z}\quad and\quad z\partial_q f(z)=f(z)*\frac{z}{(1-z)(1-qz)}$$

this implies

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z)=\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z)*\frac{z}{1-z} \tag{16}$$

$$z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z))=\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z)*\frac{z}{(1-z)(1-qz)}. \tag{17}$$

Now, substitute (16) and (17) into (15), we have

$$\frac{1}{z}\left[\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z)*\left(\frac{(1-qe^{i\theta})z}{(1-z)(1-qz)}-(1+[1-\alpha(1+q)]e^{i\theta})h(z)\right)\right]\neq 0,$$

$$\frac{e^{i\theta}(e^{-i\theta}-q)}{z}\left[\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z)*\left(\frac{z}{(1-z)(1-qz)}-\frac{(e^{-i\theta}+[1-\alpha(1+q)])h(z)}{e^{-i\theta}-q}\right)\right]\neq 0$$

which lead to (12), which proves the 'if' part.

Conversely, because the assumption (12) holds for all N, it follows that $\frac{1}{z}.\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f(z) \neq 0$, hence the function $I(z) = \frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)}$ is analytic in $\mathbb{U}$, and we shown in 'if' part that the assumption (13) is equivalent to

$$I(z) \neq \frac{1 + \Big(1 - \alpha(1+q)\Big)e^{i\theta}}{1 - qe^{i\theta}} \qquad (z \in \mathbb{U}; 0 \leq \theta \leq 2\pi). \tag{18}$$

If we denote by

$$\Gamma(z) = \frac{1 + \Big(1 - \alpha(1+q)\Big)z}{1 - qz} \qquad (z \in \mathbb{U}),$$

the relation (18) shows that $I(\mathbb{U}) \cap \Gamma(\mathbb{U}) = \phi$. Thus, the simply-connected domain $I(\mathbb{U})$ is included in $\mathbb{C}\backslash\Gamma(\partial\mathbb{U})$. Since $\Gamma(z)$ is univalent and $I(0) = \Gamma(0)$, then $I(z) \prec \Gamma(z)$ which represent the relation (6), hence $f \in \mathcal{S}^{*(k)}_q(n, \alpha)$.

3. The q-Integral Representation

In this section, we give the q-integral representation of functions f for the class $\mathcal{S}^{*(k)}_q(n, \alpha)$.

Theorem 7. *Let $f \in \mathcal{S}^{*(k)}_q(n, \alpha)$, then we have*

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k = z.\exp\left\{\frac{log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z}\frac{1 + \Big(1 - \alpha(1+q)\Big)\omega(t)}{t(1 - q\omega(t))}d_qt\right\}$$

where f_k is defined in (5)*, $\omega(z)$ is analytic with $\omega(0) = 0$, $|\omega(z)| < 1$.*

Proof. Let $f \in \mathcal{S}^{*(k)}_q(n, \alpha)$, then by Theorem 1, we have

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(z)} = \frac{1 + \Big(1 - \alpha(1+q)\Big)\omega(z)}{1 - q\omega(z)},$$

where $\omega(z)$ is analytic with $\omega(0) = 0$ and $|\omega(z)| < 1$, substituting z by $\varepsilon^\gamma z$ ($\gamma = 0, 1, ..., k-1$)

$$\frac{\varepsilon^\gamma z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(\varepsilon^\gamma z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f_k)(\varepsilon^\gamma z)} = \frac{1 + \Big(1 - \alpha(1+q)\Big)\omega(\varepsilon^\gamma z)}{1 - q\omega(\varepsilon^\gamma z)},$$

we know that $f_k(\varepsilon^\gamma z) = \varepsilon^\gamma f_k(z)$, and summing for $\gamma = 0, 1, ..., k-1$

$$\frac{z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)} = \frac{1}{k}\sum_{\gamma=0}^{k-1} \frac{1+\Big(1-\alpha(1+q)\Big)\omega(\varepsilon^\gamma z)}{1-q\omega(\varepsilon^\gamma z)}. \tag{19}$$

From the last equality we have

$$\frac{\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)}{(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\gamma=0}^{k-1} \frac{1+\Big(1-\alpha(1+q)\Big)\omega(\varepsilon^\gamma z)}{z(1-q\omega(\varepsilon^\gamma z))}.$$

Apply Jackson's q-integral, we have

$$\frac{q-1}{\log\ q}\log\left\{\frac{\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k}{z}\right\} = \frac{1}{k}\sum_{\gamma=0}^{k-1}\int_0^z \frac{1+\Big(1-\alpha(1+q)\Big)\omega(\varepsilon^\gamma \zeta)}{\zeta(1-q\omega(\varepsilon^\gamma \zeta))} d_q\zeta,$$

$$\log\left\{\frac{\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k}{z}\right\} = \frac{\log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z} \frac{1+\Big(1-\alpha(1+q)\Big)\omega(t)}{t(1-q\omega(t))} d_q t,$$

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k = z.exp\left\{\frac{\log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z} \frac{1+\Big(1-\alpha(1+q)\Big)\omega(t)}{t(1-q\omega(t))} d_q t\right\}.$$

Theorem 8. *Let $f \in \mathcal{S}_q^{*(k)}(n,\alpha)$, then we have*

$$\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f(z) = \int_0^z exp\left\{\frac{\log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z} \frac{1+\Big(1-\alpha(1+q)\Big)\omega(t)}{t(1-q\omega(t))} d_q t\right\}$$
$$\times \frac{1+\Big(1-\alpha(1+q)\Big)\omega(\zeta)}{1-q\omega(\zeta)} d_q\zeta, \tag{20}$$

where f_k is defined in (5), *$\omega(z)$ is analytic with $\omega(0) = 0$, $|\omega(z)| < 1$.*

Proof. From Theorem 2, we have

$$z\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f)(z) = (\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda} f_k)(z).\frac{1+\Big(1-\alpha(1+q)\Big)\omega(z)}{1-q\omega(z)},$$

$$\partial_q(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z) = exp\left\{\frac{\log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z}\frac{1+\Big(1-\alpha(1+q)\Big)\omega(t)}{t(1-q\omega(t))}d_qt\right\}$$

$$\times\frac{1+\Big(1-\alpha(1+q)\Big)\omega(z)}{1-q\omega(z)}.$$

Apply q-Jackson's integral of both sides to get

$$(\mathsf{D}^n_{q,\mu,\delta,\kappa,\lambda}f)(z) = \int_0^z exp\left\{\frac{\log\ q}{(q-1)k}\sum_{\gamma=0}^{k-1}\int_0^{\varepsilon^\gamma z}\frac{1+\Big(1-\alpha(1+q)\Big)\omega(t)}{t(1-q\omega(t))}d_qt\right\}$$

$$\times\frac{1+\Big(1-\alpha(1+q)\Big)\omega(\zeta)}{1-q\omega(\zeta)}d_q\zeta.$$

Acknowledgements. The work presented here was partially supported by UKM grant:GUP-2019-032.

References

[1] S. Agrawal, S. K. Sahoo, *A generalization of starlike functions of order alpha*, Hokkaido Mathematical Journal 46, 1 (2017), 15-27.

[2] B. Ahmad, M. Farooq, R. Khan, *On A Subclass of Analytic Close-to-Convex Functions in q-Analogue Associated with Janowski Functions*, J. Classical Anal. 13, 1 (2018), 83-93.

[3] O. P. Ahuja, A. Çetinkaya, Y. Polatoglu, *Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions*, J. of Computational Analysis and Applications 26,1 (2019).

[4] H. Aldweby, M. Darus, *Coefficient estimates of classes of q-starlike and q-convex functions*, Advanced Studies in Contemporary Mathematics 26, 1 (2016), 21-26.

[5] S. Elhaddad, M. Darus, H. Aldweby, *On certain subclasses of analytic functions involving differential operator*, *Jñānābha* 48 (2018), 53-62.

[6] A. Alsoboh, M. Darus, *New Subclass of Analytic Functions Defined by q-Differential Operator with Respect to k-Symmetric Points*, International Journal of Mathematics and Computer Science 14, 3 (2019). 761-773.

[7] A. Alsoboh and M. Darus, *On Fekete-Szegö Problems for Certain Subclasses of Analytic Functions Defined by Differential Operator Involving q-Ruscheweyh Operator*, Journal of function space 2019 (2019), Article ID: 8459405, 8 pages.

[8] A. Alsoboh, M. Darus, *On Fekete-Szego problem associated with q-derivative operator*, J. Phys.: Conf. Ser. 1212, 2019 (2019) , 012003.

[9] M. E. H. Ismail, E.Merkes, D. Styer, *A generalization of starlike functions*, Complex Variables, Theory and Application: An International Journal 14,(1-4) 1990, 77-84.

[10] F. H. Jackson, *On q-functions and a certain difference operator*, Earth and Environmental Science Transactions of the Royal Society of Edinburgh. 46,2 (1909), 253-281.

[11] F. H. Jackson, *On q-difference integrals*, Quart. J. Pure Appl. Math. 41, (1910), 193-203.

[12] O. Kwon, Y. Sim, *A Certain Subclass of Janowski Type Functions associated with k-Symmetric Points*, Communications of the Korean Mathematical Society 28, 1 (2013), 143-154.

[13] M. S. Liu, *On a class of p-valent close-to-convex functions of order β and type α*, J. Math. Study 30, 1 (1997), 102-104.

[14] S. S. Miller and P. T. Mocanu, *Differential subordinations: theory and applications*, CRC Press. 2000.

[15] A. Mohammed, M. Darus, *A generalized operator involving the q-hypergeometric function*, Matematički vesnik 65, 4 (2014), 454-465.

[16] K. I. Noor, S. Riaz, *Generalized q-starlike functions*, Studia Scientiarum Mathematicarum Hungarica 54, 4 (2017), 509-522.

[17] T. M. Seoudy, M. K. Aouf, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Inequal. 10, 1 (2016), 135-145.

Abdullah Alsoboh
Department of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi, Selangor, Malaysia
email: *p92712@siswa.edu.ukm.my*

Maslina Darus (Corresponding author)
Department of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi, Selangor, Malaysia
email: *maslina@ukm.edu.my*