ALMOST CONFORMAL RICCI SOLITONS ON ALMOST COKÄHLER MANIFOLDS

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Abstract. The object of the present paper is to classify almost conformal Ricci solitons, almost conformal Ricci solitons with the potential vector field is collinear to the reeb vector field ξ and finally, almost conformal gradient Ricci solitons on almost CoKähler manifolds with ξ belongs to (k, μ)-nullity distribution. In this paper, we prove that such manifolds with V is contact vector field and $Q\phi = \phi Q$ is η-Einstein and it is Einstein when the potential vector field is pointwise collinear to the reeb vectoer field ξ. We also derive so many delightful results. Moreover, we prove that a (k, μ)-almost CoKähler manifolds admitting almost conformal gradient Ricci solitons is isometric to a sphere.

1. Introduction

In 1982, R. S. Hamilton [18] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [6]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein ([8],[9]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [14] who discusses some aspects of it. Recently, the notion of almost Ricci soliton have introduced [24] by Piagoli, Riogoli, Rimoldi and Setti.

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([10], [11], [16], [19], [20], [21], [28], [29]) and many others.

In [15], during 2003-2004, Fischer developed the notion of conformal Ricci flow which is a generalization of the classical Ricci flow. The conformal Ricci flow on a $2n + 1$-dimensional smooth closed connected oriented manifold M is defined by the following equation:

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{2n+1}) = -pg \tag{3}$$

and $r(g) = -1$,

where p is a scalar non-dynamical field which depends on time, $r(g)$ is the scalar curvature of the manifold.

In 2015, Basu and Bhattacharyya [2] introduced the concept of conformal Ricci soliton by the equation

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{4}$$

where λ is constant. Conformal Ricci soliton is the generalization of Ricci soliton.

Pigola et al. first introduced [24] the notion of almost Ricci soliton in 2010. In 2014, Sharma has also studied [26] the almost Ricci soliton and has also done some gloriuos research works. Recently, in 2018, Ghosh and Patra also have been studied [17] the almost Ricci solitons on contact geometry. In Riemannian manifold (M, g), almost Ricci soliton is defiend by the equation

$$\pounds_V g + 2S = 2\lambda g, \tag{5}$$

where λ is a smooth function on M. The almost Ricci soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative.

Recently in [13], Dutta, Basu and Bhattacharyya have been introduced the notion of almost conformal Ricci soliton by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{6}$$

where λ is a smooth function on M. The almost conformal Ricci soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative. An almost conformal Ricci soliton is called conformal Ricci soliton if λ is constant. An almost conformal Ricci soliton is said to be almost conformal gradient Ricci soliton if the potential vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = Df$, where D is the gradient operator of g on M^{2n+1}. For convanience, we denote $(M^{2n+1}, g, Df, \lambda)$ as a almost conformal gradient Ricci soliton with potential function f.

In [1], Barros and Ribeiro proved that a compact almost Ricci soliton with constant scalar curvature is isometric to an Euclidean sphere. In this connection, a theorem has also been proved by Wang, Gomes and Xia in [27] for k-almost Ricci soiton which is given as follows:

Theorem 1. *[27] Let $(M^n, g, V, \beta, \lambda), n \geq 3$ be a non-trivial β-almost Ricci soliton with constant scalar curvature r. If M^n is compact, then it is isometric to a standard sphere $S^n(c)$ of radius $c = \sqrt{\frac{2n(2n+1)}{r}}$.*

The above Theorem will be used in later to prove our results.

In the present paper, after introduction, we study almost CoKähler manifolds. In section 3, we characterize almost conformal Ricci solitons on almost CoKähler manifolds and prove several important results. In the next section we study almost conformal Ricci solitons on almost CoKähler manifolds with the potential vector field is pointwise collinear to the reeb vector field ξ. Finally, in section 5, we consider almost conformal gradient Ricci solitons on almost CoKähler manifolds.

2. Almost CoKähler manifolds

In the present section, we give some well known definitions and basic formulae on Almost CoKaehler manifolds which will be very useful in the next sections. An almost contact structure on a $(2n + 1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1, 1)$-type tensor field, ξ is a global vector field and η is a 1-form satisfying ([3], [4])

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{7}$$

Here also holds

$$\phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{8}$$

If an almost contact manifold admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

for any vector fields X, Y, then the manifold is called an almost contact metric metric manifold. In such a manifold we can define a fundamental 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y), \tag{10}$$

for any vector fields X, Y. An almost contact metric manifold is said to be an almost CoKähler manifold if both η and Φ are closed. That is, $d\eta = 0$ and $d\Phi = 0$. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be normal if the almost complex structure J on $M \times \mathcal{R}$ defined by (pp. 80 of [4])

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a real valued function defined on $M \times \mathcal{R}$, is integrable. Moreover, if an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is normal, then it is said to be a CoKähler manifold. In addition an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is CoKähler if and only if $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$.

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost CoKähler manifold. Let us consider two operators h and l which are defined by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(., \xi)\xi$, where R denotes the curvature tensor and $\pounds$ is the Lie differentiation. These operators are symmetric of type $(1, 1)$ and satisfies ([7], [12] [22]) the following

$$h\xi = h'\xi = 0, \qquad \text{Tr}\, h = \text{Tr}\, h' = 0, \qquad h\phi = -\phi h, \tag{11}$$

where $h' = h \cdot \phi$. Also in an almost CoKähler manifold, we have ([7], [12] [22])

$$\nabla_X\xi = h'X = h\phi X, \tag{12}$$

$$\phi l\phi - l = 2h^2, \tag{13}$$

for any vector fields X.

A (k, μ)-contact metric manifold is a generalization of Sasakian and K-contact manifold. In [5] Blair, Koufogiorgos and Papantoniou introduced and studied the notion of (k, μ)-nullity distribution on contact metric manifolds $M^{2n+1}(\phi, \xi, \eta, g)$. A contact metric manifold M^{2n+1} whose curvature tensor satisfies

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

for all vector fields X, Y on M^{2n+1}, where $h = \frac{1}{2}\pounds_\xi\phi$ ($\pounds$ denotes the Lie derivative of ϕ along ξ and $k, \mu \in \mathbb{R}$ is known as (k, μ)-contact manifold and ξ is said to belongs to the (k, μ)-nullity distribution. Several authors have studied ([23], [25]) the (k, μ)-contact metric manifold and obtain some interesting results. When k, μ are smooth functions, it is said to be the generalized (k, μ)-nullity distribution. Thus we have the following:

Definition 1. *An almost CoKähler manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a (k, μ)-almost CoKähler manifold if ξ satisfies the equation*

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \qquad (14)$$

for all vector fields $X, Y \in \chi(M^{2n+1})$ and k, μ are real constants.

In a consequence of (14), we have $l = -k\phi^2 + \mu h$. In view of this, from (13) we deduce

$$h^2 = k\phi^2 \qquad (15)$$

and also we obtain

$$S(X, \xi) = 2nk\eta(Y), \qquad (16)$$

$$Q\xi = 2nk\xi. \qquad (17)$$

Definition 2. *([17]) A vector field V on a contact manifold is said to be a contact vector field if it preserve the contact form η, that is*

$$\pounds_V \eta = \psi\eta, \qquad (18)$$

for some smooth function ψ on M. When $\psi = 0$ on M, the vector field V is called a strict contact vector field.

Now we state a well known Lemma:

Lemma 2. *(Poincare Lemma): In a Riemannian manifold $d^2 = 0$, where d is the exterior differential operator, that is,*

$$g(\nabla_X \mathrm{grad}\zeta, Y) = g(\nabla_Y \mathrm{grad}\zeta, X), \qquad (19)$$

for any two vector fields X, Y and for any smooth function ζ.

3. Almost conformal Ricci solitons on (k, μ)-Almost CoKähler manifolds

In this section we characterize almost conformal Ricci solitons on almost CoKähler manifolds with the potential vector field is a contact vector field. Then we obtain

$$
\begin{aligned}
(\pounds_V d\eta)(X, Y) &= \pounds_V d\eta(X, Y) - d\eta(\pounds_V X, Y) - d\eta(X, \pounds_V Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \phi\pounds_V Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \pounds_V \phi Y - (\pounds_V \phi)Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \pounds_V \phi Y) + g(X, (\pounds_V \phi)Y) \\
&= (\pounds_V g)(X, \phi Y) + g(X, (\pounds_V \phi)Y), \qquad (20)
\end{aligned}
$$

for any vector fields X and Y on M. Then using (6) in (20) we get

$$
\begin{aligned}
(\pounds_V d\eta)(X,Y) &= -2S(X,\phi Y) + [2\lambda - (p + \frac{2}{2n+1})]g(X,\phi Y) \\
&\quad + g(X, (\pounds_V \phi)Y),
\end{aligned}
\tag{21}
$$

for any vector fields X and Y on M. As V is a contact vector field, from (18) we have

$$
\pounds_V d\eta = d\pounds_V \eta = (d\psi) \wedge \eta + \psi(d\eta),
\tag{22}
$$

from which it follows that

$$
(\pounds_V d\eta)(X,Y) = \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X, \phi Y).
\tag{23}
$$

for any vector fields X and Y on M. In view of (21) and (23) we infer

$$
\begin{aligned}
&-2S(X,\phi Y) + [2\lambda - (p + \frac{2}{2n+1})]g(X,\phi Y) + g(X, (\pounds_V \phi)Y) \\
&= \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X, \phi Y)
\end{aligned}
\tag{24}
$$

and hence we get

$$
\begin{aligned}
2(\pounds_V \phi)Y &= 4Q\phi Y + 2[\psi - 2\lambda + (p + \frac{2}{2n+1})]\phi Y \\
&\quad + \eta(Y)D\psi - d\psi(Y)\xi.
\end{aligned}
\tag{25}
$$

for any vector field Y on M. Substituting $Y = \xi$ in (25) yields

$$
2(\pounds_V \phi)\xi = D\psi - (\xi\psi)\xi.
\tag{26}
$$

The equation (6) can be exhibited as

$$
g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]G(X,Y),
\tag{27}
$$

for any vector fields X and Y on M. Tracing the above equation gives

$$
2\operatorname{div} V = -[2\lambda - (p + \frac{2}{2n+1})]2r + (2n+1)[2\lambda - (p + \frac{2}{2n+1})].
\tag{28}
$$

Let Ω be the volume form of M, that is,

$$
\Omega = \eta \wedge (d\eta)^n \neq 0.
\tag{29}
$$

Taking Lie derivative of the foregoing equation along the vector field V and applying the formula $\mathcal{L}_V \Omega = (\mathrm{div}V)\Omega$ and using (18) and (22) yields

$$(\mathrm{div}V)\Omega = (n+1)\psi\Omega, \tag{30}$$

and hence

$$\mathrm{div}V = (n+1)\psi. \tag{31}$$

With help of (28), from (31) it follows that

$$r = -(n+1)\psi + (n+\frac{1}{2})[2\lambda - (p+\frac{2}{2n+1})]. \tag{32}$$

The equivalent form of almost conformal Ricci soliton equation is given by

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) = [2\lambda - (p+\frac{2}{2n+1})]g(X,Y), \tag{33}$$

for any vector fields X and Y on M. Putting $X = Y = \xi$ in the last equation and using (16) we get

$$2g(\mathcal{L}_V \xi, \xi) = 4nk - 2\lambda + (p+\frac{2}{2n+1})]. \tag{34}$$

Replacing Y by ξ in the equation (33) and then using (16) and (18) we obtain

$$\mathcal{L}_V \xi = [\psi - 2\lambda + (p+\frac{2}{2n+1}) + 4nk]\xi, \tag{35}$$

for any vector fields X and Y on M. Making use of (35) we find

$$(\mathcal{L}_V \phi) = 0. \tag{36}$$

Applying (36) in (26) we have

$$D\psi = (\xi\psi)\xi, \tag{37}$$

from which it follows that

$$d\psi(Y) = (\xi\psi)\xi \tag{38}$$

and hence

$$d\psi = (\xi\psi)\eta. \tag{39}$$

Taking exterior derivative of (39) and using (19) we get

$$d(\xi\psi) \wedge \eta + (\xi\psi)d\eta = 0. \tag{40}$$

Taking wedge product of (40) with η gives

$$(\xi\psi)\eta \wedge d\eta = 0. \tag{41}$$

As $\eta \wedge (d\eta)^n$ is the volume element, then $\eta \wedge d\eta \neq 0$ and hence

$$\xi\psi = 0. \tag{42}$$

With help of (42), from (39) we have

$$d\psi = 0 \tag{43}$$

and hence ψ becomes a constant. Integrating (31) and applying the **Divergence theorem** we infer

$$\psi = 0. \tag{44}$$

Therefore, V becomes a strict contact vector field. Thus we are in a position to state the following:

Definition 3. *Let M^{2n+1} be a (k,μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is a contact vector field, then V is a strict contact vector field.*

By the virtue of (34) and (35) we get

$$2\psi = 2\lambda - (p + \frac{2}{2n+1}) - 4nk \tag{45}$$

which implies that

$$2\lambda = (p + \frac{2}{2n+1}) + 4nk. \tag{46}$$

Case I: For $k = 0$, then $\lambda > 0$ and the almost conformal Ricci soliton is shrinking.
Case II: For $k > 0$, then $\lambda > 0$ and the almost conformal Ricci soliton is shrinking.
Case III: For $k < 0$, if $p + \frac{2}{2n+1} > 4nk$, then $\lambda > 0$ and the almost conformal Ricci soliton is shrinking.
Case IV: For $k < 0$, if $p + \frac{2}{2n+1} < 4nk$, then $\lambda < 0$ and the almost conformal Ricci soliton is expanding.
Case V: For $p + \frac{2}{2n+1} + 4nk = 0$, then $\lambda = 0$ and the almost conformal Ricci soliton is steady.
Hence we can conclude the following:

Theorem 3. *Let M^{2n+1} be a (k,μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. Then the following relations holds:*

(a) For $k = 0$, the almost conformal Ricci soliton is shrinking.

(b) For $k > 0$, the almost conformal Ricci soliton is shrinking.

(c) For $k < 0$ and $p + \frac{2}{2n+1} > 4nk$, the almost conformal Ricci soliton is shrinking.

(d) For $k < 0$ and $p + \frac{2}{2n+1} < 4nk$, the almost conformal Ricci soliton is expanding.

(e) For $p + \frac{2}{2n+1} + 4nk = 0$ the almost conformal Ricci soliton is steady.

By the help of (45) and using the fact that ψ is constant, the equation (25) reduces to

$$2(\pounds_V \phi)Y = 4Q\phi Y + [-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\phi Y, \tag{47}$$

for any vector field Y on M. Also using (45) we have

$$2\pounds_V \eta = [2\lambda - (p + \frac{2}{2n+1}) - 4nk]\eta. \tag{48}$$

Now we have

$$(\pounds_V \phi)Y = \pounds_V \phi Y - \phi(\pounds_V Y) \tag{49}$$

Substituting $Y = \phi Y$ in (49) we obtain

$$(\pounds_V \phi)\phi Y = -\pounds_V Y + \pounds_V \eta(Y)\xi + \eta(Y)\pounds_V \xi - \phi(\pounds_V \phi Y). \tag{50}$$

Operating ϕ on (49) we get

$$\phi(\pounds_V \phi)Y = \phi(\pounds_V \phi Y) + \pounds_V Y - \eta(\pounds_V Y)\xi. \tag{51}$$

On addition of (50) and (51) we obtain

$$\phi(\pounds_V \phi)Y + (\pounds_V \phi)\phi Y = (\pounds_V \eta)(Y)\xi + 2\eta(Y)\pounds_V \xi. \tag{52}$$

Multiplying both sides of (52) by 2 and using (18) and (45)

$$2\phi(\pounds_V \phi)Y + 2(\pounds_V \phi)\phi Y = 0. \tag{53}$$

Making use of (47) in above equation

$$4\phi Q\phi Y + [-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\phi^2 Y$$

$$+4Q\phi^2 Y + [-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\phi^2 Y = 0. \tag{54}$$

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Let us assume that $Q\phi = \phi Q$. Then using (17) we deduce

$$
\begin{aligned}
QY &= \frac{1}{4}[2\lambda - (p + \frac{2}{2n+1}) + 4nk]Y \\
&+ \frac{1}{4}[-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\eta(Y)\xi,
\end{aligned}
\tag{55}
$$

which shows that the manifold is η-Einstein and hence we can state that

Theorem 4. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold with $Q\phi = \phi Q$. If g is an almost conformal Ricci soliton with potential vector field V such that V is contact vector field, then the manifold is η-Einstein.*

Taking covarient derivative of (55) with respect to an arbitrary vector field X we obtain

$$
\begin{aligned}
(\nabla_X Q)Y &= \frac{1}{2}(X\lambda)Y - \frac{1}{2}(X\lambda)\eta(Y)\xi \\
&+ \frac{1}{4}[-2\lambda + (p + \frac{2}{2n+1}) - 4nk]g(h'X, Y)\xi \\
&+ \frac{1}{4}[-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\eta(Y)h'X.
\end{aligned}
\tag{56}
$$

Inner product of (56) with Z entails

$$
\begin{aligned}
g((\nabla_X Q)Y, Z) &= \frac{1}{2}(X\lambda)g(Y, Z) - \frac{1}{2}(X\lambda)\eta(Y)\eta(Z) \\
&+ \frac{1}{4}[-2\lambda + (p + \frac{2}{2n+1}) - 4nk]g(h'X, Y)\eta(Z) \\
&+ \frac{1}{4}[-2\lambda + (p + \frac{2}{2n+1}) - 4nk]\eta(Y)g(h'X, Z).
\end{aligned}
\tag{57}
$$

Contracting X, Z and Y, Z in the preceeding equation respectively we get

$$
Yr = Y\lambda - (\xi\lambda)\eta(Y)
\tag{58}
$$

and

$$
Xr = n(X\lambda).
\tag{59}
$$

In view of (58) and (59) we infer

$$
(1 - n)(X\lambda) = (\xi\lambda)\eta(X).
\tag{60}
$$

Putting $X = \xi$ in (60) we have

$$
\xi\lambda = 0.
\tag{61}
$$

In a consequence of (61), from (60) we get

$$X\lambda = 0, \tag{62}$$

which implies that λ is constant, provided $n > 1$ and hence the almost conformal Ricci soliton becomes a conformal Ricci soliton. Thus we have the following:

Theorem 5. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V such that V is a contact vector field and $Q\phi = \phi Q$. Then the almost conformal Ricci solitons becomes a conformal Ricci soliton.*

Using the fact that λ is constant, from (59) we have

$$Xr = 0, \tag{63}$$

which implies that r is costant. Then by the virtue of (63) and the **Theorem 1.1** we can state the following:

Theorem 6. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V such that V is a contact vector field and $Q\phi = \phi Q$. Then the manifold is isometric to a sphere $S^{2n+1}(c)$ of radius $c = \sqrt{\frac{2(2n+1)(4n+3)}{r}}$.*

4. Almost conformal Ricci solitons on almost CoKähler manifolds with potential vector field is pointwise collinear to ξ

This section is devoted to study the almost conformal Ricci solitons on almost CoKähler manifolds with potential vector field is pointwise collinear to the reeb vector field ξ. Then we have

$$V = \rho\xi, \tag{64}$$

where ρ is a smooth function on M.
Taking covariant derivative of (64) with respect to an arbitrary vector field X we find

$$\nabla_X V = (X\rho)\xi + \rho h' X. \tag{65}$$

Applying (65) on (27) we get

$$(X\rho)\eta(Y) + \rho g(h'X, Y) + (Y\rho)\eta(X) + \rho g(X, h'Y) + 2S(X, Y)$$
$$= [2\lambda - (p + \frac{2}{2n+1})]g(X, Y). \tag{66}$$

Replacing Y by ξ in (66) we infer

$$X\rho + (\xi\rho)\eta(X) + 4nk\eta(X) = [2\lambda - (p + \frac{2}{2n+1})]\eta(X). \tag{67}$$

Substituting $X = \xi$ in the above equation we obtain

$$\xi\rho = \lambda - \frac{1}{2}(p + \frac{2}{2n+1}) - 4nk. \tag{68}$$

Using (68) in (67) we infer

$$X\rho = [\lambda - \frac{1}{2}(p + \frac{2}{2n+1}) + 4nk]\eta(X) \tag{69}$$

from which it follows that

$$d\rho = [\lambda - \frac{1}{2}(p + \frac{2}{2n+1}) + 4nk]\eta. \tag{70}$$

Taking exterior differentiation of (70) and using (19) yields

$$(d\lambda) \wedge \eta + [\lambda - \frac{1}{2}(p + \frac{2}{2n+1})]d\eta = 0. \tag{71}$$

Taking wedge product of (71) with η gives

$$2\lambda = (p + \frac{2}{2n+1}) - 8nk. \tag{72}$$

Therefore, we can conclude the following:

Theorem 7. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is pointwise collinear to the reeb vector field ξ, then the almost conformal Ricci solitons is shrinking, steady or expanding according as $p + \frac{2}{2n+1}$ is greater than, equal to or less than $8nk$.*

Making use of (72) in (69) we get

$$X\rho = 0, \tag{73}$$

for any vector field X on M from which shows that ρ is constant. Thus we have the following:

Theorem 8. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is pointwise collinear to the reeb vector field ξ, then V is costant multiple of ξ.*

Since, ρ is constant, then from (66) we have

$$2\rho g(h'X, Y) + 2S(X, Y) = 2\lambda - (p + \frac{2}{2n+1})g(X, Y), \tag{74}$$

for any vector fields X, Y on M. Let us assume that $X \in [\lambda]' = \{X : h'X = \lambda X\}$, that is, λ is the eigen value of h'. Then the foregoing equation assigns

$$S(X, Y) = [\lambda(1 - \rho) - (p + \frac{2}{2n+1})]g(X, Y), \tag{75}$$

for all vector fields X and Y on M which entails that the manifold is Einstein. Then we can state our next theorem as follows:

Theorem 9. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is pointwise collinear to the reeb vector field ξ, then the manifold is Einstein.*

Again, taking wedge product of (71) with $d\eta$ returns

$$d\lambda = 0, \tag{76}$$

which determines λ is constant. Thus we are in a position to state that

Theorem 10. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is pointwise collinear to the reeb vector field ξ, then the almost conformal Ricci solitons becomes conformal Ricci solitons.*

Contracting X and Y in (75) yields

$$r = \lambda - (n + \frac{1}{2})p - 1, \tag{77}$$

which ensures that r is constant, as λ is constant. Hence following the **Theorem 1.1** we can conclude that

Theorem 11. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact (k, μ)-almost CoKähler manifold admitting almost conformal Ricci solitons with potential vector field V. If V is pointwise collinear to the reeb vector field ξ and any vector field X on M belongs to $[\lambda]'$, then the manifold is isometric to the sphere $S^{2n+1}(c)$ of radius $c = \sqrt{\frac{2(2n+1)(4n+3)}{r}}$.*

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5. Almost conformal gradient Ricci solitons on almost CoKählar manifolds

This section deals with the study of an almost conformal gradient Ricci soliton on almost CoKählar manifolds. Then we have

$$V = Df, \tag{78}$$

where f is a smooth function on M and D denotes the gradient operator. Then using **Poincare Lemma**, from the equation (27) we get

$$\nabla_X Df = [\lambda - \frac{1}{2}(p + \frac{2}{2n+1})]X - QX, \tag{79}$$

for any vector field X on M. Taking covariant derivative of (79) with respect to an arbitrary vector field Y

$$\nabla_Y \nabla_X Df = [\lambda - \frac{1}{2}(p + \frac{2}{2n+1})]\nabla_Y X - \nabla_Y QX + (Y\lambda)X, \tag{80}$$

for any vector fields X, Y on M. By the virtue of (79) and (80) we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + \{(X\lambda)Y - (Y\lambda)X\}. \tag{81}$$

Thus we have the following:

Lemma 12. *Let M^{2n+1} be an (k,μ)-almost CoKähler manifolds admitting almost conformal gradient Ricci solitons. Then the curvature tensor R of type $(1,3)$ can be expressed as follows:*
$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + \{(X\lambda)Y - (Y\lambda)X$$

Taking covarient derivative of (17) with respect to any vector field X on M we have

$$(\nabla_X Q)\xi = 2nkh'X - Qh'X. \tag{82}$$

Taking inner product of (81) with ξ and using (82) we get

$$g(R(X,Y)Df, \xi) = S(h'X, Y) - S(X, h'Y) + \{(X\lambda)\eta(Y) - (Y\lambda)\eta(X)\}, \tag{83}$$

for any vector fields X and Y on M.
Again, with the help of (14) we obtain

$$\begin{aligned}
g(R(X,Y)Df, \xi) &= -g(R(X,Y)\xi, Df) \\
&= -k\{(Xf)\eta(Y) - (Yf)\eta(X)\} \\
&\quad -\mu\{\eta(Y)g(Df, hX) - \eta(X)g(Df, hY)\}, \tag{84}
\end{aligned}$$

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for any vector fields X, Y on M.
Compairing (83) and (84) we have

$$
\begin{aligned}
& S(h'X, Y) - S(X, h'Y) + \{(X\lambda)\eta(Y) - (Y\lambda)\eta(X)\} \\
= {} & -k\{(Xf)\eta(Y) - (Yf)\eta(X)\} \\
& -\mu\{\eta(Y)g(Df, hX) - \eta(X)g(Df, hY)\},
\end{aligned}
\tag{85}
$$

for any vector fields X, Y on M.
Substituting $X = hX$ and $Y = h^2 Y$ in the last equation we infer

$$
Q\phi X - \phi Q X = 0.
\tag{86}
$$

Let $\{e_i, \phi e_i, \xi\}, i = 1, 2, 3, ..., n$, be an orthonormal $\phi-$basis of M such that $Qe_i = \sigma_i e_i$. Then we have $Q\phi e_i = \sigma_i \phi e_i$. Substituting e_i for X in the last equation we get

$$
Q\phi e_i = \sigma_i \phi e_i.
\tag{87}
$$

Making use of ϕ-basis and (17) we obtain

$$
\begin{aligned}
r & = g(Q\xi, \xi) + \sum_{i=1}^{n} [g(Qe_i, e_i) + g(Q\phi e_i, \phi e_i)] \\
& = 2nk + 2\sum_{i=1}^{n} \sigma_i.
\end{aligned}
\tag{88}
$$

As σ_i are the eigen values, $\sum_{i=1}^{n} \sigma_i$ is constant and hence r is constant. Thus, following the **Theorem 1.1** we can state our last theorem as follows:

Theorem 13. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact (k, μ)-almost CoKähler manifold admitting almost conformal gradient Ricci solitons. Then M is isometric to a sphere $S^{2n+1}(c)$ of radius $c = \sqrt{\frac{2(n+1)(4n+3)}{r}}$.*

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