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SOME STABILITY AND CONVERGENCE RESULTS FOR AGARWAL-O'REGAN-SAHU AND JUNGCK-DAS-DEBATA ITERATIVE PROCESSES

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Abstract. In the paper of the author [M. O. Olatinwo; Stability and convergence results for Das-Debata type iterative process using simultaneous set of contractive conditions of integral type, Bulletin of the Allahabad Mathematical Society 30 (1) (2015), 75-94], the Jungck-Das-Debata iterative process involving the nonself-mappings R, S and T was introduced. The $(R,\ S,\ T)-$stability of iterative processes was then initiated in the same article. The convergence result was also proved for the Jungck-Das-Debata iterative process. In the present article, we shall obtain some stability and strong convergence results for the iterative processes of Agarwal-O'Regan-Sahu, Jungck-Das-Debata and a new Jungck-Ishikawa type iteration for a new class of contractive conditions. Our results generalize and extend amongst others the stability results of [S. L. Singh, C. Bhatnagar, S. N. Mishra; Stability of Jungck-type iterative procedures, Int. J. Math. & Math. Sc. 19 (2005), 3035-3043] as well as stability and convergence results contained in [V. Berinde; Iterative approximation of fixed points, Springer-Verlag, Berlin Heidelberg (2007)] and a host of others in the literature.

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1. Introduction

In Olatinwo [27], the $(R,\ S,\ T)-$stability of iterative process was introduced for three nonselfmappings and strong convergence result was also obtained for the Jungck-Das-Debata iterative process by employing a simultaneous set of contractive conditions. However, in the present article, we shall obtain some stability and convergence theorems for Agarwal *et al.*, Jungck-Das-Debata iterative processes and a new Jungck-Ishikawa type for a new class of contractive conditions. Our stability results are generalizations and extensions of those of [5, 13, 14, 25, 30, 32, 33, 35], while the convergence results extend and generalize some results of [6, 7, 19, 20].

Let $(E,\ d)$ be a complete metric space and $S,\ T\colon Y \to E$ two non-self mappings.

Definition 1. *[34] Let Y and E be two nonempty sets and $S,\ T\colon Y \to E$ two mappings. Then, an element $x^* \in Y$ is a coincidence point of S and T if and only if $Sx^* = Tx^*$. Denote the set of the coincidence points of S and T by $C(S,\ T)$.*

The following general definition for the concept of stability of iterative processes involving three nonselfmappings is contained in Olatinwo [27]:

Definition 2. *[27] Let $(E,\ d)$ be a complete metric space and $R,\ S,\ T\colon Y \to E$ three nonselfmappings, $T(Y) \subseteq R(Y),\ S(Y) \subseteq R(Y)$ and z a coincidence point of $R,\ S$ and T, that is, $Rz = Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{Rx_n\}_{n=0}^{\infty} \subset E$ generated by the iterative procedure*

$$Rx_{n+1} = f(S,\ T,\ x_n),\ n = 0, 1, \cdots, \tag{1}$$

converge to p, where f is some function. Let $\{Ry_n\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence, and set $\epsilon_n = d(Ry_{n+1}, f(S,\ T,\ y_n)),\ (n = 0, 1, \cdots)$. Then, the iterative procedure (1) will be called $(R,\ S,\ T)-$stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} Ry_n = p$.

Remark 1. *(i) The definition above reduces to that of stability of iterative processes in the sense of Harder and Hicks [13] when $Y = E,\ R = S = I$ (identity operator) and coincidence point of $R,\ S,\ T$ reduces to the fixed point of T.*
(ii) The definition also reduces to that of stability of iterative processes in the sense of Singh et al. [35] if $R = S$.
In (1), putting $R = S$,

$$Sx_{n+1} = f(S,\ T,\ x_n) \equiv h(T,\ x_n) = (1-\alpha_n)Sx_n + \alpha_n Tx_n,\ \alpha_n \in [0, 1], \tag{2}$$

$n = 0, 1, 2, \cdots$, where h is some function, then we obtain the Jungck-Mann iterative process of Singh et al. [35].

Jungck [17] established that two mappings S and T satisfying

$$d(Tx, Ty) \leq ad(Sx, Sy),\ \forall\ x,\ y \in E,\ a \in [0, 1), \tag{3}$$

have a unique common fixed point in complete metric space E, provided that S and T commute, $T(Y) \subseteq S(Y)$ and S is continuous.

In the normed space setting, we also have the following well-known iterative processes:
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \; n = 0, \; 1, \; 2, \cdots, \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iterative process (see Mann [22]).
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \end{array}\right\} \; n = 0, 1, \cdots, \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_\mathrm{n}\}_{n=0}^{\infty}$ are sequences in $[0,1]$, is called the Ishikawa iterative process (see Ishikawa [15]).
Let $x_0 \in E$. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \end{array}\right\} \; n = 0, 1, \cdots, \tag{6}$$

with $\{\alpha_n\}_{n=0}^{\infty}, \; \{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ was introduced in 2007 by Agarwal *et al.* [1]. In this paper, the iterative process defined in (6) will be called the *Agarwal-O'Regan-Sahu iterative process.* However, the authors of the paper [1] named (6) as $S-$ iteration process.
In 2009, the following iterative process was defined to establish some results in a normed space setting: For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} x_{n+1} = P((1-\alpha_n)z_n + \alpha_n T_1(PT_1)^{n-1}z_n) \\ z_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n) \end{array}\right\} \; n = 0, 1, 2, \cdots, \tag{7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. The iterative process will be called the *Thaiwan iterative process* (see Thaiwan [36]).
There are several other iterative processes available in the literature. However, we refer to the author [24, 25] for some recently introduced iterative algorithms and variants of contractive mappings.

Kannan [18] established an extension of the Banach's fixed point theorem by using the following contractive definition: For a selfmap T, there exists $\beta \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right], \; \forall \; x, \; y \in E. \tag{8}$$

Chatterjea [9] used the following contractive condition: For a selfmap T, there exists $\gamma \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right], \; \forall \; x, \; y \in E. \tag{9}$$

By combining (2), (8) and (9), Zamfirescu [37] established a nice generalization of the Banach's fixed point theorem.

Several stability results established in metric space and normed linear space are available in the literature. Some of the various authors whose contributions are of momentous impact to the study of stability of the fixed point iterative procedures are Beg and Abbas [4], Berinde [5, 8], Harder and Hicks [13], Jachymski [16], Osilike and Udomene [30], Ostrowski [31], Rhoades [32, 33] and Singh et al [35]. In Harder and Hicks [13], the contractive definition stated in (2) was used to prove a stability result for the Kirk's iterative process. The first stability result on $T-$ stable mappings was proved by Ostrowski [31] for the Picard iteration using condition (2).

However, Singh *et al.* [35] established some stability results for Jungck and Jungck-Mann iteration processes by employing two contractive definitions both of which generalize those of Osilike and Udomene [30] but independent of that of Imoru and Olatinwo [14]. Singh *et al.* [35] obtained stability results for Jungck and Jungck-Mann iterative procedures in metric space using both the contractive definition (3) and the following: For $S,\ T\colon Y \to E$ and some $a \in [0,1)$, we have

$$d(Tx,Ty) \leq ad(Sx,Sy) + Ld(Sx,Tx),\ \forall\ x,\ y \in Y,\ L \geq 0. \tag{10}$$

Let $(E, ||.||)$ be a normed space and Y an arbitrary set. Let $R,\ S,\ T\colon Y \to E$ be three mappings such that $T(Y) \subseteq R(Y),\ S(Y) \subseteq R(Y),\ R(Y)$ is a complete subspace of E and R is injective. Then, for $x_0 \in Y$, define the sequence $\{Rx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$\left.\begin{array}{l} Rx_{n+1} = (1-\alpha_n)Rx_n + \alpha_n Tv_n \\ Rv_n = (1-\beta_n)Rx_n + \beta_n Sx_n \end{array}\right\},\ n = 0,1,2,\cdots, \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\mathrm{n}}\}_{\mathrm{n}=0}^{\infty}$ are sequences in $[0,1]$. The iterative process defined in (11) was the *Jungck-Das-Debata* iterative process introduced in Olatinwo [27] to establish some stability and strong convergence results for nonself-mappings.

Remark 2. *(i) We are using this medium to say that the iterative process defined in (11) should be called Jungck-Das-Debata iterative process rather than Jungck-Das-Debata type iterative process for which it was originally nomenclated in [27]. This is simply because Jungck-Das-Debata iterative process had not been formulated before the one in [27].*
(ii) Jungck-Das-Debata iterative process reduces to several special cases. In particular, the iterative processes of Das and Debata [11], Ishikawa [15], Jungck [17], Mann [22], Jungck-Mann (in Singh et al. [35]), Jungck-Ishikawa (in [25]) and some others are such special cases. For detail, see [27].

Apart from the Agarwal-O'Regan-Sahu iterative process defined in (6) and Jungck-Das-Debata iterative process given in (11), we shall employ in addition, the following *Jungck-Ishikawa type* iteration stated below:
Again, with $(E, \|.\|)$ as a normed space and Y an arbitrary set. Suppose $S,\ T\colon Y \to E$ are mappings such that $T(Y) \subseteq S(Y),\ S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$\left.\begin{array}{l} Sx_{n+1} = (1-\alpha_n)Sr_n + \alpha_n Tr_n \\ Sr_n = (1-\beta_n)Sx_n + \beta_n Tx_n \end{array}\right\},\ n = 0, 1, 2, \cdots, \tag{12}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\mathrm{n}}\}_{\mathrm{n}=0}^{\infty}$ are sequences in $[0, 1]$.

Remark 3. *(i) The Jungck-Ishikawa type iterative process is a new iterative scheme which is independent of Jungck-Ishikawa iterative process.*
(ii) If $\beta_n = 0$ in (12), we obtain the Jungck-Mann iterative process defined in (2) since S is injective.
(iii) The iterative process given in (12) also reduces to Mann iterative process if $Y = E,\ \beta_n = 0$ and $S = I =$Identity mapping.
(iv) Similarly, Jungck and Picard iterations can be obtained from (12).

We shall employ the following contractive conditions which are stated in the metric forms: Let $(E,\ d)$ be a metric space and Y an arbitrary set.
(i) For $T\colon E \to E$, there exist real numbers $L \geq 0,\ k \in [0, 1)$ and a monotone increasing function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$, such that

$$d(Tx, Ty) \leq [\phi(d(x, Tx)) + kd(x, y)]e^{Ld(x,Tx)},\ \forall\ x,\ y \in E. \tag{13}$$

(ii) For $S,\ T\colon Y \to E$, with $T(Y) \subseteq S(Y)$, there exist real numbers $\eta \geq 0,\ k \in [0, 1)$, such that $\forall\ x,\ y \in Y$, we have

$$d(Tx, Ty) \leq [\Phi(d(Sx, Tx)) + kd(Sx, Sy)]e^{\eta d(Sx,Tx)}, \tag{14}$$

where $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function such that $\Phi(0) = 0$.
(iii) For $R,\ S,\ T\colon Y \to E$, with $T(Y) \subseteq R(Y),\ S(Y) \subseteq R(Y)$, there exist real numbers $L \geq 0,\ M \geq 0,\ a,\ b \in [0, 1)$, such that $\forall\ x,\ y \in Y$, we have

$$\left.\begin{array}{l} d(Tx, Ty) \leq [\Phi(d(Rx, Tx)) + ad(Rx, Ry)]e^{Ld(Rx,Tx)} \\ d(Sx, Sy) \leq [\Psi(d(Rx, Sx)) + bd(Rx, Ry)]e^{Md(Rx,Sx)} \end{array}\right\}, \tag{15}$$

where $\Phi,\ \Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ are monotone increasing functions such that
$\Phi(0) = \Psi(0) = 0$.
The contractive inequality conditions in (15) are a set of simultaneous contractive conditions.

Remark 4. *(i) We shall use contractive conditions (13), (14) and (15 to obtain our results. Since metric is induced by norm, we have* $d(u, v) = \|u - v\|, \ \forall \ u, \ v \in X$, *where* (X, d) *is a metric space and* $(X, \|\cdot\|)$ *is a normed linear space.*
(ii) The contractive conditions (15) are independent of those of Olatinwo [27].
(iii) The contractive condition (14) is independent of that of Olatinwo [23], but it is more general than that of Olatinwo [14].
(iv) If in (15), $L = \eta = 0, \ \Phi(u) = Nu$, *or,* $\Psi(u) = Nu, \ N \geq 0, \ u \in \mathbb{R}^+$, *and* $S = T$, *then we obtain the contractive condition employed by Singh et al. [35].*
(v)The contractive condition (14) is reducible to those employed in [35].
(vi) Similarly, contractive conditions (13), (14) and (15) reduce to some other well-known contractive conditions in the literature.

We shall require the following lemma in the sequel.

Lemma 1. *[5, 8] If* δ *is a real number such that* $0 \leq \delta < 1$, *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \ n = 0, 1, \cdots,$$

we have $\lim_{n\to\infty} u_n = 0$.

Remark 5. *Recall that for* $T \colon E \to E$, *the fixed point set denoted by* $F(T)$ *is defined by* $F(T) = \{v \in E \mid Tv = v\}$.

2. Stability Results

Theorem 2. *Let* $(E, \|.\|)$ *be a normed linear space and* $T \colon E \to E$ *a mapping satisfying contractive condition (13). Suppose* T *has a fixed point* q *For* $x_0 \in E$, *let* $\{x_n\}_{n=0}^{\infty}$ *defined by (6) be the Agarwal-O'Regan-Sahu iterative process where* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ *are sequences in* $[0, 1]$ *such that* $0 < \alpha \leq \alpha_n, \ 0 < \beta \leq \beta_n$ $(n = 0, \ 1, \ 2, \cdots,)$. *Let* $\phi \colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a monotone increasing function such that* $\phi(0) = 0$. *Then, the Agarwal-O'Regan-Sahu iterative process is* $T-$*stable.*

Proof. Suppose that $\{y_n\}_{n=0}^{\infty} \subset E, \ \epsilon_n = \|y_{n+1} - (1 - \alpha_n)Ty_n - \alpha_n T s_n\|$,
$s_n = (1 - \beta_n)y_n + \beta_n T y_n$ and let $\lim_{n\to\infty} \epsilon_n = 0$. Then, using the contractive condition

(13) gives

$$\begin{aligned}\|y_{n+1}-q\| &\leq \|y_{n+1}-(1-\alpha_n)Ty_n-\alpha_nTs_n\|+\|(1-\alpha_n)Ty_n+\alpha_nTs_n-q\|\\ &=\epsilon_n+\|(1-\alpha_n)Ty_n+\alpha_nTs_n-[1-\alpha_n+\alpha_n]q\|\\ &=\|(1-\alpha_n)(Ty_n-q)+\alpha_n(Ts_n-q)\|+\epsilon_n\\ &\leq (1-\alpha_n)\|Tq-Ty_n\|+\alpha_n\|Tq-Ts_n\|+\epsilon_n\\ &\leq (1-\alpha_n)[\phi(\|q-Tq\|)+k\|q-y_n\|]e^{L\|q-Tq\|}\\ &\quad+\alpha_n[\phi(\|q-Tq\|)+k\|q-s_n\|]e^{L\|q-Tq\|}+\epsilon_n\\ &=k(1-\alpha_n)\|y_n-q\|+k\alpha_n\|q-s_n\|+\epsilon_n. \end{aligned} \tag{16}$$

Also, we have by using the contractive condition (13) again that

$$\begin{aligned}\|q-s_n\| &=\|(1-\beta_n+\beta_n)q-(1-\beta_n)y_n-\beta_nTy_n\|=\|(1-\beta_n)(q-y_n)+\beta_n(q-Ty_n)\|\\ &=\leq (1-\beta_n)\|y_n-q\|+\beta_n\|Tq-Ty_n\|\\ &\leq (1-\beta_n)\|y_n-q\|+\beta_n[\phi(\|q-Tq\|)+k\|y_n-q\|]e^{L\|q-Tq\|}\\ &=(1-\beta_n)\|y_n-q\|+k\beta_n\|y_n-q\|=[1-(1-k)\beta_n]\|y_n-q\|. \end{aligned} \tag{17}$$

Using (17) in (16) gives

$$\begin{aligned}\|y_{n+1}-q\| &\leq k(1-\alpha_n)\|y_n-q\|+k\alpha_n[1-(1-k)\beta_n]\|y_n-q\|+\epsilon_n\\ &=k[1-\alpha_n+\alpha_n-(1-k)\alpha_n\beta_n]\|y_n-q\|+\epsilon_n\\ &=k[1-(1-k)\alpha_n\beta_n]\|y_n-q\|+\epsilon_n\\ &\leq k[1-(1-k)\alpha\beta]\|y_n-q\|+\epsilon_n. \end{aligned} \tag{18}$$

Since $0\leq k[1-(1-k)\alpha\beta]<1$, applying Lemma 1 in (18) yields
$\lim\limits_{n\to\infty}\|y_{n+1}-q\|=0 \;\Leftrightarrow\; \lim\limits_{n\to\infty}y_n=q$.

Conversely, let $\lim\limits_{n\to\infty}y_n=q$. Then, we have by using the contractive condition (13) again that

$$\begin{aligned}\epsilon_n &=\|y_{n+1}-(1-\alpha_n)Ty_n-\alpha_nTs_n\|\\ &\leq \|y_{n+1}-q\|+(1-\alpha_n)\|Tq-Ty_n\|+\alpha_n\|Tq-Ts_n\|\\ &\leq \|y_{n+1}-q\|+k(1-\alpha_n)\|y_n-q\|+k\alpha_n\|p-s_n\|. \end{aligned} \tag{19}$$

Using (17) again in (19) yields

$$\begin{aligned}\epsilon_n &\leq \|y_{n+1}-q\|+k(1-\alpha_n)\|y_n-q\|+k\alpha_n[1-(1-k)\beta_n]\|y_n-q\|\\ &=\|y_{n+1}-q\|+k[1-(1-k)\alpha_n\beta_n]\|y_n-q\|\\ &\leq \|y_{n+1}-q\|+k[1-(1-k)\alpha\beta]\|y_n-q\|\to\ 0\ \text{ as } n\to\infty. \end{aligned}$$

Theorem 3. *Let $(E,||.||)$ be a normed linear space and Y an arbitrary set. Suppose that $S,\ T:Y\to E$ are mappings such that $T(Y)\subseteq S(Y),\ S(Y)$a complete subspace* *of E and S is an injective mapping. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy contractive condition (14). Let $\Phi\colon I\!R^+ \to I\!R^+$ be a monotone increasing function such that $\Phi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty} \subset E$ be the Jungck-Ishikawa type iterative process defined by (12) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$* and *$\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that $0 < \alpha \leq \alpha_n$ and $0 < \beta \leq \beta_n$ $(n = 0, 1, 2, \cdots)$. Then, the Jungck-Ishikawa type iterative process is $(S,T)-$stable.*

Proof. Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset E,\ \epsilon_n = \|Sy_{n+1} - (1-\alpha_n)Sv_n - \alpha_n Tv_n\|$, $Sv_n = (1-\beta_n)Sy_n + \beta_n Ty_n$ and let $\lim\limits_{n\to\infty} \epsilon_n = 0$. Then, using the contractive condition (14) gives

$$\begin{aligned}
\|Sy_{n+1} - p\| &\leq \|Sy_{n+1} - (1-\alpha_n)Sv_n - \alpha_n Tv_n\| + \|(1-\alpha_n)Sv_n + \alpha_n Tv_n - p\| \\
&= \|(1-\alpha_n)(Sv_n - p) + \alpha_n(Tv_n - p)\| + \epsilon_n \\
&\leq (1-\alpha_n)\|p - Sv_n\| + \alpha_n\|Tz - Tv_n\| + \epsilon_n \\
&\leq (1-\alpha_n)\|p - Sv_n\| + \alpha_n[\Phi(\|Sz - Tz\|) + k|\|Sz - Sv_n\||]e^{\eta\|Sz - Tz\|} + \epsilon_n \\
&= [1 - (1-k)\alpha_n]\|Sz - Sv_n\| + \epsilon_n. \qquad (20)
\end{aligned}$$

Again, we have by using the contractive condition (14) that

$$\begin{aligned}
\|Sz - Sv_n\| &= \|(1-\beta_n+\beta_n)Sz - (1-\beta_n)Sy_n - \beta_n Ty_n\| \\
&\leq (1-\beta_n)\|Sy_n - Sz\| + \beta_n\|Tz - Ty_n\| \\
&\leq (1-\beta_n)\|Sy_n - p\| + \beta_n[\Phi(\|Sz - Tz\|) + k|\|Sz - Sy_n\||]e^{\eta\|Sz - Tz\|} \\
&= (1-\beta_n)\|Sy_n - p\| + k\beta_n\|Sz - Sy_n\| \\
&= [1 - (1-k)\beta_n]\|Sy_n - p\|. \qquad (21)
\end{aligned}$$

Using (21) in (20) gives

$$\begin{aligned}
\|Sy_{n+1} - p\| &\leq [1 - (1-k)\alpha_n][1 - (1-k)\beta_n]\|Sy_n - p\| + \epsilon_n \\
&\leq [1 - (1-k)\alpha][1 - (1-k)\beta]\|Sy_n - p\| + \epsilon_n \\
&= \eta\|Sy_n - p\| + \epsilon_n, \qquad (22)
\end{aligned}$$

where $0 \leq \eta = [1-(1-k)\alpha][1-(1-k)\beta] < 1$, since $0 \leq [1-(1-k)\alpha] < 1$ and $0 \leq [1-(1-k)\beta] < 1$.
Therefore, we obtain from (22) and Lemma1 that
$\lim\limits_{n\to\infty} \|Sy_{n+1} - p\| = 0 \ \Leftrightarrow\ \lim\limits_{n\to\infty} Sy_n = p$. Conversely, let $\lim\limits_{n\to\infty} Sy_n = p$. Then, we obtain by using the contractive condition (14) that

$$\begin{aligned}
\epsilon_n &\leq \|Sy_{n+1} - p\| + (1-\alpha_n)\|p - Sv_n\| + \alpha_n\|p - Tv_n\| \\
&= \|Sy_{n+1} - p\| + (1-\alpha_n)\|p - Sv_n\| + \alpha_n\|Tz - Tv_n\| \\
&\leq \|Sy_{n+1} - p\| + (1-\alpha_n)\|Sz - Sv_n\| + k\alpha_n\|Sz - Sv_n\| \\
&= \|Sy_{n+1} - p\| + [1 - (1-k)\alpha_n]\|Sz - Sv_n\|. \qquad (23)
\end{aligned}$$

Using (21) again in (23) yields

$$\begin{aligned}\epsilon_n &\leq \|Sy_{n+1} - p\| + [1-(1-k)\alpha_n][1-(1-k)\beta_n]\|Sy_n - p\| \\ &\leq \|Sy_{n+1} - p\| + [1-(1-k)\alpha][1-(1-k)\beta]\|Sy_n - p\| \to 0 \ \text{ as } n \to \infty.\end{aligned}$$

Theorem 4. *Let $(E, ||.||)$ be a normed linear space and Y an arbitrary set. Suppose that $R,\ S,\ T : Y \to E$ are mappings such that $S(Y) \subseteq R(Y),\ T(Y) \subseteq R(Y),\ R(Y)$ a complete subspace of E* and R *is an injective mapping. Let z be a coincidence point of $R,\ S$ and T (that is, $Rz = Sz = Tz = p$). Suppose that $R,\ S$ and T satisfy contractive conditions (15). Let $\Phi,\ \Psi\colon I\!R^+ \to I\!R^+$ be monotone increasing functions such that $\Phi(0) = \Psi(0) = 0$. For $x_0 \in Y$, let $\{Rx_n\}_{n=0}^{\infty} \subset E$ be the Jungck-Das-Debata iterative process defined by (11) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that $0 < \alpha \leq \alpha_n$ and $0 < \beta \leq \beta_n$ $(n = 0, 1, 2, \cdots)$. Then, the Jungck-Das-Debata iterative process is (R, S, T)-stable.*

Proof. Suppose that $\{Ry_n\}_{n=0}^{\infty} \subset E,\ \epsilon_n = ||Ry_{n+1} - (1-\alpha_n)Ry_n - \alpha_n Tb_n||$, $n = 0, 1, 2, \cdots$, where $Rb_n = (1-\beta_n)Ry_n + \beta_n Sy_n$, and let $\lim_{n\to\infty} \epsilon_n = 0$. Therefore, using conditions (15) and the iterative algorithm (11), we obtain

$$\begin{aligned}\|Ry_{n+1} - p\| &\leq \|Ry_{n+1} - (1-\alpha_n)Ry_n - \alpha_n Tb_n\| + (1-\alpha_n)\|Ry_n - p\| + \alpha_n\|Tb_n - p\| \\ &= (1-\alpha_n)\|Ry_n - p\| + \alpha_n\|Tz - Tb_n\| + \epsilon_n \\ &\leq (1-\alpha_n)\|Ry_n - p\| + \alpha_n[\Phi(\|Rz - Tz\|) + a\|Rz - Rb_n\|]e^{L\|Rz-Tz\|} + \epsilon_n \\ &= (1-\alpha_n)\|Ry_n - p\| + a\alpha_n\|Rz - Rb_n\| + \epsilon_n. \qquad (24)\end{aligned}$$

Again, using the contractive conditions (15) as well as algorithm (11) give

$$\begin{aligned}||Rz - Rb_n|| &= \|Rz - (1-\beta_n)Ry_n - \beta_n Sy_n\| \\ &\leq (1-\beta_n)\|Rz - Ry_n\| + \beta_n\|Rz - Sy_n\| \\ &= (1-\beta_n)\|Ry_n - p\| + \beta_n\|Sz - Sy_n\| \\ &\leq (1-\beta_n)\|Ry_n - p\| + \beta_n[\Psi(\|Rz - Sz\|) + b\|Rz - Ry_n\|)]e^{M\|Rz-Tz\|} \\ &= [1-(1-b)\beta_n]\|Ry_n - p\|. \qquad (25)\end{aligned}$$

Using (25) in (24) yields

$$\begin{aligned}||Ry_{n+1} - p|| &\leq [1-(1-a)\alpha_n - (1-b)a\alpha_n\beta_n]\|Ry_n - p\| + \epsilon_n \\ &\leq [1-(1-a)\alpha - (1-b)a\alpha\beta]\|Ry_n - p\| + \epsilon_n. \qquad (26)\end{aligned}$$

Therefore, using Lemma 1 in (26) yields $\lim_{n\to\infty} ||Ry_n - p|| = 0$. That is, $\lim_{n\to\infty} Ry_n = p$.

Conversely, let $\lim_{n\to\infty} Ry_n = p$. Then, by using the contractive definitions (15) and algorithm (11) again, we have

$$\begin{aligned}\epsilon_n &= \|Ry_{n+1} - (1-\alpha_n)Ry_n - \alpha_n Tb_n\| \\ &\leq \|Ry_{n+1} - p\| + (1-\alpha_n)\|Ry_n - p\| + \alpha_n\|p - Tb_n\| \\ &= \|Ry_{n+1} - p\| + (1-\alpha_n)\|Ry_n - p\| + \alpha_n\|Tz - Tb_n\| \\ &\leq \|Ry_{n+1} - p\| + (1-\alpha_n)\|Ry_n - p\| + a\alpha_n\|Rz - Rb_n\|. \end{aligned} \tag{27}$$

By using (25) again in (27), we obtain

$$\begin{aligned}\epsilon_n &\leq \|Ry_{n+1} - p\| + [1 - (1-a)\alpha_n - (1-b)a\alpha_n\beta_n]\|Ry_n - p\| \\ &\leq \|Ry_{n+1} - p\| + [1 - (1-a)\alpha - (1-b)a\alpha\beta]\|Ry_n - p\| \to \ 0 \text{ as } n \to \infty,\end{aligned}$$

from which it follows that $\lim_{n\to\infty} \epsilon_n = 0$.

Remark 6. *Theorem 2.1 is more general than Theorem 3.1 contained in Olatinwo [14] but independent of Theorem 3.2 of the same article, and this statement is valid for a host of corresponding results in the literature. Theorem 2.2 and Theorem 2.3 are generalizations and extensions of some results in [5, 8, 13, 14, 27, 30, 32, 33, 35], while Theorem 2.3 is independent of Theorem 3.1 of the author Olatinwo [27].*

3. Convergence Results

Theorem 5. *Let $(E, \|.\|)$ be an arbitrary Banach space, K a closed convex subset of E and $T\colon K \to K$ an operator satisfying (13). For $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ defined by (6) be the Agarwal-O'Regan-Sahu iterative process with $\alpha_n,\ \beta_n \in [0,1]$ such that $0 < \alpha \leq \alpha_n,\ 0 < \beta \leq \beta_n,\ (n = 0, 1, 2, \cdots,)$. Then, the Agarwal-O'Regan-Sahu iteration $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Proof. Firstly, we shall establish that T has a unique fixed point by using condition (13) as follows: Suppose not. Then, there exist $u,\ v \in F(T),\ u \neq v$, and $\|u-v\| > 0$. Therefore, we obtain that

$$\begin{aligned}0 < \|u - v\| = \|Tu - Tv\| &\leq [\phi(\|u - Tu\|) + k\|u - v\|]e^{L\|u-Tu\|} \\ &= \phi(0) + k\|u - v\| = k\|u - v\|,\end{aligned}$$

from which we have that $(1-k)\|u-v\| \leq 0$. That is, since $k \in [0, 1),\ 1-k > 0$, but $\|u - v\| \leq 0$ (which is a contradiction since norm is nonnegative).
Therefore, we have $\|u - v\| = 0 \iff u = v$ (thus, proving the uniqueness of the fixed point of T).

We now prove that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point u of T using condition (13).

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)Tx_n+\alpha_nTb_n-p\| \\ &\le (1-\alpha_n)\|Tp-Tx_n\|+\alpha_n\|Tp-Tb_n\| \\ &\le (1-\alpha_n)[\phi(\|p-Tp\|)+k\|p-x_n\|]e^{L\|p-Tp\|} \\ &\quad +\alpha_n[\phi(\|p-Tp\|)+k|\|p-b_n\|]e^{L\|p-Tp\|} \\ &= k(1-\alpha_n)\|x_n-p\|+k\alpha_n\|p-b_n\|. \end{aligned} \tag{28}$$

Now, we have that

$$\begin{aligned}\|p-b_n\| &= \|p-(1-\beta_n)x_n\beta_nTx_n\| = \|(1-\beta_n)(p-x_n)+\beta_n(p-Tx_n)\| \\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|Tp-Tx_n\| \le [1-(1-k)\beta_n]\|x_n-p\|. \end{aligned} \tag{29}$$

Using (19) in (18) gives

$$\begin{aligned}\|x_{n+1}-p\| &\le k(1-\alpha_n)\|x_n-p\|+k\alpha_n[1-(1-k)\beta_n]\|x_n-p\| \\ &= k[1-(1-k)\alpha_n\beta_n]\|x_n-p\| \\ &\le k[1-(1-k)\alpha\beta]\|x_n-p\| = \gamma\|x_n-p\| \text{ (where } \gamma = k[1-(1-k)\alpha\beta]) \\ &\le \gamma^2\|x_{n-1}-p\| \le \gamma^3\|x_{n-2}-p\| \le \cdots \le \gamma^{n+1}\|x_0-p\| \to 0 \text{ as } n\to\infty, \end{aligned} \tag{30}$$

since $0 \le \gamma < 1$.
Hence, it follows from (30) that Agarwal-O'Regan-Sahu iterative process $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p of T.

Theorem 6. *Let $(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that $S,\ T\colon Y \to E$ are mappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E and S is an injective mapping. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy contractive condition (14). Let $\Phi\colon I\!R^+ \to I\!R^+$ be a monotone increasing function such that $\Phi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty} \subset E$ be the Jungck-Ishikawa type iterative process defined by (12), where $\{\alpha_n\}_{n=0}^{\infty}$* and *$\{\beta_\mathrm{n}\}_{\mathrm{n}=0}^{\infty}$ are sequences in $[0,1]$ such that $0 < \alpha \le \alpha_n$, and $0 < \beta \le \beta_n$ $(n = 0, 1, 2, \cdots)$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.*

Proof. We shall now use contractive condition (14) to establish that S and T have a unique coincidence point z (i.e. $Sz = Tz = p$): Injectivity of S is sufficient. Let $C(S,\ T)$ be the set of coincidence points of $R,\ S$ and T. Suppose that there exist $z_1,\ z_2 \in C(S,\ T)$, that is, $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$.
If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$.
If $p_1 \ne p_2$, then we have by the contractiveness condition (14) for S and T that

$$\begin{aligned}\|p_1-p_2\| &= \|Tz_1-Tz_2\| \\ &\le [\Phi(\|Sz_1-Tz_1\|)+k\|Sz_1-Sz_2\|]e^{\eta\|Sz_1-Tz_1\|} \\ &= k\|Sz_1-Sz_2\| = k\|p_1-p_2\|, \end{aligned}$$

from which it follows that $(1-k)\|p_1 - p_2\| \leq 0$. Now, $1-k > 0,\ k \in [0,1)$ but $\|p_1 - p_2\| \leq 0$ (which is a contradiction since norm is nonnegative). Therefore, it follows that $\|p_1 - p_2\| = 0$.
That is, $p_1 = p_2 \implies Sz_1 = Sz_2$.
We have by the injectivity of S that $Sz_1 = Sz_2 \implies z_1 = z_2$.
Hence, proving the uniqueness of the coincidence point of S and T.

We now establish that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p using contractive condition (14). Therefore, we obtain by using the iterative scheme (12) as well as the contractive condition (14) that

$$\begin{aligned}
\|Sx_{n+1} - p\| &= \|(1-\alpha_n)Sr_n + \alpha_n Tr_n - p\| = (1-\alpha_n)\|(Sr_n - p) + \alpha_n(Tr_n - p)\| \\
&\leq (1-\alpha_n)\|Sr_n - p\| + \alpha_n\|Tz - Tr_n\| \\
&\leq (1-\alpha_n)\|Sr_n - Sz\| + \alpha_n[\varphi(\|Sz - Tz\|) + k|\|Sz - Sr_n\|]e^{L\|Sz-Tz\|} \\
&= [1-(1-k)\alpha_n]\|Sz - Sr_n\|. \qquad (31)
\end{aligned}$$

Also, we obtain by using the contractive condition (14) again that

$$\begin{aligned}
\|Sz - Sr_n\| &\leq (1-\beta_n)\|Sz - Sx_n\| + \beta_n\|Sz - Tx_n\| \\
&= (1-\beta_n)\|Sz - Sx_n\| + \beta_n\|Tz - Tx_n\| \\
&\leq (1-\beta_n)\|Sx_n - p\| + k\beta_n\|Sz - Sx_n\| \\
&= (1-\beta_n)\|Sx_n - p\| + k\beta_n\|p - Sx_n\| \\
&= [1-(1-k)\beta_n]\|Sx_n - p\|. \qquad (32)
\end{aligned}$$

Using (32) in (31) gives

$$\begin{aligned}
\|Sx_{n+1} - p\| &\leq [1-(1-k)\alpha_n][1-(1-k)\beta_n]\|Sx_n - p\| \\
&\leq [1-(1-k)\alpha][1-(1-k)\beta]\|Sy_n - p\| \\
&= \eta\|Sx_n - p\| \text{ (where } \eta = [1-(1-k)\alpha][1-(1-k)\beta]) \\
&\leq \eta^2\|Sx_{n-1} - p\| \leq \eta^3\|Sx_{n-2} - p\| \\
&\leq \cdots \leq \eta^{n+1}\|Sx_0 - p\| \to 0 \text{ as } n \to \infty, \qquad (33)
\end{aligned}$$

where $0 \leq \eta < 1$, since $0 \leq [1-(1-k)\alpha] < 1$ and $0 \leq [1-(1-k)\beta] < 1$.
Therefore, it follows from (33) that our new iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 7. *Let $(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that $R,\ S,\ T\colon Y \to E$ are mappings such that $S(Y) \subseteq R(Y)$, $T(Y) \subseteq R(Y)$, $R(Y)$ a complete subspace of E,* and R *is an injective operator. Let z be a coincidence point of S and T (that is, $Rz = Sz = Tz = p$). Suppose that R, S and T satisfy the contractive conditions (15). Let $\Phi,\ \Psi\colon I\!R^+ \to I\!R^+$ be monotone increasing* *functions such that* $\Phi(0) = \Psi(0) = 0$. *For* $x_0 \in Y$, *let* $\{Rx_n\}_{n=0}^{\infty}$ *be the Jungck-Das-Debata type iterative process defined by (11), where* $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\rm n}\}_{\rm n=0}^{\infty}$ *are sequences in* $[0,1]$ *such that* $0 < \alpha \le \alpha_n$, $0 < \beta \le \beta_n$ $(n = 0, 1, 2, \cdots)$. *Then,* $\{Rx_n\}_{n=0}^{\infty}$ *converges strongly to* p.

Proof. We shall now use conditions (15) to establish that R, S and T have a unique coincidence point z (i.e. $Rz = Sz = Tz = p$): Injectivity of R is sufficient. Let $C(R, S, T)$ be the set of coincidence points of R, S and T. Suppose that there exist z_1, $z_2 \in C(R, S, T)$, that is, $Rz_1 = Sz_1 = Tz_1 = p_1$ and $Rz_2 = Sz_2 = Tz_2 = p_2$.
If $p_1 = p_2$, then $Rz_1 = Rz_2$ and since R is injective, it follows that $z_1 = z_2$.
If $p_1 \neq p_2$, then we have by the contractiveness conditions (15) for R, S and T that

$$\begin{aligned} \|p_1 - p_2\| &= \|Tz_1 - Tz_2\| \\ &\le [\Phi(\|Rz_1 - Tz_1\|) + a\|Rz_1 - Rz_2\|]e^{L\|Rz_1 - Tz_1\|} \\ &= a\|Rz_1 - Rz_2\| = a\|Sz_1 - Sz_2\| \\ &\le a[\Psi(\|Rz_1 - Sz_1\|) + b\|R_1 - Tz_2\|]e^{M\|Rz_1 - Tz_1\|} \\ &= ab\|Rz_1 - Rz_2\| = ab\|p_1 - p_2\|, \end{aligned}$$

from which it follows that $(1 - ab)\|p_1 - p_2\| \le 0$. Now, $1 - ab > 0$, a, $b \in [0, 1)$, but $\|p_1 - p_2\| \le 0$ (which is a contradiction since norm is nonnegative). Therefore, $\|p_1 - p_2\| = 0$,
from which it follows that $p_1 = p_2 \implies Rz_1 = Rz_2$,
and by the injectivity of R, we have that $Rz_1 = Rz_2 \implies z_1 = z_2$.
Thus, proving the uniqueness of the coincidence point of R, S and T.

We now show that $\{Rx_n\}_{n=0}^{\infty}$ converges strongly to p using contractive conditions (15). Therefore, we obtain by using the iterative scheme (11) as well as the contractive conditions (15) that

$$\begin{aligned} \|Rx_{n+1} - p\| &\le (1 - \alpha_n)\|Rx_n - p\| + \alpha_n\|Tz - Tv_n\| \\ &\le (1 - \alpha_n)\|Rx_n - p\| + \alpha_n[\Phi(\|Rz - Tz\|) + a\|Rz - Rv_n\|]e^{L\|Rz - Tz\|} \\ &= (1 - \alpha_n)\|Rx_n - p\| + a\alpha_n\|Rz - Rv_n\|. \end{aligned} \tag{34}$$

In a similar manner, using the contractive conditions (15) and the iterative scheme (11) again yield

$$\|Rz - Rv_n\| \le (1 - \beta_n + b\beta_n)\|Rx_n - p\| = [1 - (1 - b)\beta_n]\|Rx_n - p\|. \tag{35}$$

Using (35) in (34) yields

$$\begin{aligned} \|Rx_{n+1} - p\| &\le [1 - (1 - a)\alpha_n - (1 - b)a\alpha_n\beta_n]\|Rx_n - p\| \\ &\le [1 - (1 - a)\alpha - (1 - b)a\alpha\beta]\|Rx_n - p\| = \eta\|Rx_n - p\|, \end{aligned} \tag{36}$$

where $0 \le \eta = 1-(1-a)\alpha-(1-b)a\alpha\beta < 1$ since $\alpha,\ \beta \in [0,1)$.
Therefore, we obtain from (36) that

$$\|Rx_{n+1}-p\| \le \eta\|Rx_n-p\| \le \eta^2\|Rx_{n-1}-p\| \le \cdots \le \eta^{n+1}\|Rx_0-p\| \to\ 0 \text{ as } n\to\infty,$$

from which it follows that $\lim_{n\to\infty}\|Rx_n-p\|=0$. That is, $\lim_{n\to\infty}Rx_n=p$.

Remark 7. *Theorem 3.3 generalizes and extends a multitude of results in the literature. Particularly, Theorem 3.3 generalizes and extends both Theorem 1 and Theorem 2 of Berinde [6], Theorem 2 and Theorem 3 of Kannan [19], Theorem 3 of Kannan [20], Theorem 4.9 of Berinde [8]), Theorem 5.6 of Berinde [8] and indeed, some results of the author [25].*

Example 1. *Let $E=I\!R$, $Y=[0,1]\subset I\!R$ with the Euclidean norm. Let the operators $R,\ S,\ T\colon Y\to E$ be defined by $Rx=1-4x,\ Sx=x^2,\ Tx=x^3,\ \forall\ x\in Y$. Also, define $\Phi,\ \Psi\colon [0,6]\to[0,6]$ by*

$$\Phi(t)=\begin{cases}\frac{3t^2}{4}, & \text{if } t\in[0,3)\\ \frac{t}{2}, & \text{if } t\in[3,6]\end{cases}$$

and

$$\Psi(t)=\begin{cases}\frac{t^3}{2}, & \text{if } t\in[0,2)\\ \frac{2t}{3}, & \text{if } t\in[2,6]\end{cases}$$

We now show that $R,\ S,\ T$ satisfy the set of contractive conditions (15) and that the iterative algorithm defined in (11) is stable as well as convergent.

Solution. *Here, both Φ and Ψ are monotone increasing in $[0,5]$.*
$T(Y)=[0,1]\subset R(Y)=[-3,1],\ S(Y)=[0,1]\subset R(Y)$. *Now,* $\forall\ x,\ y\in Y$,
$\|Tx-Ty\|=|x^3-y^3|=|x-y||x^2+xy+y^2|\le 3|x-y|,\ ||Rx-Ry||=4|x-y|$,
$\|Sx-Sy\|=|x^2-y^2|=|x+y||x-y|\le 2|x-y|,\ \|Rx-Tx\|=|1-4x-x^3|$, *and*
$\|Rx-Sx\|=|1-4x-x^2|$.
Therefore, for $L=1,\ a=\frac{3}{4},\ x=0,\ y=1$, we obtain from above that $\forall\ x,\ y\in Y$,

$$\begin{aligned}[\Phi(\|Rx-Tx\|)+a\|Rx-Ry\|]e^{L\|Rx-Tx\|} &\ge \Phi(|1-4x-x^3|)+4a|x-y|\ (\textit{since } e^{L\|Rx-Tx\|}>0)\\ &=\tfrac{15}{4}=\tfrac{15}{4}|x-y|\ge 3|x-y|\ge\|Tx-Ty\|.\end{aligned}$$

Similarly, for $M=0,\ b=\frac{1}{2},\ x=0,\ y=1$, we have $\forall\ x,\ y\in Y$,

$$\begin{aligned}[\Psi(\|Rx-Sx\|)+b\|Rx-Ry\|]e^{M\|Rx-Sx\|} &= \Psi(|1-4x-x^2|)+4b|x-y|=\tfrac{5}{2}\\ &=\tfrac{5}{2}|x-y|\ge 2|x-y|\ge\|Sx-Sy\|.\end{aligned}$$

Thus, it follows from above that $R,\ S,\ T$ satisfy the contractive conditions (15).

Consequently, substituting for Rx_n, Sx_n, Tx_n in (11), then the mappings R, S and T have a unique coincidence point, which can be found as limit of a stable Jungck-Das-Debata iterative algorithm

$$x_{n+1} = \frac{1}{4}\left[1 - (1-\alpha_n)(1-4x_n) - \frac{\alpha_n}{64}\left\{1 - (1-\beta_n)(1-4x_n) - \beta_n x_n^2\right\}^3\right], \qquad (37)$$

with $x_0 \in Y$, for any sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ such that $0 < \alpha \leq \alpha_n$ and $0 < \beta \leq \beta_n$ $(n = 0, 1, 2, \cdots)$.
More accurately, with $x_0 \in (0,1]$, then the sequence $\{x_n\}$ defined by (37) above provides a stable iterative algorithm, for any sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ such that $0 < \alpha \leq \alpha_n$ and $0 < \beta \leq \beta_n$ $(n = 0, 1, 2, \cdots)$.
In particular, with $\alpha_n = \frac{1}{10}$, $\beta_n = \frac{1}{8}$, then Eqn. (37) reduces to

$$x_{n+1} = \frac{1}{40} + \frac{9}{10}x_n - \frac{1}{2560}\left(\frac{1}{8} + \frac{7}{2}x_n - \frac{1}{8}x_n^2\right)^3, \qquad (38)$$

which converges to the coincidence point $z = x_{191} = 0.2463283675$, for $x_0 = 1.0$.

Remark 8. *The part of Solution of Example 3.5 involving Eqn. (37) and Eqn. (38) is also available in Example 3.1 which is contained in Olatinwo [27]. Since the operators R, S, T satisfying the contractive conditions (15) in the present article also satisfy the contractive conditions (14)in [27], then this is a vindication of the fact that the contractive conditions (15) are independent of the contractive conditions (14) in [27]. In addition, we refer to Olatinwo and Postolache [29] for more study and examples.*

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] R. P. Agarwal, O. O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Journal of Nonlinear and Convex Analysis 8, 1 (2007), 6179-6189.

[2] M. Akram, A. A. Zafar, A. A. Siddiqui, *A general class of contractions: $A-$contractions*, Novi Sad J. Math. 38, 1 (2008), 25-33.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math. 3, (1922), 133-181.

[4] I. Beg and M. Abbas, *Equivalence and stability of random fixed point iterative procedures*, J. Appl. Math. Stoch. Anal. (2006), Article ID 23297, 1-19.

[5] V. Berinde, *On the stability of some fixed point procedures*, Bul. Stiint. Univ. Baia Mare, Ser. B, Matematica-Informatica 18, 1 (2002), 7-14.

[6] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi-contractive operators*, Acta Math. Univ. Comenianae LXXIII, 1 (2004), 119-126.

[7] V. Berinde, *A convergence theorem for Mann iteration in the class of Zamfirescu operators*, Analele Universitatii de Vest, Timisoara, Seria Matematica-Informatica 45, 1 (2007), 33-41.

[8] V. Berinde, *Iterative approximation of fixed points*, Springer-Verlag, Berlin Heidelberg (2007).

[9] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. 10, (1972), 727-730.

[10] Lj. B. Ciric, *Some recent results in metrical fixed point theory*, University of Belgrade (2003).

[11] G. Das and J. P. Debata, *Fixed points of quasi-nonexpansive mappings*, Indian J. Pure and Appl. Math. 17, (1986), 1263-1269.

[12] F. Khojasteh, E. Karapinar, S. Radenovic, *$\theta-$metric space: A generalization*, Mathematical Problems in Engineering, 2013, Article ID 504609, 7 pages.

[13] A. M. Harder, T. L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japonica 33, 5 (1988), 693-706.

[14] C. O. Imoru, M. O. Olatinwo, *On the stability of Picard and Mann iteration processes*, Carpathian J. Math. 19, 2 (2003), 155-160.

[15] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. 44, 1 (1974), 147-150.

[16] J. R. Jachymski, *An extension of A. Ostrowski's theorem on the round-off stability of iterations*, Aequ. Math. 53, (1997), 242-253.

[17] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly 83, 4 (1976), 261-263.

[18] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 10, (1968), 71-76.

[19] R. Kannan, *Some results on fixed points III,* Fund. Math. 70, 2 (1971), 169-177.

[20] R. Kannan, *Construction of fixed points of a Class of Nonlinear Mappings*, J. Math. Anal. Appl. 41, (1973), 430-438.

[21] V. Karakaya, K. Dogan, F. Gürsoy, M. Ertürk, *Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces*, Abstract and Applied Analysis, 2013, Article ID 560258, 9 pages.

[22] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 44, (1953), 506-510.

[23] M. O. Olatinwo, *Some stability results for Picard and Mann iteration processes using contractive condition of integral type*, Creative Math. & Inf. 19, 1 (2010), 57-64.

[24] M. O. Olatinwo, *On almost surely $T-$stability and convergence results for two random iterative algorithms in separable Banach space*, Octogon Mathematical Magazine 18, 1 (2010), 73-83.

[25] M. O. Olatinwo, *Some stability and strong convergence results for the Jungck-Ishikawa iteration process*, Creative Math. & Inf. 17, (2008), 33-42.

[26] M. O. Olatinwo, *Some stability results in complete metric space*, Acta Univ. Palacki. Olomuc. Fac. rer. nat. Mathematica 48, (2009), 83-92.

[27] M. O. Olatinwo, *Stability and convergence results for Das-Debata type iterative process using simultaneous set of contractive conditions of integral type*, Bulletin of the Allahabad Mathematical Society 30, 1 (2015), 75-94.

[28] M. O. Olatinwo, *Stability and convergence results for Picard, Mann, Ishikawa and Jungck type iterative algorithms for Akram-Zafar-Siddiqui type contraction mappings*, Nonlinear Analysis Forum 21, 1 (2016), 65-75.

[29] M. O. Olatinwo, M. Postolache, *Stability results for Jungck-type iterative processes in convex metric spaces*, Applied Mathematics and Computation 218, (2012), 6727-6732.

[30] M. O. Osilike, A. Udomene, *Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings*, Indian J. Pure Appl. Math. 30, 12 (1999), 1229-1234.

[31] A. M. Ostrowski, *The round-off stability of iterations*, Z. Angew. Math. Mech. 47 (1967), 77-81.

[32] B. E. Rhoades, Fixed point theorems and stability results for fixed point iteration procedures, Indian J. Pure Appl. Math. 21, 1 (1990), 1-9.

[33] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures II*, Indian J. Pure Appl. Math. 24, 11 (1993), 691-703.

[34] I. A. Rus, A. Petrusel, G. Petrusel, *Fixed point theory*, 1950-2000, Romanian Contributions, House of the Book of Science, Cluj Napoca, (2002).

[35] S. L. Singh, C. Bhatnagar, S. N. Mishra, *Stability of Jungck-type iterative procedures*, Int. J. Math. & Math. Sc. 19, (2005), 3035-3043.

[36] S. Thianwan, *Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space*, Journal of Computational and Applied Mathematics 224, 2 (2009), 688-695.

[37] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. 23, (1972), 292-298.

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