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INTEGRAL REPRESENTATIONS OF GENERALIZED CLASSES OF CONCAVE UNIVALENT FUNCTIONS DEFINED BY SALAGEAN OPERATOR

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Abstract. In this research work, we prove a new integral representations for the generalized classes of concave univalent functions defined by Salagean operator denoted by $C_n(0)$, $C_n(p)$ and $C_n(\alpha)$, using a function of positive real part.Our results unify the ealier ones.

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1. Introduction

The study of concave univalent functions was introduced in [2], where a meromorphic and injective function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

denoted by C_0,was considered in a neighborhood of the origin and map the unit disk denoted as $\mathbb{U} = \{|z| < 1\}$ onto a concave domain $\mathbb{E}$ which is the exterior of a convex domain.

Avkhadiev and Wirths, studied the inner and the outer radius of the ring domain which is the domain of variability of a_2 for such function f and that $f \in C_0$ implies that

$$\Phi(z) = z + 2\frac{f'(z)}{f''(z)}. \tag{2}$$

is holomorphic in $\mathbb{U}$ and maps $\mathbb{U}$ to itself.

This concept was further studied by researchers (see [1], [4], [8]), where concave univalent function was classified in to three different classes define as follows:

Definition 1[8]
A meromorphic, univalent function f is said to be in the class $C_o(0)$, has a simple pole at the origin and the representation

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n. \tag{3}$$

Definition 2[8]
A meromorphic, univalent function f is said to be in the class $C_o(p)$ for $p \in (0,1)$ has a simple pole at p.

Definition 3[8]
An analytic, univalent function f of the form (1) is said to be in the class $C_o(\alpha)$, if $f(1) = \infty$ and an opening angle of $f(\mathbb{E})$ at ∞ is less than or equal to $\alpha\pi$.

The geometric properties of the functions in the above definitions were given in [5, 7, 8] as follows:

Theorem 1. *Let $f : \mathbb{U} \to \mathbb{E}$, $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be a meromorphic function. The function f is said to be in the class $C_o(0)$ if and only if the inequality*

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) < 0, z \in \mathbb{U}. \tag{4}$$

holds

Theorem 2. *Let $f : \mathbb{U} \to \mathbb{E}$ be a meromorphic function. The function f is said to be in the class $C_o(p)$, if and only if for $z \in \mathbb{U}$*

$$Re\left(1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < 0. \tag{5}$$

Theorem 3. *Let $\alpha \in (1,2]$. An analytic function f with $f(0) = f'(0) - 1 = 0$ is said to be in the class $C_o(\alpha)$, if and only if for $z \in \mathbb{U}$*

$$Re\left(1 + z\frac{f''(z)}{f'(z)} - \frac{\alpha+1}{2}\frac{1+z}{1-z}\right) < 0. \tag{6}$$

A factor $\frac{2}{\alpha-1}$ has to be added to the characterization in case a normalization is required and this was considered in [3], which showed that an analytic function f maps the unit disk $\mathbb{U}$ onto a concave domain $\mathbb{E}$ of angle $\pi\alpha$ if and only if $Rep(z) > 0$, $z \in \mathbb{U}$, where

$$p(z) = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right]. \tag{7}$$

The salagean differential operator denoted as D^n is define as $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, $D^n f(z) = D(D^{n-1}f(z)), (n \in \mathbb{N} = 1, 2, ...)$ and its integral operator define as $I^0 f(z) = f(z)$, $I^1 f(z) = \int_0^z \frac{f(t)}{t}dt$, $I^n f(z) = I(I^{n-1}f(z)), (n \in \mathbb{N})$. Both appeared in [10].

We use the above operator to define new classes of concave univalent function.

Definition 4
Let $f : \mathbb{U} \to \mathbb{E}$, $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be a meromorphic function. The function f is said to be in the class $C_n(0)$ if and only if the inequality

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < 0, z \in \mathbb{U}, n \geq 1. \tag{8}$$

holds.

Definition 5
Let $f : \mathbb{U} \to \mathbb{E}$ be a meromorphic function. Then the function f is said to be in the class $C_n(p)$, if and only if for $z \in \mathbb{U}$, $n \geq 1$.

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < .0 \tag{9}$$

Definition 6
An analytic function f with $f(0) = f'(0) - 1 = 0$ is said to be in the class $C_n(\alpha)$, if and only if for $z \in \mathbb{U}$, $n \geq 1$ and $\alpha \in (1, 2]$

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{\alpha+1}{2}\frac{1+z}{1-z}\right) < 0. \tag{10}$$

We note that the geometric inequalities of the classes $C_n(0)$, $C_n(p)$ and $C_n(\alpha)$ belong to the class P which is of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{11}$$

with $Rep(z) > 0$ and that

$$p(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t), \mu(t)(0 \leq t \leq 2\pi). \tag{12}$$

which is known as Herglotz formula see([9]). It has been shown in [8], that the function $\varphi : \mathbb{U} \to \mathbb{U}$, expressed as $z \to \frac{1+z\varphi(z)}{1-z\varphi(z)}$, holomorphic in $\mathbb{U}$, maps the unit disk onto itself, normalized by $0 \to 1$ and that

$$\frac{1+z\varphi(z)}{1-z\varphi(z)} = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t). \tag{13}$$

In the next section, we prove the integral representations for the classes $C_n(0)$, $C_n(p)$ and $C_n(\alpha)$ using the function of positive real part .

$$p(z) = \frac{1+z\varphi(z)}{1-z\varphi(z)}. \tag{14}$$

2. Main Results

Theorem 4. *Let $n \in \mathbb{N}$, $f : \mathbb{U} \to \mathbb{E}$, where $f(z) = \frac{1}{z} + \sum_{k=0} a_k z^k$ be a meromorphic function. $f \in C_n(0)$ if and only if there exists a function $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic in $\mathbb{U}$, such that for $z \in \mathbb{U}$, then*

$$f(z) = I_n \left\{ \frac{1}{z} exp \left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)} dt \right) \right\}. \tag{15}$$

Proof. The function $f \in C_n(0)$, if and only if there exists the function φ such that

$$\frac{D^{n+1}f(z)}{D^n f(z)} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$

From the relation

$$D^{n+1}f(z) = z(D^n f(z))'$$

We have that

$$\frac{z(D^n f(z))'}{D^n f(z)} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{z(D^n f(z))'}{D^n f(z)} + 1 = -\frac{2z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{1}{z} + \frac{(D^n f(z))'}{D^n f(z)} = -\frac{2\varphi(z)}{1-z\varphi(z)}$$

$$\frac{d}{dz}log(z(D^n f(z))) = -\frac{2\varphi(z)}{1-z\varphi(z)}$$
$$log(z(D^n f(z)) = -\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt$$
$$z(D^n f(z)) = exp\left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right)$$
$$D^n f(z) = \frac{1}{z}exp\left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right).$$
$$f(z) = I_n\left\{\frac{1}{z}exp\left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right)\right\}.$$

Conversely, if $\varphi : \mathbb{U} \to \mathbb{U}$ is holomorphic function, the function

$$f(z) = I_n\left\{\frac{1}{z}exp\left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right)\right\}. \tag{16}$$

Corollary 5. *If $n = 1$, then we have*

$$f(z) = \int_0^z \left\{\frac{1}{s^2}exp\left(-\int_0^s \frac{2\varphi(t)}{1-t\varphi(t)}dt\right)\right\}. \tag{17}$$

which is the result obtained in [8].

Theorem 6. *Let $p \in (0,1)$, $n \in \mathbb{N}$, $f : \mathbb{D} \to \mathbb{E}$ be a meromorphic function. $f \in C_n(p)$ if and only if there exists a function $\varphi : \mathbb{U} \to \bar{\mathbb{U}}$ holomorphic in $\mathbb{U}$, such that for $z \in \mathbb{U}$, then*

$$f(z) = I_n\left\{\frac{z}{(z-p)^2(1-pz)^2}exp\left\{-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right\}\right\}. \tag{18}$$

Proof. Let $p \in (0,1)$. The function $f \in C_n(p)$, if and only if there exist the function φ such that

$$\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{z+p}{z-p} - \frac{1+zp}{1-zp} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$
$$\frac{D^{n+1}f(z)}{D^n f(z)} + \left(\frac{2z}{z-p} - 1\right) - \left(\frac{2zp}{1-zp} + 1\right) + 1- = -\frac{2z\varphi(z)}{1-z\phi(z)}$$
$$\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{2z}{z-p} - 1 - \frac{2zp}{1-zp} - 1 + 1 = -\frac{2z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{2z}{z-p} - 1 - \frac{2zp}{1-zp} = -\frac{2z\varphi(z)}{1-z\varphi(z)}$$

From the relation

$$D^{n+1}f(z) = z(D^n f(z))^{'}$$

then

$$\frac{z(D^n f(z))^{'}}{D^n f(z)} + \frac{2z}{z-p} - 1 - \frac{2zp}{1-zp} = -\frac{2z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{(D^n f(z))^{'}}{D^n f(z)} + \frac{2}{z-p} - \frac{1}{z} - \frac{2p}{1-zp} = -\frac{2\varphi(z)}{1-z\varphi(z)}$$

$$\frac{d}{dz}\left\{logD^n f(z) + 2log(z-p) + 2log(1-pz) - logz\right\} = -\frac{2\varphi(z)}{1-z\varphi(z)}$$

$$log\frac{D^n f(z)(z-p)^2(1-pz)^2}{z} = -\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt$$

$$D^n f(z) = \frac{z}{(z-p)^2(1-pz)^2}exp\left(-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right)$$

$$f(z) = I_n\left\{\frac{z}{(z-p)^2(1-pz)^2}exp\left\{-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right\}\right\}.$$

Conversely, if $\varphi : \mathbb{U} \to \mathbb{U}$ is holomorphic function, the function

$$f(z) = I_n\left\{\frac{z}{(z-p)^2(1-pz)^2}exp\left\{-\int_0^z \frac{2\varphi(t)}{1-t\varphi(t)}dt\right\}\right\}.$$

Corollary 7. *If* $n = 1$,*then*

$$f(z) = \int_0^z \left\{\frac{1}{(s-p)^2(1-sp)^2}exp\left\{-\int_0^s \frac{2\phi(t)}{1-t\phi(t)}dt\right\}.\right\} \tag{19}$$

which is the result obtained in [8].

Theorem 8. *Let* $\alpha \in (1,2]$, $n \in \mathbb{N}$ *and* f *be an analytic function with* $f(0) = f^{'}(0) - 1 = 0$. *Then* $f \in C_n(\alpha)$ *if and only if there exists a function* $\varphi : \mathbb{U} \to \mathbb{U}$ *holomorphic in* $\mathbb{U}$, *such that for* $z \in \mathbb{U}$ *then*

$$f(z) = I_n\left\{\frac{z}{(1-z)^{\alpha+1}}exp\left(-(\alpha-1)\int_0^z \frac{\varphi(t)}{1-t\varphi(t)}dt\right)\right\}. \tag{20}$$

Proof. The function $f \in C_n(\alpha)$, if and only if there exist a function φ such that

$$-\frac{2}{\alpha-1}\left[\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{\alpha+1}{2}\frac{1+z}{1-z}\right] = \frac{1+z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{2}{\alpha-1}\left[\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{\alpha+1}{2}\frac{1+z}{1-z}\right] = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{2}{\alpha-1}\left[\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{\alpha+1}{2}\left[\frac{2z}{1-z}+1\right]\right] = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{2}{\alpha-1}\left[\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{(\alpha+1)z}{1-z} - \frac{\alpha+1}{2}\right] = -\frac{1+z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{2}{\alpha-1}\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{2z(\alpha+1)}{(\alpha-1)(1-z)} - \frac{\alpha+1}{\alpha-1} + 1 = -\frac{1+z\varphi(z)}{1-z\varphi(z)} + 1$$

$$\frac{2}{\alpha-1}\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{2z(\alpha+1)}{(\alpha-1)(1-z)} - \frac{2}{\alpha-1} = -\frac{2z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{z(\alpha+1)}{(1-z)} - 1 = -(\alpha-1)\frac{z\varphi(z)}{1-z\varphi(z)}$$

From the relation

$$D^{n+1}f(z) - z(D^n f(z))'$$

then

$$\frac{z(D^n f(z))'}{D^n f(z)} - \frac{z(\alpha+1)}{(1-z)} - 1 = -(\alpha-1)\frac{z\varphi(z)}{1-z\varphi(z)}$$

$$\frac{(D^n f(z))'}{D^n f(z)} - \frac{(\alpha+1)}{(1-z)} - \frac{1}{z} = -(\alpha-1)\frac{\varphi(z)}{1-z\varphi(z)}$$

$$\frac{d}{dz}\left[log D^n f(z) + (\alpha+1)log(1-z) - log z\right] = -(\alpha-1)\frac{\varphi(z)}{1-z\varphi(z)}$$

$$log\frac{D^n f(z)(1-z)^{\alpha+1}}{z} = -(\alpha-1)\int_0 \frac{\varphi(z)}{1-z\varphi(z)}$$

$$\frac{D^n f(z)(1-z)^{\alpha+1}}{z} = exp\left\{-(\alpha-1)\int_0^z \frac{\varphi(t)}{1-t\varphi(t)}dt\right\}$$

$$D^n f(z)(1-z)^{\alpha+1} = zexp\left\{-(\alpha-1)\int_0^z \frac{\varphi(t)}{1-t\varphi(t)}dt\right\}$$

$$f(z) = I_n\left\{\frac{z}{(1-z)^{\alpha+1}}exp\left(-(\alpha-1)\int_0^z \frac{\varphi(t)}{1-t\varphi(t)}dt\right)\right\}.$$

Conversely, if $\varphi : \mathbb{U} \to \mathbb{U}$ is holomorphic function, the function

$$f(z) = I_n \left\{ \frac{z}{(1-z)^{\alpha+1}} exp\left(-(\alpha-1) \int_0^z \frac{\varphi(t)}{1-t\varphi(t)} dt \right) \right\}. \tag{21}$$

Corollary 9. *If $n = 1$, then*

$$f(z) = \int_0^z \left\{ \frac{1}{(1-s)^{\alpha+1}} exp\left(-(\alpha-1) \int_0^s \frac{\varphi(t)}{1-t\varphi(t)} dt \right) \right\}. \tag{22}$$

which is the result obtained in [8]

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