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GENERALIZED VISCOSITY APPROXIMATION METHOD FOR EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new iterative scheme by the generalized viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves and extends some recent results.

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1. Introduction

Let H be a real Hilbert space, let A be a bounded operator on H. In this paper, we assume that A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \forall x \in H$. Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbf{R}$. The equilibrium problem for $\phi : C \times C \to \mathbf{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0 \text{ for all } v \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(\phi)$. The equilibrium problem (1) includes as special cases numerous problems in physics, optimization and economics. Some authors have proposed some useful methods for solving the equilibrium problem (1); see [6], [10] and [18].

A mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Let $F(T)$ denote the fixed points set of T. Also, a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$, where $\alpha \in [0, 1)$ is a constant. In 2000, Mudafi [15] proved the following strong convergence theorem.

Theorem 1. *[15] Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive self-mapping on C such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)$$

for all $n \geq 1$, where $\varepsilon_n \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \text{and} \quad \lim_{n\to\infty} |\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}| = 0.$$

Then, the sequence $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Such a method for approximation of fixed points is called the viscosity approximation method.

Finding an optimal point in the intersection F of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g.,[2] and [5]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbf{R}$ over F is of wide interdisciplinary interest and practical importance see, e.g., [1], [4], [8] and [24]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [11] and [24]. The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by inner product of H) from any given point a in H is the simplest case of our problem.

Yao et al. [22] introduced the iterative sequence:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) W_n x_n \text{ for all } \ n \geq 0.$$

where f is a contraction on H, $A : H \to H$ is a strongly positive bounded linear operator, $\gamma > 0$ is a constant, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Under very mild conditions on the parameters, it was proved that the sequence $\{x_n\}$ converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^* \rangle \leq 0 \text{ for all } \ x^* \in F,$$

which is the optimality condition for minimization problem

$$\min_{x\in F}\frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

On the other hand, Ceng and Yao [7] introduced an iterative scheme by

$$\begin{cases} \phi(u_n,x)+\frac{1}{r_n}\langle x-u_n,u_n-x_n\rangle\geq 0, \quad \text{for all } x\in C,\\ y_n=(1-\gamma_n)x_n+\gamma_n W_n u_n,\\ x_{n+1}=\beta_n W_n y_n+\alpha_n f(y_n)+(1-\beta_n-\alpha_n)x_n, \end{cases} \tag{2}$$

where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n\leq 1$ and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1,T_2,\dots,T_n,\dots$ and $\lambda_1,\lambda_2,\dots,\lambda_n,\dots$.

Razani and Yazdi [16], motivated by Yao et al. [22] and Ceng and Yao [7], introduced a new iterative scheme by the viscosity approximation method:

$$\begin{cases} \phi(u_n,x)+\frac{1}{r_n}\langle x-u_n,u_n-x_n\rangle\geq 0, \quad \text{for all } x\in C,\\ y_n=(1-\gamma_n)x_n+\gamma_n W_n u_n,\\ x_{n+1}=\alpha_n\gamma f(y_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n, \end{cases} \tag{3}$$

where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$, f is a contraction, A is a strongly positive bounded linear operator, $\gamma>0$ is a constant and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1,T_2,\dots,T_n,\dots$ and $\lambda_1,\lambda_2,\dots,\lambda_n,\dots$ such that the common fixed points set $F:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. They proved the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by (3) converge strongly to $p\in F$, where $p=P_{\bigcap_{n=1}^{\infty}F(T_n)\bigcap EP(\phi)}(I-A+\gamma f)(p)$.

Moreover, Duan and He [9] combined a sequence of contractive mappings $\{f_n\}$ and proposed a generalized viscosity approximation method. They considered the following iterative algorithm:

$$x_{n+1}=\alpha_n f_n(x_n)+(1-\alpha_n)Tx_n,$$

where T is a nonexpansive mapping and $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved the sequence $\{x_n\}$ converges strongly to $p\in F(T)$ which is a unique solution of a variational inequality.

In this paper, inspired by above results, we introduce a new iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves the main results of [7] and [16].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and the norm $\|.\|$. We denote weak convergence and strong convergence by notation $\rightharpoonup$ and $\rightarrow$, respectively. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \;\Leftrightarrow\; \langle x - z, z - y \rangle \geq 0 \text{ for all } y \in C.$$

Now, we collect some lemmas which will be used in the proofs for the main results.

Lemma 2. *[3] Let C be a nonempty closed convex subset of H and $\phi : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle x - z, z - x \rangle \geq \; 0 \textit{ for all } y \in C.$$

Lemma 3. *[6] Assume that $\phi : C \times C \rightarrow \mathbb{R}$ satisfies $(A_1) - (A_4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r x = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \textit{ for all } y \in C\}$$

for all $x \in H$. Then, the following hold:
(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(iii) $F(T_r) = EP(\phi)$;
(iv) $EP(\phi)$ is closed and convex.

Lemma 4. *[14] Assume A is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 5. *[20] Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\lambda \in [0, 1]$,*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|.$$

Lemma 6. *[19] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 7. *[21] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n v_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} v_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n v_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 8. *[15] Assume A is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 9. *[12] Each Hilbert space H satisfies Opial's condition, i. e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for each $y \in H$ with $x \neq y$.

Let H be a real Hilbert space and A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let f be a contraction of C into itself with constant $\alpha \in [0,1)$ and $0 < \alpha\gamma < \overline{\gamma}$ where γ is some constant. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of H into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{4}$$

Such a mapping W_n is called the $W-$mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$; see [13].

Lemma 10. *[17] Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 1. *[23] It can be known from Lemma 10 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$*

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \varepsilon.$$

Remark 2. *[23] Using Lemma 10, one can define mapping $W : C \to C$ as follows:*

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for all $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 1 that for an arbitrary $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \varepsilon.$$

This implies that $\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0$.

Throughout this paper, we always assume that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$.

Lemma 11. *[17] Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Main result

In this section, we prove the following strong convergence theorem for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Suppose the contractive mapping sequence $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C.

Theorem 12. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \times C \to \mathbf{R}$ be a bifunction satisfying $(A_1) - (A_4)$, A be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$ and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi) \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ is a real sequence satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(iv) $0 < \liminf_{n\to\infty} r_n$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Let $\{f_n\}$ be a sequence of ρ_n-contractive self-maps of C with

$$0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1.$$

Assume $x_0 \in C$, $0 < \gamma < \frac{\overline{\gamma}}{\rho_u}$ where γ is some constant, $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. If one define $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in C$, then the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases} \phi(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0 \text{ for all } x \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n W_n u_n, \\ x_{n+1} = \alpha_n \gamma f_n(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n, \end{cases} \tag{5}$$

converge strongly to $x^* \in F$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)(x^*)$.

Proof. Let $Q = P_F$. Then

$$\begin{aligned} &\|Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y)\| \\ &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ &\leq \|(I - A)(x) - (I - A)(y)\| + \gamma\|f(x) - f(y)\| \\ &\leq (1 - \overline{\gamma})\|x - y\| + \gamma\alpha\|x - y\| \\ &= (1 - (\overline{\gamma} - \gamma\alpha))\|x - y\| \end{aligned}$$

for all $x, y \in F$. Therefore, $Q(I - A + \gamma f)$ is a contraction of F into itself. So, there exists a unique element $x^* \in F$ such that $x^* = Q(I - A + \gamma f)(x^*) = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)(x^*)$. Note that from the condition (i), we may assume, without loss of generality, $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\alpha_n A)x,x\rangle &= (1-\beta_n)-\alpha_n\langle Ax,x\rangle\\ &\geq 1-\beta_n-\alpha_n\|A\|\geq 0,\end{aligned} \tag{6}$$

that is to say $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}&\|(1-\beta_n)I-\alpha_n A\|\\ &=\sup\{\langle((1-\beta_n)I-\alpha_n A)x,x\rangle : x\in H, \|x\|=1\}\\ &=\sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\leq 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Let $p\in F$. From the definition of T_r, we know that $u_n=T_{r_n}x_n$. It follows that

$$\|u_n-p\|=\|T_{r_n}x_n-T_{r_n}p\|\leq\|x_n-p\|,$$

and hence

$$\begin{aligned}\|y_n-p\| &= \|(1-\gamma_n)(x_n-p)+\gamma_n(W_nu_n-p)\|\\ &\leq (1-\gamma_n)\|x_n-p\|+\gamma_n\|W_nu_n-p\|\\ &\leq (1-\gamma_n)\|x_n-p\|+\gamma_n\|u_n-p\|\\ &\leq (1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\|=\|x_n-p\|.\end{aligned}$$

First, we claim that $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, from (4), (3) and (6), we obtain

$$\begin{aligned}&\|x_{n+1}-p\|\\ =\;&\|\alpha_n(\gamma f_n(y_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(W_ny_n-p)\|\\ \leq\;&(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f_n(y_n)-Ap\|\\ \leq\;&(1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\gamma\|f_n(y_n)-f_n(p)\|+\alpha_n\|\gamma f_n(p)-Ap\|\\ \leq\;&(1-\alpha_n(\overline{\gamma}-\rho_n\gamma))\|x_n-p\|+\alpha_n\|\gamma f_n(p)-Ap\|.\end{aligned} \tag{7}$$

By induction, $\|x_n-p\|\leq\max\{\|x_0-p\|,\frac{1}{\overline{\gamma}-\rho_u\gamma}\|\gamma f_n(p)-Ap\|\},\quad n\geq 1$. From the uniform convergence of $\{f_n\}$ on any bounded subset of C, we conclude $\{f_n(p)\}$ is bounded. Hence $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{y_n\}$, $\{f_n(y_n)\}$, $\{W_nu_n\}$ and $\{W_ny_n\}$.

Define

$$x_{n+1}=(1-\beta_n)z_n+\beta_nx_n,\quad n\geq 0.$$

Then

$$
\begin{aligned}
z_{n+1}-z_n = &\ \tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\
= &\ \tfrac{\alpha_{n+1}\gamma f_n(y_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\
&\ -\tfrac{\alpha_n\gamma f_n(y_n)+((1-\beta_n)I-\alpha_n A)W_n y_n}{1-\beta_n}\\
= &\ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f_n(y_{n+1}) - \tfrac{\alpha_n}{1-\beta_n}\gamma f_n(y_n) + W_{n+1}y_{n+1}\\
&\ -W_n y_n + \tfrac{\alpha_n}{1-\beta_n}AW_n y_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}y_{n+1}\\
= &\ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f_n(y_{n+1}) - AW_{n+1}y_{n+1}] + \tfrac{\alpha_n}{1-\beta_n}[AW_n y_n -\\
&\ \gamma f_n(y_n)] + W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n,
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
&\|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\
\le\ & \|y_{n+1}-y_n\|\\
=\ & \|(1-\gamma_{n+1})x_{n+1}+\gamma_{n+1}W_{n+1}u_{n+1}-(1-\gamma_n)x_n-\gamma_n W_n u_n\|\\
\le\ & (1-\gamma_{n+1})\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\|\\
& +\gamma_{n+1}\|W_{n+1}u_{n+1}-W_n u_n\|+|\gamma_{n+1}-\gamma_n|\|W_n u_n\|\\
\le\ & (1-\gamma_{n+1})\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\|\\
& +\gamma_{n+1}(\|W_{n+1}u_{n+1}-W_{n+1}u_n\|+\|W_{n+1}u_n-W_n u_n\|)\\
& +|\gamma_{n+1}-\gamma_n|\|W_n u_n\|.
\end{aligned} \tag{9}
$$

From (4), Since T_i and $U_{n,i}$ are nonexpansive, we have for each $n \ge 1$

$$
\begin{aligned}
\|W_{n+1}u_n-W_n u_n\| =\ & \|\lambda_1 T_1 U_{n+1,2}u_n-\lambda_1 T_1 U_{n,2}u_n\|\\
\le\ & \lambda_1\|U_{n+1,2}u_n-U_{n,2}u_n\|\\
=\ & \lambda_1\|\lambda_2 T_2 U_{n+1,3}u_n-\lambda_2 T_2 U_{n,3}u_n\|\\
\le\ & \lambda_1\lambda_2\|U_{n+1,3}u_n-U_{n,3}u_n\|\\
\le\ & \dots\\
\le\ & \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}u_n-U_{n,n+1}u_n\|\\
\le\ & M\textstyle\prod_{i=1}^n \lambda_i,
\end{aligned} \tag{10}
$$

and similarly

$$
\|W_{n+1}y_n-W_n y_n\| \le \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}y_n-U_{n,n+1}y_n\| \le M\prod_{i=1}^n \lambda_i, \tag{11}
$$

for some constant $M \geq 0$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$,

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C, \tag{12}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \text{for all } y \in C. \tag{13}$$

Putting $y = u_{n+1}$ in (12) and $y = u_n$ in (13), we obtain

$$\phi(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$

and

$$\phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A_2)

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0,$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, we may assume that there exists a real number r such that $0 < r < r_n$ for all $n \geq 0$. Therefore

$$\begin{aligned} & \|u_{n+1} - u_n\|^2 \\ \leq\ & \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \tfrac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ \leq\ & \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}. \end{aligned}$$

So

$$\begin{aligned} \|u_{n+1} - u_n\| \leq\ & \|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\| \\ \leq\ & \|x_{n+1} - x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L, \end{aligned} \tag{14}$$

where $L = \sup\{\|u_n - x_n\| : n \geq 0\}$. Substituting (10) and (14) in (9), we have

$$\begin{aligned} & \|W_{n+1}y_{n+1} - W_{n+1}y_n\| \\ \leq\ & (1 - \gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| \\ & \quad +\gamma_{n+1}(\|x_{n+1} - x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L) \\ & \quad +\gamma_{n+1}M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} - \gamma_n|\|W_n u_n\| \\ \leq\ & \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L \\ & \quad +M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} - \gamma_n|\|W_n u_n\|. \end{aligned} \tag{15}$$

Combining (8),(11)and (15), we obtain

$$\begin{aligned}
& \|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\
\leq\ & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
& \quad + \tfrac{\alpha_n}{1-\beta_n}(\|AW_ny_n\| + \|\gamma f_n(y_n)\|) \\
& \quad + \|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_ny_n\| \\
& \quad - \|x_{n+1}-x_n\| \\
\leq\ & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
& \quad + \tfrac{\alpha_n}{1-\beta_n}(\|AW_ny_n\| + \|\gamma f_n(y_n)\|) \\
& \quad + [\|x_{n+1}-x_n\| + |\gamma_{n+1}-\gamma_n|\|x_n\| \\
& \quad + \tfrac{1}{r}|r_n - r_{n+1}|L + M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} \\
& \quad - \gamma_n|\|W_nu_n\|] + M\textstyle\prod_{i=1}^n \lambda_i - \|x_{n+1}-x_n\| \\
\leq\ & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
& \qquad + \tfrac{\alpha_n}{1-\beta_n}(\|AW_ny_n\| + \|\gamma f_n(y_n)\|) \\
& \qquad + |\gamma_{n+1}-\gamma_n|\|x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L \\
& \qquad + |\gamma_{n+1}-\gamma_n|\|W_nu_n\| + 2M\textstyle\prod_{i=1}^n \lambda_i.
\end{aligned} \tag{16}$$

Thus it follows from (16) and condition $(i)-(iv)$ that (noting that $0<\lambda_i\leq b<1$ for all $i\geq 1$)

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence by Lemma 6, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

From (14) and $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$, $\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0$. From (5),

$$\|x_n - W_ny_n\| \leq \|x_{n+1}-x_n\| + \alpha_n\|\gamma f_n(y_n) - AW_ny_n\| + \beta_n\|x_n - W_ny_n\|.$$

That is $\|x_n - W_n y_n\| \leq \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f_n(y_n) - AW_n y_n\|$. It follows that

$$\lim_{n\to\infty} \|x_n - W_n y_n\| = 0. \tag{17}$$

For $p \in F$, since T_r is firmly nonexpansive, we have

$$\begin{aligned} \|u_n - p\|^2 = & \|T_{r_n}x_n - T_{r_n}p\|^2 \leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\ = & \langle u_n - p, x_n - p\rangle = \tfrac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 \\ & -\|x_n - u_n\|^2), \end{aligned}$$

and hence $\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2$. Therefore

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ = & \|\alpha_n \gamma f_n(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n - p\|^2 \\ = & \|(1-\beta_n)(W_n y_n - p) + \beta_n(x_n - p) + \alpha_n \gamma f_n(y_n) \\ & -\alpha_n AW_n y_n\|^2 \\ = & \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 + \|\beta_n(x_n - p) \\ & +(1-\beta_n)(W_n y_n - p)\|^2 + 2\alpha_n\langle\beta_n(x_n - p) \\ & +(1-\beta_n)(W_n y_n - p), \gamma f_n(y_n) - AW_n y_n\rangle \\ \leq & \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n)\|W_n y_n \\ & -p\|^2 + 2\alpha_n(1-\beta_n)\langle W_n y_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ & +2\alpha_n\beta_n\langle x_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ \leq & (1-\beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 \\ & +2\alpha_n(1-\beta_n)\langle W_n y_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ & +2\alpha_n\beta_n\langle x_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ = & (1-\beta_n)\|(1-\gamma_n)(x_n - p) + \gamma_n(W_n u_n - p)\| + \beta_n\|x_n - p\|^2 \\ & +\alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 \\ & +2\alpha_n(1-\beta_n)\langle W_n y_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ & +2\alpha_n\beta_n\langle x_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ \leq & (1-\beta_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\beta_n)\gamma_n\|u_n - p\|^2 + \beta_n\|x_n \\ & -p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 + 2\alpha_n(1-\beta_n)\langle W_n y_n - p, \\ & \gamma f_n(y_n) - AW_n y_n\rangle + 2\alpha_n\beta_n\langle x_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ \leq & (1-\beta_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\beta_n)\gamma_n(\|x_n - p\|^2 \\ & -\|x_n - u_n\|^2) + \beta_n\|x_n - p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 \\ & +2\alpha_n(1-\beta_n)\langle W_n y_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ & +2\alpha_n\beta_n\langle x_n - p, \gamma f_n(y_n) - AW_n y_n\rangle \\ \leq & \|x_n - p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 + 2\alpha_n(1-\beta_n)(\|x_n - p\| \\ & \|\gamma f_n(y_n) - AW_n y_n\|) + 2\alpha_n\beta_n\|x_n - p\|\|\gamma f_n(y_n) - AW_n y_n\| \\ & -(1-\beta_n)\gamma_n\|x_n - u_n\|^2 \\ = & \|x_n - p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 + 2\alpha_n(\|x_n - p\| \\ & \|\gamma f_n(y_n) - AW_n y_n\|) - (1-\beta_n)\gamma_n\|x_n - u_n\|^2. \end{aligned}$$

Thus

$$\begin{aligned}
& (1-\beta_n)\gamma_n\|x_n-u_n\|^2 \\
\le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|\gamma f_n(y_n)-AW_ny_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f_n(y_n)-AW_ny_n\| \\
=\ & (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|) \\
& +\alpha_n^2\|\gamma f_n(y_n)-AW_ny_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f_n(y_n)-AW_ny_n\| \\
\le\ & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
& +\alpha_n^2\|\gamma f_n(y_n)-AW_ny_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f_n(y_n)-AW_ny_n\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$ and $\liminf_{n\to\infty}\gamma_n>0$, it is easy to see that $\liminf_{n\to\infty}(1-\beta_n)\gamma_n>0$. So

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{18}$$

Observe that

$$\begin{aligned}
\|y_n-u_n\| \le\ & \|y_n-x_n\|+\|x_n-u_n\| \\
\le\ & \gamma_n\|W_nu_n-x_n\|+\|x_n-u_n\| \\
\le\ & \gamma_n\|W_nu_n-W_ny_n+W_ny_n-x_n\|+\|x_n-u_n\| \\
\le\ & \gamma_n[\|y_n-u_n\|+\|W_ny_n-x_n\|]+\|x_n-u_n\|,
\end{aligned}$$

and hence $(1-\gamma_n)\|y_n-u_n\|\le\|W_ny_n-x_n\|+\|x_n-u_n\|$. So, from (17),(18) and $\limsup_{n\to\infty}\gamma_n<1$,

$$\lim_{n\to\infty}\|y_n-u_n\|=0 \tag{19}$$

and so $\lim_{n\to\infty}\|x_n-y_n\|=0$. Since

$$\begin{aligned}
\|W_nu_n-u_n\| \le\ & \|W_nu_n-W_ny_n\|+\|W_ny_n-x_n\|+\|x_n-u_n\| \\
\le\ & \|y_n-u_n\|+\|W_ny_n-x_n\|+\|x_n-u_n\|,
\end{aligned}$$

we also have $\lim_{n\to\infty}\|W_nu_n-u_n\|=0$. On the other hand, observe that

$$\|Wu_n-u_n\|\le\|W_nu_n-Wu_n\|+\|W_nu_n-u_n\|. \tag{20}$$

It follows from (20) and Remark 2, we obtain $\lim_{n\to\infty}\|Wu_n-u_n\|=0$.

Next, we claim that

$$\limsup_{n\to\infty}\langle\gamma f(x^*)-Ax^*,x_n-x^*\rangle\le 0, \tag{21}$$

where $x^*=P_{F(W)\bigcap EP(\phi)}(I-A+\gamma f)x^*$. First, we can choose a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\lim_{j\to\infty}\langle\gamma f(x^*)-Ax^*,u_{n_j}-x^*\rangle=\limsup_{n\to\infty}\langle\gamma f(x^*)-Ax^*,u_n-x^*\rangle.$$

Since $\{u_{n_j}\}$ is bounded, there exists a subsequence of $\{u_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume $u_{n_j} \rightharpoonup w$. From $\|Wu_n - u_n\| \to 0$, $Wu_{n_j} \rightharpoonup w$. Now, we show $w \in EP(\phi)$. By $u_n = T_{r_n}x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C.$$

From (A_2), $\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n)$, and hence $\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle \geq \phi(y, u_{n_j})$. Since $\frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup w$, from (A_4), we get

$$\phi(y, w) \leq 0 \text{ for all } y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\phi(y_t, w) \leq 0$. So, from (A_1) and (A_4),

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, w) \leq t\phi(y_t, y),$$

and so $\phi(y_t, y) \geq 0$. From (A_3), $\phi(w, y) \geq 0$ for all $y \in C$, and hence $w \in EP(\phi)$. Next, we show $w \in F(W)$. Assume $w \notin F(W)$. Since $u_{n_j} \rightharpoonup w$ and $Ww \neq w$, from Lemma 9 we have

$$\begin{array}{rl} & \liminf_{j\to\infty} \|u_{n_j} - w\| \\ < & \liminf_{j\to\infty} \|u_{n_j} - Ww\| \\ \leq & \liminf_{j\to\infty}(\|u_{n_j} - Wu_{n_j}\| + \|Wu_{n_j} - Ww\|) \\ \leq & \liminf_{j\to\infty} \|u_{n_j} - w\|. \end{array}$$

This is a contradiction. So, $w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, $w \in F$. Since $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)x^*$, we obtain

$$\begin{array}{rl} & \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \\ = & \lim_{j\to\infty}\langle \gamma f(x^*) - Ax^*, x_{n_j} - x^*\rangle \\ = & \lim_{j\to\infty}\langle \gamma f(x^*) - Ax^*, u_{n_j} - x^*\rangle \\ = & \langle \gamma f(x^*) - Ax^*, w - x^*\rangle \leq 0. \end{array}$$

From (17),

$$\begin{array}{l} \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, W_n y_n - x^*\rangle \\ = \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \leq 0. \end{array} \tag{22}$$

Finally, we prove that $\{x_n\}$ converges strongly to $x^* = P_{F(W)\bigcap EP(\phi)}(I - A + \gamma f)x^*$.

Indeed, from (3),

$$\begin{aligned}
& \|x_{n+1}-x^*\|^2 \\
= \ & \|\alpha_n(\gamma f_n(y_n)-Ax^*)+\beta_n(x_n-x^*) \\
& \quad +((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*)\|^2 \\
= \ & \alpha_n^2\|\gamma f_n(y_n)-Ax^*\|^2+\|\beta_n(x_n-x^*) \\
& \quad +((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*)\|^2 \\
& \quad +2\beta_n\alpha_n\langle x_n-x^*,\gamma f_n(y_n)-Ax^*\rangle \\
& \quad +2\alpha_n\langle((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*),\gamma f_n(y_n)-Ax^*\rangle \\
\leq \ & ((1-\beta_n-\alpha_n\overline{\gamma})\|W_ny_n-x^*\|+\beta_n\|x_n-x^*\|)^2 \\
& \quad +\alpha_n^2\|\gamma f_n(y_n)-Ax^*\|^2+2\beta_n\alpha_n\gamma\langle x_n-x^*,f_n(y_n)-f_n(x^*)\rangle \\
& \quad +2\beta_n\alpha_n\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \\
& \quad +2(1-\beta_n)\gamma\alpha_n\langle W_ny_n-x^*,f_n(y_n)-f_n(x^*)\rangle \\
& \quad +2(1-\beta_n)\alpha_n\langle W_ny_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \\
& \quad -2\alpha_n^2\langle A(W_ny_n-x^*),\gamma f_n(y_n)-Ax^*\rangle,
\end{aligned}$$

Which implies that

$$\begin{aligned}
& \|x_{n+1}-x^*\|^2 \\
\leq \ & [(1-\alpha_n\overline{\gamma})^2+2\rho_u\beta_n\alpha_n\gamma+2\rho_u(1-\beta_n)\alpha_n\gamma]\|x_n-x^*\|^2 \\
& \quad +2\beta_n\alpha_n\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \\
& \quad +\alpha_n^2\|\gamma f_n(y_n)-Ax^*\|^2 \\
& \quad +2(1-\beta_n)\alpha_n\langle W_ny_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \\
& \quad -2\alpha_n^2\langle A(W_ny_n-x^*),\gamma f_n(y_n)-Ax^*\rangle \\
\leq \ & [1-2\alpha_n(\overline{\gamma}-\rho_u\gamma)]\|x_n-x^*\|^2+\alpha_n^2\overline{\gamma}^2\|x_n-x^*\|^2 \\
& \quad +2\beta_n\alpha_n\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle+\alpha_n^2\|\gamma f_n(y_n)-Ax^*\|^2 \\
& \quad +2(1-\beta_n)\alpha_n\langle W_ny_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \\
& \quad +2\alpha_n^2\|\gamma f_n(y_n)-Ax^*\|\|A(W_ny_n-x^*)\| \\
= \ & [1-2\alpha_n(\overline{\gamma}-\rho_u\gamma)]\|x_n-x^*\|^2+\alpha_n\{\alpha_n(\overline{\gamma}^2\|x_n-x^*\|^2 \\
& \quad +\|\gamma f_n(y_n)-Ax^*\|^2+2\|\gamma f_n(y_n)-Ax^*\|\|A(W_ny_n-x^*)\|) \\
& \quad +2\beta_n\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle+ \\
& \quad +2(1-\beta_n)\langle W_ny_n-x^*,\gamma f_n(x^*)-Ax^*\rangle\}.
\end{aligned} \tag{23}$$

By Schwartzs inequality,

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \leq \ & \lim_{n\to\infty}\gamma\|x_n-x^*\|\|f_n(x^*)-f(x^*)\| \\
& +\limsup_{n\to\infty}\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle.
\end{aligned}$$

From (21),

$$\limsup_{n\to\infty}\langle x_n-x^*,\gamma f_n(x^*)-Ax^*\rangle \leq 0. \tag{24}$$

Since $\{x_n\}, \{f_n(y_n)\}$ and $\{W_n y_n\}$ are bounded, we can take a constant $M_1 \geq 0$ such that $\overline{\gamma}^2\|x_n - x^*\|^2 + \|\gamma f_n(y_n) - Ax^*\|^2 + 2\|\gamma f_n(y_n) - Ax^*\|\|A(W_n y_n - x^*)\| \leq M_1$, for all $n \geq 0$. From (23),

$$\|x_{n+1} - x^*\|^2 \leq [1 - 2\alpha_n(\overline{\gamma} - \rho_u\gamma)]\|x_n - x^*\|^2 + \alpha_n\xi_n, \tag{25}$$

where $\xi_n = 2\beta_n\langle x_n - x^*, \gamma f_n(x^*) - Ax^*\rangle + 2(1 - \beta_n)\langle W_n y_n - x^*,$ $\gamma f_n(x^*) - Ax^*\rangle + \alpha_n M_1$. By (i), (22) and (24), we get $\limsup_{n\to\infty} \xi_n \leq 0$. Now applying Lemma 7 to (25) concludes that $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Taking $f_n = f$ for all $n \in \mathbb{N}$ where f is a contraction on C into itself in Theorem 12, we get

Remark 3. *Theorem 12 is a generalization of [16, Theorem 2.11].*

Remark 4. *Let $T_n x = x$ for all $n \in \mathbb{N}$ and for all $x \in C$ in (4). Then, $W_n x = x$ for all $x \in C$ in Theorem 12. Therefore, Theorem 12 is a generalization of [16, Corollary 2.12].*

Remark 5. *Let $\phi(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 12, then Theorem 12 is a generalization of [16, Corollary 2.13].*

Remark 6. *Let $A = I$ (identity map) with constant $\overline{\gamma} = 1$, $\gamma = 1$ and $\eta_n = 1 - \alpha_n - \beta_n$ in Theorem 12, then Theorem 12 is a generalization of [7, Theorem 3.1].*

4. Numerical Test

In this section, we give an example to illustrate the scheme (5) given in Theorem 12.

Example 3.1 *Let $C = [-1, 1] \subset H = \mathbb{R}$ and define $\phi(x, y) = -5x^2 + xy + 4y^2$. It is easy to see verify that ϕ satisfies the conditions $(A_1) - (A_4)$. From Lemma 2.2, T_r is single-valued for all $r > 0$. Now, we deduce a formula for $T_r(x)$. For any $y \in [-1, 1]$ and $r > 0$, we have*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Leftrightarrow 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2 \geq 0.$$

Set $G(y) = 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4r, b = (r + 1)z - x$ and $c = xz - (5r + 1)z^2$. So its discriminate $\Delta = b^2 - 4ac$ is

$$\begin{aligned} \Delta = {} & [(r + 1)z - x]^2 - 16r(xz - (5r + 1)z^2) \\ = {} & (r + 1)^2 z^2 - 2(r + 1)xz + x^2 - 16rxz + (80r^2 + 16r)z^2 \\ = {} & [(9r + 1)z - x]^2. \end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C$, this is true if and only if $\Delta \leq 0$. That is, $[(9r+1)z - x]^2 \leq 0$. Therefore, $z = \frac{x}{9r+1}$, which yields $T_r(x) = \frac{x}{9r+1}$. So, from Lemma 3, we get $EP(\phi) = \{0\}$. Let $\alpha_n = \frac{1}{n}, \beta_n = \frac{n}{3n+1}, \lambda_n = \beta \in (0,1), \gamma_n = \frac{1}{2}, r_n = 1, T_n = I$, for all $n \in \mathbb{N}$, $Ax = x$ with coefficient $\overline{\gamma} = 1$, $f_n(x) = \frac{n}{3n+1}x$ and $\gamma = \frac{1}{2}$. Hence, $F = \bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi) = \{0\}$. Also, $W_n = I$. Indeed, from (4), we have

$$\begin{aligned}
W_1 = U_{1,1} = &\ \lambda_1 T_1 U_{1,2} + (1-\lambda_1)I = \lambda_1 T_1 + (1-\lambda_1)I,\\
W_2 = U_{2,1} = &\ \lambda_1 T_1 U_{2,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{2,3} + (1-\lambda_2)I)\\
= &\ \lambda_1\lambda_2 T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I,\\
W_3 = U_{3,1} = &\ \lambda_1 T_1 U_{3,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{3,3} + (1-\lambda_2)I)\\
&\quad +(1-\lambda_1)I\\
= &\ \lambda_1\lambda_2 T_1 T_2 U_{3,3} + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I\\
= &\ \lambda_1\lambda_2 T_1 T_2(\lambda_3 T_3 U_{3,4} + (1-\lambda_3)I) + \lambda_1(1-\lambda_2)T_1\\
&\quad +(1-\lambda_1)I\\
= &\ \lambda_1\lambda_2\lambda_3 T_1 T_2 T_3 + \lambda_1\lambda_2(1-\lambda_3)T_1 T_2 + \lambda_1(1-\lambda_2)T_1\\
&\quad +(1-\lambda_1)I.
\end{aligned}$$

By computing in this way by (4), we obtain

$$\begin{aligned}
W_n = U_{n,1} = &\ \lambda_1\lambda_2\ldots\lambda_n T_1 T_2 \ldots T_n\\
&\quad +\lambda_1\lambda_2\ldots\lambda_{n-1}(1-\lambda_n)T_1 T_2\ldots T_{n-1}\\
&\quad +\lambda_1\lambda_2\ldots\lambda_{n-2}(1-\lambda_{n-1})T_1 T_2\ldots T_{n-2}\\
&\quad +\lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I.
\end{aligned}$$

Since $T_n = I, \lambda_n = \beta$ for all $n \in \mathbb{N}$, we get

$$W_n = (\beta^n + \beta^{n-1}(1-\beta) + \ldots + \beta(1-\beta) + (1-\beta))I = I.$$

Then, from Lemma 7, the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by

$$\begin{cases} u_n = T_{r_n} x_n = \frac{1}{10}x_n,\\ y_n = \frac{1}{2}x_n + \frac{1}{2}W_n u_n = \frac{11}{20}x_n,\\ x_{n+1} = \frac{84n^2 - 33n^2 - 22}{40n(3n+1)}x_n, \end{cases} \tag{26}$$

converges strongly to $0 \in F$, where $0 = P_F(\frac{1}{6}I)(0)$.

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