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Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum

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Abstract

Given the Hermitian, symmetric and symplectic ensembles, it is shown that the probability that the spectrum belongs to one or several intervals satisfies a nonlinear PDE. This is done for the three classical ensembles: Gaussian, Laguerre and Jacobi. For the Hermitian ensemble, the PDE (in the boundary points of the intervals) is related to the Toda lattice and the KP equation, whereas for the symmetric and symplectic ensembles the PDE is an inductive equation, related to the so-called Pfaff-KP equation and the Pfaff lattice. The method consists of inserting time-variables in the integral and showing that this integral satisfies integrable lattice equations and Virasoro constraints.

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0. Introduction

Consider weights of the form $\rho(z)dz := e^{-V(z)}dz$ on an interval $F = [A, B] \subseteq \mathbb{R}$, with rational logarithmic derivative and subjected to the following boundary conditions:

$$-\frac{\rho'}{\rho} = V' = \frac{g}{f} = \frac{\sum_0^\infty b_i z^i}{\sum_0^\infty a_i z^i}, \quad \lim_{z\to A,B} f(z)\rho(z)z^k = 0 \text{ for all } k \geq 0, \tag{0.0.1}$$

together with a disjoint union of intervals,

$$E = \bigcup_1^r [c_{2i-1}, c_{2i}] \subseteq F \subseteq \mathbb{R}. \tag{0.0.2}$$

The data (0.0.1) and (0.0.2) define an algebra of differential operators

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i}. \tag{0.0.3}$$

Let $\mathcal{H}_n$, $\mathcal{S}_n$ and $\mathcal{T}_n$ denote the Hermitian ($M = \bar{M}^\top$), symmetric ($M = M^\top$) and "symplectic" ensembles ($M = \bar{M}^\top$, $M = J\bar{M}J^{-1}$), respectively. Traditionally, the latter is called the "symplectic ensemble," although the matrices involved are not symplectic! These conditions guarantee the reality of the spectrum of M. Then, $\mathcal{H}_n(E)$, $\mathcal{S}_n(E)$ and $\mathcal{T}_n(E)$ denote the subsets of $\mathcal{H}_n$, $\mathcal{S}_n$ and $\mathcal{T}_n$ with spectrum in the subset $E \subseteq F \subseteq \mathbb{R}$. The aim of this paper is to find PDEs for the probabilities

(0.0.4)

$$\begin{aligned} P_n(E) :&= P_n(\text{ all spectral points of } M \in E) \\ &= \frac{\int_{\mathcal{H}_n(E),\ \mathcal{S}_n(E) \text{ or } \mathcal{T}_n(E)} e^{-tr\ V(M)} dM}{\int_{\mathcal{H}_n(F),\ \mathcal{S}_n(F) \text{ or } \mathcal{T}_n(F)} e^{-tr\ V(M)} dM} \\ &= \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)} dz_k}{\int_{F^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)} dz_k}, \qquad \beta = 2, 1, 4 \text{ respectively,} \end{aligned}$$

for the Gaussian, Laguerre and Jacobi weights. The probabilities involve parameters β, a, b (see (0.1.1), (0.2.1) and (0.3.2)) and

$$\delta_{1,4}^\beta := 2\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 = \begin{cases} 0 & \text{for } \beta = 2 \\ 1 & \text{for } \beta = 1, 4. \end{cases}$$

The method used to obtain these PDEs involves inserting time-parameters into the integrals, appearing in (0.0.4) and to notice that the integrals obtained satisfy

- Virasoro constraints: linear PDEs in t and the boundary points of E, and
- integrable hierarchies:

ensemble	β	lattice
Hermitian	$\beta = 2$	Toda
symmetric	$\beta = 1$	Pfaff
symplectic	$\beta = 4$	Pfaff

As a consequence of a duality (explained in Theorem 1.1) between β-Virasoro generators under the map $\beta \mapsto 4/\beta$, the PDEs obtained have a remarkable property: the coefficients Q and Q_i in the PDEs are functions of the variables n, β, a, b, and have the invariance property under the map

$$n \to -2n, \; a \to -\frac{a}{2}, \; b \to -\frac{b}{2};$$

to be precise,

$$Q_i(-2n, \beta, -\frac{a}{2}, -\frac{b}{2})\Big|_{\beta=1} = Q_i(n, \beta, a, b)|_{\beta=4}\,. \tag{0.0.5}$$

Important remark. For $\beta = 2$, the probabilities satisfy PDEs in the boundary points of E, whereas in the case $\beta = 1, 4$, the equations are inductive. Namely, for $\beta = 1$ (resp. $\beta = 4$), the probabilities P_{n+2} (resp. P_{n+1}) are given in terms of P_{n-2} (resp. P_{n-1}) and a differential operator acting on P_n.

0.1. *Hermitian, symmetric and symplectic Gaussian ensembles.* Given the disjoint union $E \subset \mathbb{R}$ and the weight e^{-bz^2}, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}.$$

Also, define the *invariant* polynomials (in the sense of (0.0.5))

$$Q = 12b^2 n\left(n + 1 - \frac{2}{\beta}\right), \quad Q_2 = 4(1 + \delta_{1,4}^{\beta}) b \left(2n + \delta_{1,4}^{\beta}(1 - \frac{2}{\beta})\right)$$

and

$$Q_1 = \left(2 - \delta_{1,4}^{\beta}\right) \frac{b^2}{\beta}.$$

THEOREM 0.1. *The following probabilities for* ($\beta = 2, 1, 4$)

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^{\beta} \prod_{k=1}^n e^{-bz_k^2} dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^{\beta} \prod_{k=1}^n e^{-bz_k^2} dz_k}, \tag{0.1.1}$$

satisfy the PDE's ($F := F_n = \log P_n$):

(0.1.2)

$$\delta^{\beta}_{1,4} Q \left(\frac{P_{n-{}^{2}_{1}} P_{n+{}^{2}_{1}}}{P_n^2} - 1 \right) \text{ with index } \left\{ \begin{array}{ll} 2 & \text{when } n \text{ is even and } \beta = 1 \\ 1 & \text{when } n \text{ is arbitrary and } \beta = 4 \end{array} \right.$$

$$= \left(\mathcal{B}_{-1}^4 + (Q_2 + 6\mathcal{B}_{-1}^2 F)\mathcal{B}_{-1}^2 + 4Q_1(3\mathcal{B}_0^2 - 4\mathcal{B}_{-1}\mathcal{B}_1 + 6\mathcal{B}_0) \right) F.$$

0.2. *Hermitian, symmetric and symplectic Laguerre ensembles.* Given the disjoint union $E \subset \mathbb{R}^+$ and the weight $z^a e^{-bz}$, the $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+2} \frac{\partial}{\partial c_i}.$$

Also define the polynomials, again respecting the duality (0.0.5),

$$Q = \left\{ \begin{array}{ll} \frac{3}{4} n(n-1)(n+2a)(n+2a+1), & \text{for } \beta = 1 \\ \\ \frac{3}{2} n(2n+1)(2n+a)(2n+a-1), & \text{for } \beta = 4 \end{array} \right. ,$$

$$Q_2 = \left(3\beta n^2 - \frac{a^2}{\beta} + 6an + 4(1 - \frac{\beta}{2})a + 3 \right) \delta^{\beta}_{1,4} + (1-a^2)(1-\delta^{\beta}_{1,4}),$$

$$Q_1 = \left(\beta n^2 + 2an + (1 - \frac{\beta}{2})a \right), \qquad Q_0 = b(2 - \delta^{\beta}_{1,4})(n + \frac{a}{\beta}),$$

$$Q_{-1} = \frac{b^2}{\beta} \left(2 - \delta^{\beta}_{1,4} \right).$$

THEOREM 0.2. *The following probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k} dz_k}{\int_{\mathbb{R}_+^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k} dz_k} \tag{0.2.1}$$

satisfy the PDE[1]: ($F := F_n = \log P_n$)

(0.2.2)

$$\delta^{\beta}_{1,4} Q \left(\frac{P_{n-{}^{2}_{1}} P_{n+{}^{2}_{1}}}{P_n^2} - 1 \right)$$

$$= \Big(\mathcal{B}_{-1}^4 - 2(\delta^{\beta}_{1,4}+1)\mathcal{B}_{-1}^3$$
$$+ (Q_2 + 6\mathcal{B}_{-1}^2 F - 4(\delta^{\beta}_{1,4}+1)\mathcal{B}_{-1}F)\mathcal{B}_{-1}^2 - 3\delta^{\beta}_{1,4}(Q_1 - \mathcal{B}_{-1}F)\mathcal{B}_{-1}$$
$$+ Q_{-1}(3\mathcal{B}_0^2 - 4\mathcal{B}_1\mathcal{B}_{-1} - 2\mathcal{B}_1) + Q_0(2\mathcal{B}_0\mathcal{B}_{-1} - \mathcal{B}_0) \Big) F.$$

[1]with the same convention on the indices $n \pm 2$ and $n \pm 1$, as in (0.1.2)

0.3. *Hermitian, symmetric and symplectic Jacobi ensembles.* In terms of $E \subset [-1,1]$ and the Jacobi weight $(1-z)^a(1+z)^b$, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1}(1-c_i^2)\frac{\partial}{\partial c_i}.$$

Setting $b_0 = a-b,\ b_1 = a+b$, we introduce the new variables, which themselves have the invariance property (0.0.5):

$$r = \frac{4}{\beta}(b_0^2 + (b_1+2-\beta)^2) \qquad s = \frac{4}{\beta}b_0(b_1+2-\beta)$$

$$q_n = \frac{4}{\beta}(\beta n + b_1 + 2 - \beta)(\beta n + b_1),$$

and the following polynomials in $q = q_n, r, s$, thus *invariant* under the map (0.0.5):

$$\begin{aligned} Q &= \frac{3}{16}\left((s^2 - qr + q^2)^2 - 4(rs^2 - 4qs^2 - 4s^2 + q^2 r)\right), \\ Q_1 &= 3s^2 - 3qr - 6r + 2q^2 + 23q + 24, \\ Q_2 &= 3qs^2 + 9s^2 - 4q^2 r + 2qr + 4q^3 + 10q^2, \\ Q_3 &= 3qs^2 + 6s^2 - 3q^2 r + q^3 + 4q^2, \\ Q_4 &= 9s^2 - 3qr - 6r + q^2 + 22q + 24 = Q_1 + (6s^2 - q^2 - q). \end{aligned} \tag{0.3.1}$$

THEOREM 0.3. *The following probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a(1+z_k)^b dz_k}{\int_{[-1,1]^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a(1+z_k)^b dz_k} \tag{0.3.2}$$

satisfy the PDE ($F = F_n = \log P_n$):

for $\beta = 2$:

(0.3.3)

$$\begin{aligned} &\Big(2\mathcal{B}_{-1}^4 + (q-r+4)\mathcal{B}_{-1}^2 - (4\mathcal{B}_{-1}F - s)\mathcal{B}_{-1} + 3q\mathcal{B}_0^2 - 2q\mathcal{B}_0 + 8\mathcal{B}_0\mathcal{B}_{-1}^2 \\ &\quad -4(q-1)\mathcal{B}_1\mathcal{B}_{-1} + (4\mathcal{B}_{-1}F - s)\mathcal{B}_1 + 2(4\mathcal{B}_{-1}F - s)\mathcal{B}_0\mathcal{B}_{-1} + 2q\mathcal{B}_2\Big)F \\ &\quad\quad +4\mathcal{B}_{-1}^2 F\left(2\mathcal{B}_0 F + 3\mathcal{B}_{-1}^2 F\right) = 0 \end{aligned}$$

for $\beta = 1, 4$:

(0.3.4)

$$Q\left(\frac{P_{n+\frac{2}{1}}P_{n-\frac{2}{1}}}{P_n^2}-1\right)$$

$$= (q+1)\Big(4q\mathcal{B}_{-1}^4 + 12(4\mathcal{B}_{-1}F - s)\mathcal{B}_{-1}^3 + 2\,(q+12)\,(4\mathcal{B}_{-1}F - s)\mathcal{B}_0\mathcal{B}_{-1}$$

$$+3q^2\mathcal{B}_0^2 - 4\,(q-4)\,q\mathcal{B}_1\mathcal{B}_{-1} + q(4\mathcal{B}_{-1}F - s)\mathcal{B}_1 + 20q\mathcal{B}_0\mathcal{B}_{-1}^2 + 2q^2\mathcal{B}_2\Big)F$$

$$+\Big(Q_2\mathcal{B}_{-1}^2 - sQ_1\mathcal{B}_{-1} + Q_3\mathcal{B}_0\Big)F + 48(\mathcal{B}_{-1}F)^4 - 48s(\mathcal{B}_{-1}F)^3 + 2Q_4(\mathcal{B}_{-1}F)^2$$

$$+12\,q^2(\mathcal{B}_0F)^2 + 16\,q\,\,(2\,q-1)\,(\mathcal{B}_{-1}^2F)(\mathcal{B}_0F) + 24\,(q-1)\,q(\mathcal{B}_{-1}^2F)^2$$

$$+24\Big(2\mathcal{B}_{-1}F - s\Big)\Big((q+2)\mathcal{B}_0F + (q+3)\mathcal{B}_{-1}^2F\Big)\mathcal{B}_{-1}F.$$

0.4. ODE*s, when E has one boundary point.* Assume the set E consists of one boundary point $c = x$, besides the boundary of the full range. In that case the PDEs in the previous section lead to ODEs in x:

(1) *Gaussian* $(n \times n)$ *matrix ensemble* (for the function $\beta = 2, 1, 4$):

$$f_n(x) = \frac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$$

satisfies

(0.4.1)
$$\delta_{1,4}^{\beta}Q\left(\frac{P_{n-\frac{2}{1}}P_{n+\frac{2}{1}}}{P_n^2}-1\right)$$
$$= f_n''' + 6f_n'^2 + \left(4\frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2) + Q_2\right)f_n' - 4\frac{b^2x}{\beta}(\delta_{1,4}^{\beta}-2)f_n.$$

(2) *Laguerre ensemble* (for $\beta = 2, 1, 4$): all eigenvalues λ_i satisfy $\lambda_i \ge 0$ and

$$f_n(x) = x\frac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$$

satisfies (with $f := f_n(x)$)

(0.4.2)
$$\delta_{1,4}^{\beta}Q\left(\frac{P_{n-\frac{2}{1}}P_{n+\frac{2}{1}}}{P_n^2}-1\right) - \left(3\delta_{1,4}^{\beta}f - \frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2) - Q_0x - 3\delta_{1,4}^{\beta}Q_1\right)f$$
$$= x^3f''' - (2\delta_{1,4}^{\beta}-1)x^2f'' + 6x^2f'^2$$
$$-x\left(4(\delta_{1,4}^{\beta}+1)f - \frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2) - 2Q_0x - Q_2 + 2\delta_{1,4}^{\beta}+1\right)f'.$$

(3) *Jacobi ensemble*: all eigenvalues λ_i satisfy $-1 \le \lambda_i \le 1$ and

$$f_n(x) = (1-x^2)\frac{d}{dx}\log P_n(\max_i \lambda_i \le x)$$

satisfies (with $f := f_n(x)$):

for $\beta = 2$:

$$2(x^2-1)^2 f''' + 4(x^2-1)\left(xf'' - 3f'^2\right) + \left(16xf - q(x^2-1) - 2sx - r\right) f' - f\,(4f - qx - s) = 0 \tag{0.4.3}$$

for $\beta = 1, 4$:

$$\begin{aligned} &Q\left(\frac{P_{n+\frac{2}{1}}P_{n-\frac{2}{1}}}{P_n^2} - 1\right) \\ = \;& 4(q+1)(x^2-1)^2\Big(-q(x^2-1)f''' + (12f - qx - 3s)f'' + 6q(q-1)f'^2\Big) \\ & - (x^2-1)f'\Big(24f(q+3)(2f-s) + 8fq(5q-1)x - q(q+1)(qx^2+2sx+8) + Q_2\Big) \\ & + f\Big(48f^3 + 48f^2(qx+2x-s) + 2f\left(8q^2x^2 + 2qx^2 - 12qsx - 24sx + Q_4\right) \\ & \quad - q(q+1)x(3qx^2 + sx - 2qx - 3q) + Q_3x - Q_1 s\Big). \end{aligned} \tag{0.4.4}$$

For $\beta = 2$, $f_n(x)$ satisfies a third-order equation (of the so-called Chazy-type) with quadratic nonlinearity in f_n'. Then f_n also satisfies an equation, which is second-order in f and quadratic in f'', which after some rescaling can be put in a canonical form. Namely,

Gauss	$g_n(z) = b^{-1/2} f_n(zb^{-1/2}) + \frac{2}{3}nz,$
Laguerre,	$g_n(z) = f_n(z) + \frac{b}{4}(2n+a)z + \frac{a^2}{4},$
Jacobi	$g_n(z) := -\frac{1}{2} f_n(x)\vert_{x=2z-1} - \frac{q}{8}z + \frac{q+s}{16}$

satisfies the respective canonical equations of Cosgrove [11] and Cosgrove-Scoufis [12],

- $g''^2 = -4g'^3 + 4(zg'-g)^2 + A_1 g' + A_2,$ (Painlevé IV)
- $(zg'')^2 = (zg'-g)\Big(-4g'^2 + A_1(zg'-g) + A_2\Big) + A_3 g' + A_4,$ (Painlevé V)

- $$(z(z-1)g'')^2 = (zg'-g)\Big(4g'^2 - 4g'(zg'-g) + A_2\Big) + A_1 g'^2 + A_3 g' + A_4, \qquad \text{(Painlevé VI)}$$

with coefficients which will be determined in Section 4.3. Each of these equations can be transformed into the standard Painlevé equations.

For $\beta = 1$ and 4, the inductive partial differential equations (0.1.2), (0.2.2) and (0.3.4) are new. For $\beta = 2$ and for general E, they were first computed by Adler-Shiota-van Moerbeke [7], using the method of the present paper. For $\beta = 2$ and for E having one boundary point, the equations obtained here coincide with the ones first obtained by Tracy-Widom in [20], who saw them to be Painlevé IV and V for the Gaussian and Laguerre distribution respectively. In his Louvain doctoral dissertation, J. P. Semengue, together with L. Haine [14], were led to Painlevé VI for the Jacobi ensemble, for $\beta = 2$ and E having one boundary point, upon subtracting the Tracy-Widom differential equation ([20]) from the ones computed with the Adler-Shiota-van Moerbeke method ([7]). As we shall see, the classification of Cosgrove [11] and Cosgrove-Scoufis [12], (A.3) leads directly to these results.

1. Beta-integrals

1.1. *Virasoro constraints for β-integrals.* Consider the data from (0.0.1) to (0.0.3) and the t-deformations of the integrals (0.0.4), for *general* $\beta > 0$: ($t := (t_1, t_2, ...)$ and $c = (c_1, c_2, ..., c_{2r})$)

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k\right) \text{ for } n > 0. \tag{1.1.1}$$

The main statement of this section is Theorem 1.1, whose proof will be outlined in the next subsection. In Section 5 (Appendix), we give a less conceptual proof, which is based on the invariance of the integral (1.1.2) below, under the transformation $z_i \mapsto z_i + \varepsilon f(z_i) z_i^{k+1}$ of the integration variables. The central charge (1.1.6) has already appeared in the work of Awata et al. [10].

THEOREM 1.1 (Adler-van Moerbeke [2]). *The multiple integrals*

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k\right) \textit{ for } n > 0 \tag{1.1.2}$$

and

$$I_n(t, c; \frac{4}{\beta}) := \int_{E^n} |\Delta_n(z)|^{4/\beta} \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k\right), \textit{ for } n > 0, \tag{1.1.3}$$

with $I_0 = 1$, *satisfy respectively the following Virasoro constraints*[2] *for all* $k \geq -1$:

$$(1.1.4)\quad \left(-\mathcal{B}_k + \sum_{i\geq 0}\left(a_i \ {}^{\beta}\mathbb{J}^{(2)}_{k+i,n}(t,n) - b_i \ {}^{\beta}\mathbb{J}^{(1)}_{k+i+1,n}(t,n)\right)\right) I_n(t,c;\beta) = 0,$$

$$\left(-\mathcal{B}_k + \sum_{i\geq 0}\left(a_i \ {}^{\beta}\mathbb{J}^{(2)}_{k+i,n}\Big(-\frac{\beta t}{2}, -\frac{2n}{\beta}\Big) + \frac{\beta b_i}{2} \ {}^{\beta}\mathbb{J}^{(1)}_{k+i+1,n}\Big(-\frac{\beta t}{2}, -\frac{2n}{\beta}\Big)\right)\right) I_n(t,c;\frac{4}{\beta}) = 0,$$

in terms of the coefficients a_i, b_i *of the rational function* $(-\log\rho)'$ *and the end points* c_i *of the subset* E, *as in* (0.0.1) *to* (0.0.3). *For all* $n \in \mathbb{Z}$, *the* ${}^{\beta}\mathbb{J}^{(2)}_{k,n}(t,n)$ *and* ${}^{\beta}\mathbb{J}^{(1)}_{k,n}(t,n)$ *form a Virasoro and a Heisenberg algebra respectively, interacting as follows*:

$$(1.1.5)\quad \begin{aligned} \left[{}^{\beta}\mathbb{J}^{(2)}_{k,n}, {}^{\beta}\mathbb{J}^{(2)}_{\ell,n}\right] &= (k-\ell)\ {}^{\beta}\mathbb{J}^{(2)}_{k+\ell,n} + c\left(\frac{k^3-k}{12}\right)\delta_{k,-\ell} \\ \left[{}^{\beta}\mathbb{J}^{(2)}_{k,n}, {}^{\beta}\mathbb{J}^{(1)}_{\ell,n}\right] &= -\ell\ \ {}^{\beta}\mathbb{J}^{(1)}_{k+\ell,n} + c' k(k+1)\delta_{k,-\ell} \\ \left[{}^{\beta}\mathbb{J}^{(1)}_{k,n}, {}^{\beta}\mathbb{J}^{(1)}_{\ell,n}\right] &= \frac{k}{\beta}\delta_{k,-\ell}, \end{aligned}$$

with central charge

$$(1.1.6)\quad c = 1 - 6\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 \quad \textit{and} \quad c' = \left(\frac{1}{\beta} - \frac{1}{2}\right).$$

Remark 1. The ${}^{\beta}\mathbb{J}^{(2)}_{k,n}$'s are defined as follows:

$$(1.1.7)\quad {}^{\beta}\mathbb{J}^{(2)}_{k,n} = \frac{\beta}{2}\sum_{i+j=k} :\ {}^{\beta}\mathbb{J}^{(1)}_{i,n}\ {}^{\beta}\mathbb{J}^{(1)}_{j,n} : + \left(1 - \frac{\beta}{2}\right)\left((k+1)\ {}^{\beta}\mathbb{J}^{(1)}_{k,n} - k\mathbb{J}^{(0)}_{k,n}\right).$$

Componentwise, we have

$${}^{\beta}\mathbb{J}^{(1)}_{k,n}(t,n) = {}^{\beta}J^{(1)}_k + nJ^{(0)}_k \text{ and } {}^{\beta}\mathbb{J}^{(0)}_{k,n} = nJ^{(0)}_k = n\delta_{0k}$$

and hence

$$\begin{aligned} {}^{\beta}\mathbb{J}^{(2)}_{k,n}(t,n) &= \left(\frac{\beta}{2}\right)\ {}^{\beta}J^{(2)}_k + \left(n\beta + (k+1)(1-\frac{\beta}{2})\right)\ {}^{\beta}J^{(1)}_k \\ &\quad + n\Big((n-1)\frac{\beta}{2} + 1\Big)J^{(0)}_k, \end{aligned}$$

[2]When E equals the whole range F, then the $\mathcal{B}_k$'s are absent in the formulae (1.1.4).

where

$$(1.1.8)\qquad {}^{\beta}J_k^{(1)} = \frac{\partial}{\partial t_k} + \frac{1}{\beta}(-k)t_{-k}$$

$${}^{\beta}J_k^{(2)} = \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{2}{\beta} \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{\beta^2} \sum_{-i-j=k} it_i j t_j.$$

We put n explicitly in ${}^{\beta}\mathbb{J}_{\ell,n}^{(2)}(t,n)$ to indicate that the n^{th} component contains n explicitly, besides t.

Remark 2. The Heisenberg and Virasoro generators satisfy the following *duality* properties:

$$(1.1.9)\qquad {}^{\frac{4}{\beta}}\mathbb{J}_{\ell,n}^{(2)}\Big(t,n\Big) = {}^{\beta}\mathbb{J}_{\ell,n}^{(2)}\left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right),\ n \in \mathbb{Z}$$

$${}^{\frac{4}{\beta}}\mathbb{J}_{\ell,n}^{(1)}\Big(t,n\Big) = -\frac{\beta}{2}\ {}^{\beta}\mathbb{J}_{\ell,n}^{(1)}\left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right),\ n > 0.$$

In (1.1.9), ${}^{\beta}\mathbb{J}_{\ell,n}^{(2)}(-\beta t/2, -2n/\beta)$ means that the variable n, which appears in the n^{th} component, gets replaced by $-2n/\beta$ and t by $-\beta t/2$.

1.2. *Proof*: *β-integrals as fixed points of vertex operators.* The most transparent way to prove Theorem 1.1 is via vector vertex operators, for which the β-integrals are fixed points. This is a technique which has been used by us already in [1]. Indeed, define the (vector) vertex operator $\mathbb{X}$, for $t = (t_1, t_2, ...) \in \mathbb{C}^{\infty},\ u \in \mathbb{C}$:

$$(1.2.1)\qquad \mathbb{X}_{\beta}(t,u) = \Lambda^{-1} e^{\sum_1^{\infty} t_i u^i} e^{-\beta \sum_1^{\infty} \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}} \chi(|u|^{\beta}),$$

where $\chi(z) := (1, z, z^2, \ldots)$. The vertex operator acts on vectors $f(t) = (f_0(t), f_1(t), ...)$ of functions, as follows[3]

$$\Big(\mathbb{X}_{\beta}(t,u) f(t)\Big)_n = e^{\sum_1^{\infty} t_i u^i} \Big(|u|^{\beta}\Big)^{n-1} f_{n-1}(t - \beta[u^{-1}]).$$

For the sake of convenience, in this section we introduce the following vector Virasoro generators: ${}^{\beta}\mathbb{J}_k^{(i)}(t) := (\ {}^{\beta}\mathbb{J}_{k,n}^{(i)}(t,n))_{n \in \mathbb{Z}}$.

PROPOSITION 1.2. *The multiplication operator z^k and the differential operators $\frac{\partial}{\partial z} z^{k+1}$ with $z \in \mathbb{C}^*$, acting on the vertex operator $\mathbb{X}_{\beta}(t,z)$, have realizations as commutators, in terms of the Heisenberg and Virasoro generators*

[3]For $\alpha \in \mathbb{C}$, define $[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, ...) \in \mathbb{C}^{\infty}$. The operator Λ is the shift matrix, with zeroes everywhere, except for 1's just above the diagonal, i.e., $(\Lambda v)_n = v_{n+1}$.

${}^{\beta}\mathbb{J}_k^{(1)}(t)$ *and* ${}^{\beta}\mathbb{J}_k^{(2)}(t)$:

$$
\begin{aligned}
z^k \mathbb{X}_\beta(t,z) &= \left[{}^{\beta}\mathbb{J}_k^{(1)}(t), \mathbb{X}_\beta(t,z) \right], \\
\frac{\partial}{\partial z} z^{k+1} \mathbb{X}_\beta(t,z) &= \left[{}^{\beta}\mathbb{J}_k^{(2)}(t), \mathbb{X}_\beta(t,z) \right].
\end{aligned}
\tag{1.2.2}
$$

Corollary 1.3. *Given a weight* $\rho(z)dz$ *on* $\mathbb{R}$ *satisfying* (0.0.1), *we have*

$$
\frac{\partial}{\partial z} z^{k+1} f(z) \mathbb{X}_\beta(t,z)\rho(z) = \left[\sum_{i\geq 0} \left(a_i \; {}^{\beta}\mathbb{J}_{k+i}^{(2)}(t) - b_i \; {}^{\beta}\mathbb{J}_{k+i+1}^{(1)}(t) \right), \mathbb{X}_\beta(t,z)\rho(z) \right].
\tag{1.2.3}
$$

Proof. Using (1.2.2) in the last line, compute

$$
\begin{aligned}
&\frac{\partial}{\partial z} z^{k+1} f(z) \mathbb{X}_\beta(t,z)\rho(z) \\
&= \left(\frac{\rho'(z)}{\rho(z)} f(z) \right) z^{k+1} \mathbb{X}_\beta(t,z)\rho(z) + \rho(z) \frac{\partial}{\partial z} \left(z^{k+1} f(z) \mathbb{X}_\beta(t,z) \right) \\
&= -\left(\sum_0^\infty b_i z^{k+i+1} \mathbb{X}_\beta(t,z) \right) \rho(z) + \rho(z) \frac{\partial}{\partial z} \left(\sum_0^\infty a_i z^{k+i+1} \mathbb{X}_\beta(t,z) \right) \\
&= -\left[\sum_0^\infty b_i \; {}^{\beta}\mathbb{J}_{k+i+1}^{(1)}, \mathbb{X}_\beta(t,z)\rho(z) \right] + \left[\sum_0^\infty a_i \; {}^{\beta}\mathbb{J}_{k+i}^{(2)}, \mathbb{X}_\beta(t,z)\rho(z) \right],
\end{aligned}
\tag{1.2.4}
$$

establishing (1.2.3). □

Given the weight $\rho_E(u)du = \rho(u)I_E(u)du$, with ρ and E as before, and with I_E the indicator function of E, define the integrated vector vertex operator

$$
\mathbb{Y}_\beta(t,\rho_E) := \int_E du \rho(u) \mathbb{X}_\beta(t,u),
\tag{1.2.5}
$$

and the vector operator

$$
\begin{aligned}
\mathcal{D}_k &:= \mathcal{B}_k - \mathcal{V}_k \\
&:= \sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} - \sum_{i\geq 0} \left(a_i \; {}^{\beta}\mathbb{J}_{k+i}^{(2)}(t) - b_i \; {}^{\beta}\mathbb{J}_{k+i+1}^{(1)}(t) \right),
\end{aligned}
\tag{1.2.6}
$$

consisting of a c-dependent boundary part $\mathcal{B}_k$ and a (t,n)-dependent Virasoro part $\mathcal{V}_k$.

Proposition 1.4. *The following commutation relation holds*:

$$
[\mathcal{D}_k, \mathbb{Y}_\beta(t,\rho_E)] = 0.
\tag{1.2.7}
$$

Proof. Integrating both sides of (1.2.3) over E, one computes:

(1.2.8)
$$\begin{aligned}\int_E dz\frac{\partial}{\partial z}\left(z^{k+1}f(z)\mathbb{X}_\beta(t,z)\rho(z)\right) &= \sum_1^{2r}(-1)^i c_i^{k+1}f(c_i)\mathbb{X}_\beta(t,c_i)\rho(c_i)\\ &= \sum_1^{2r} c_i^{k+1}f(c_i)\frac{\partial}{\partial c_i}\int_E \mathbb{X}_\beta(t,z)\rho(z)dz\\ &= \left[\mathcal{B}_k,\mathbb{Y}_\beta(t,\rho_E)\right];\end{aligned}$$

while on the other hand

(1.2.9)
$$\begin{aligned}&\int_E dz\left[\sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i}-b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1}\right),\mathbb{X}_\beta(t,z)\rho(z)\right]\\ &\quad=\left[\sum_{i\geq 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i}-b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1}\right),\int_E dz\rho(z)\mathbb{X}_\beta(t,z)\right]\\ &\quad=\left[\mathcal{V}_k,\mathbb{Y}_\beta(t,\rho_E)\right].\end{aligned}$$

Subtracting both expressions (1.2.8) and (1.2.9) yields, using (1.2.3),

$$0=\left[\mathcal{B}_k-\mathcal{V}_k,\mathbb{Y}_\beta(t,\rho_E)\right]=\left[\mathcal{D}_k,\mathbb{Y}_\beta(t,\rho_E)\right],$$

concluding the proof of Proposition 1.4. ☐

Proposition 1.5. *The column vector,*

$$I(t):=\left(\int_{E^n}|\Delta_n(z)|^\beta\prod_{k=1}^n e^{\sum_1^\infty t_i z_k^i}\rho(z_k)dz_k\right)_{n\geq 0}$$

is a fixed point for the vertex operator $\mathbb{Y}_\beta(t,\rho_E)$:

(1.2.10)
$$\left(\mathbb{Y}_\beta(t,\rho_E)I\right)_n=I_n,\ \ n\geq 1.$$

Proof. We have

(1.2.11)
$$\begin{aligned}I_n(t) &= \int_{\mathbb{R}^n}|\Delta_n(z)|^\beta\prod_{k=1}^n\left(e^{\sum_1^\infty t_i z_k^i}\rho_E(z_k)dz_k\right)\\ &= \int_{\mathbb{R}}du\rho_E(u)e^{\sum_1^\infty t_i u^i}|u|^{\beta(n-1)}\\ &\qquad\int_{\mathbb{R}^{n-1}}\prod_{k=1}^{n-1}\left|1-\frac{z_k}{u}\right|^\beta|\Delta_{n-1}(z)|^\beta\prod_{k=1}^{n-1}\left(e^{\sum_1^\infty t_i z_k^i}\rho_E(z_k)dz_k\right)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} du\rho_E(u) e^{\sum_1^\infty t_i u^i} |u|^{\beta(n-1)} \\
&\qquad e^{-\beta\sum_1^\infty \frac{u^{-i}}{i}\frac{\partial}{\partial t_i}} \int_{\mathbb{R}^{n-1}} |\Delta_{n-1}(z)|^\beta \prod_{k=1}^{n-1}\left(e^{\sum_1^\infty t_i z_k^i}\rho_E(z_k)dz_k\right) \\
&= \int_{\mathbb{R}} du\rho_E(u)|u|^{\beta(n-1)} e^{\sum_1^\infty t_i u^i} e^{-\beta\sum_1^\infty \frac{u^{-i}}{i}\frac{\partial}{\partial t_i}} I_{n-1}(t) \\
&= \left(\mathbb{Y}_\beta(t,\rho_E) I(t)\right)_n .
\end{aligned}
$$

It suffices to do the above argument for all $t_i > 0$, enabling one to replace $e^{\sum_1^\infty t_i z^i}$ by $|e^{\sum_1^\infty t_i z^i}|$. Then one continues the result for all $t_i \in \mathbb{C}$. □

Proof of Theorem 1.1. From Proposition 1.4 it follows that for $n \geq 1$,

$$
\begin{aligned}
0 &= \left[\mathcal{D}_k, (\mathbb{Y}_\beta(t,\rho_E))^n\right] I \\
&= \mathcal{D}_k \mathbb{Y}_\beta(t,\rho_E)^n I - \mathbb{Y}_\beta(t,\rho_E)^n \mathcal{D}_k I.
\end{aligned}
\tag{1.2.12}
$$

Taking the n^{th} component for $n \geq 1$ and $k \geq -1$, setting

$$
X_\beta(t,u) = e^{\sum t_i u^i} e^{-\beta \sum \frac{u^{-i}}{i}\frac{\partial}{\partial t_i}},
$$

and using (1.2.10), we have

$$
\begin{aligned}
0 &= \left(\mathcal{D}_k I - \mathbb{Y}_\beta(t,\rho_E)^n \mathcal{D}_k I\right)_n \\
&= (\mathcal{D}_k I)_n - \int du\rho_E(u) X_\beta(t;u)(|u|^\beta)^{n-1} \ldots \int du\rho_E(u) X_\beta(t;u)(\mathcal{D}_k I)_0 \\
&= (\mathcal{D}_k I)_n .
\end{aligned}
$$

Indeed $(\mathcal{D}_k I)_0 = 0$ for $k \geq -1$, since $I_0 = 1$ and $\mathcal{D}_k$ involves $\mathcal{B}_k, {}^\beta J_k^{(2)}, {}^\beta J_k^{(1)}$ and $J_k^{(0)}$ for $k \geq -1$:

$$
\begin{cases}
\mathcal{B}_k \text{ and } {}^\beta J_k^{(2)} \text{ are pure differentiations for } k \geq -1; \\
{}^\beta J_k^{(1)} \text{is pure differentiation, except for } k = -1; \\
{}^\beta J_{-1}^{(1)} \text{ appears with coefficient } n\beta, \text{ which vanishes for } n = 0; \\
J_k^{(0)} \text{ appears with coefficient } n((n-1)\frac{\beta}{2} + 1), \text{ vanishing for } n = 0.
\end{cases}
$$

The proof of the 2^{nd} formula in (1.1.4) follows immediately from the duality (1.1.9). □

1.3. *Examples. Example* 1 (*Gaussian β-integrals*). The weight and the a_i and b_i, as in (0.0.1), are given by (setting $b = 1$ in (0.1.1))

$$
\rho(z) = e^{-V(z)} = e^{-z^2}, \quad V' = g/f = 2z,
$$

$$
a_0 = 1, b_0 = 0, b_1 = 2, \text{ and all other } a_i, b_i = 0.
$$

From Theorem 1.1, the integrals

$$I_n = \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-z_k^2 + \sum_{i=1}^\infty t_i z_k^i} dz_k \tag{1.3.1}$$

satisfy the Virasoro constraints

(1.3.2)

$$-\mathcal{B}_k I_n = -\sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} I_n = \left(-\, {}^\beta\mathbb{J}_{k,n}^{(2)} + 2\, {}^\beta\mathbb{J}_{k+2,n}^{(1)} \right) I_n, \quad k = -1, 0, 1, \dots .$$

Introducing the following notation

$$\sigma_i = (n - \frac{i+1}{2})\beta + i + 1 - b_0 = (n - \frac{i+1}{2})\beta + i + 1,$$

and upon setting $F = \log I_n$ we find that the first three constraints have the following form:

$$\begin{aligned} -\mathcal{B}_{-1}F &= \left(2\frac{\partial}{\partial t_1} - \sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F - nt_1, \\ -\mathcal{B}_0 F &= \left(2\frac{\partial}{\partial t_2} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_i} \right) F - \frac{n}{2}\sigma_1, \\ -\mathcal{B}_1 F &= \left(2\frac{\partial}{\partial t_3} - \sigma_1 \frac{\partial}{\partial t_1} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}} \right) F. \end{aligned}$$

For later use, take linear combinations such that each expression contains the pure differentiation term $\partial F/\partial t_i$:

$$\mathcal{D}_1 = -\frac{1}{2}\mathcal{B}_{-1}, \quad \mathcal{D}_2 = -\frac{1}{2}\mathcal{B}_0, \quad \mathcal{D}_3 = -\frac{1}{2}\left(\mathcal{B}_1 + \frac{\sigma_1}{2}\mathcal{B}_{-1} \right), \tag{1.3.3}$$

which yields

$$\begin{aligned} \mathcal{D}_1 F &= \left(\frac{\partial}{\partial t_1} - \frac{1}{2}\sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{nt_1}{2}, \\ \mathcal{D}_2 F &= \left(\frac{\partial}{\partial t_2} - \frac{1}{2}\sum_{i\geq 1} it_i \frac{\partial}{\partial t_i} \right) F - \frac{n}{4}\sigma_1, \\ \mathcal{D}_3 F &= \left(\frac{\partial}{\partial t_3} - \frac{1}{2}\sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \frac{1}{4}\sigma_1 \sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{n}{4}\sigma_1 t_1. \end{aligned} \tag{1.3.4}$$

Example 2 (*Laguerre β-integrals*). Here, the weight and the a_i and b_i, as in (0.0.1), are given by (again setting $b = 1$ in (0.2.1))

$$e^{-V} = z^a e^{-z}, \quad V' = \frac{g}{f} = \frac{z-a}{z},$$

$$a_0 = 0,\ a_1 = 1,\ b_0 = -a,\ b_1 = 1, \text{ and all other } a_i, b_i = 0.$$

Thus from (1.1.4), the integrals

$$I_n = \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-z_k + \sum_{i=1}^\infty t_i z_k^i} dz_k \tag{1.3.5}$$

satisfy the Virasoro constraints, for $k \geq -1$,

$$-\mathcal{B}_k I_n = -\sum_1^{2r} c_i^{k+2} \frac{\partial}{\partial c_i} I_n = \left(-\ {}^\beta\mathbb{J}_{k+1,n}^{(2)} - a\ {}^\beta\mathbb{J}_{k+1,n}^{(1)} +\ {}^\beta\mathbb{J}_{k+2,n}^{(1)} \right) I_n. \tag{1.3.6}$$

Introducing the following notation, as before,

$$\sigma_i \ = \ (n - \frac{i+1}{2})\beta + i + 1 - b_0 = (n - \frac{i+1}{2})\beta + i + 1 + a,$$

and upon setting $F = F_n = \log I_n$, we see that the first three have the form:

$$\begin{aligned}
-\mathcal{B}_{-1}F &= \left(\frac{\partial}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) F - \frac{n}{2}(\sigma_1 + a), \\
-\mathcal{B}_0 F &= \left(\frac{\partial}{\partial t_2} - \sigma_1 \frac{\partial}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} \right) F, \\
-\mathcal{B}_1 F &= \left(\frac{\partial}{\partial t_3} - \sigma_2 \frac{\partial}{\partial t_2} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+2}} - \frac{\beta}{2} \frac{\partial^2}{\partial t_1^2} \right) F - \frac{\beta}{2} \left(\frac{\partial F}{\partial t_1} \right)^2.
\end{aligned}$$

Replacing the operators $\mathcal{B}_i$ by linear combinations $\mathcal{D}_i$, we see that

$$\begin{aligned}
\mathcal{D}_1 &= -\mathcal{B}_{-1} \\
\mathcal{D}_2 &= -\mathcal{B}_0 - \sigma_1 \mathcal{B}_{-1} \\
\mathcal{D}_3 &= -\mathcal{B}_1 - \sigma_2 \mathcal{B}_0 - \sigma_1 \sigma_2 \mathcal{B}_{-1}
\end{aligned} \tag{1.3.7}$$

yields expressions, each containing a pure derivative $\partial F / \partial t_i$

(1.3.8)

$$\begin{aligned}
\mathcal{D}_1 F &= \frac{\partial F}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial F}{\partial t_i} - \frac{n}{2}(\sigma_1 + a), \\
\mathcal{D}_2 F &= \frac{\partial F}{\partial t_2} + \sum_{i \geq 1} i t_i \left(-\sigma_1 \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_{i+1}} \right) F - \frac{n}{2}(\sigma_1 + a)\sigma_1,
\end{aligned}$$

$$\mathcal{D}_3 F = \frac{\partial F}{\partial t_3} - \sum_{i\geq 1} it_i \left(\sigma_1\sigma_2 \frac{\partial}{\partial t_i} + \sigma_2 \frac{\partial}{\partial t_{i+1}} + \frac{\partial}{\partial t_{i+2}}\right) F - \frac{n}{2}(\sigma_1 + a)\sigma_1\sigma_2$$
$$-\frac{\beta}{2}\left(\frac{\partial^2 F}{\partial t_1^2} + \left(\frac{\partial F}{\partial t_1}\right)^2\right).$$

Example 3 (*Jacobi β-integral*). The weight and the a_i and b_i, as in (0.0.1), are given by

$$\rho_{ab}(z) := e^{-V} = (1-z)^a(1+z)^b, V' = \frac{g}{f} = \frac{a-b+(a+b)z}{1-z^2},$$
$$a_0 = 1, a_1 = 0, a_2 = -1, b_0 = a-b, b_1 = a+b, \text{ and all other } a_i, b_i = 0.$$

The integrals

$$\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a(1+z_k)^b e^{\sum_{i=1}^\infty t_i z_k^i} dz_k \tag{1.3.9}$$

satisfy the Virasoro constraints ($k \geq -1$):

$$\begin{aligned} -\mathcal{B}_k I_n &= -\sum_1^{2r} c_i^{k+1}(1-c_i^2)\frac{\partial}{\partial c_i} I_n \\ &= \left({}^\beta\mathbb{J}^{(2)}_{k+2,n} - {}^\beta\mathbb{J}^{(2)}_{k,n} + b_0\, {}^\beta\mathbb{J}^{(1)}_{k+1,n} + b_1\, {}^\beta\mathbb{J}^{(1)}_{k+2,n}\right) I_n. \end{aligned} \tag{1.3.10}$$

Introducing the following notation,

$$\sigma_i = (n - \frac{i+1}{2})\beta + i + 1 + b_1,$$

and upon setting $F = F_n = \log I_n$, we see that the first four have the following form:

(1.3.11)

$$-\mathcal{B}_{-1}F = \left(\sigma_1 \frac{\partial}{\partial t_1} + \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}}\right) F + n(b_0 - t_1),$$

$$-\mathcal{B}_0 F = \left(\sigma_2 \frac{\partial}{\partial t_2} + b_0 \frac{\partial}{\partial t_1} + \sum_{i\geq 1} it_i(\frac{\partial}{\partial t_{i+2}} - \frac{\partial}{\partial t_i}) + \frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}\right) F$$
$$+\frac{\beta}{2}\left(\frac{\partial F}{\partial t_1}\right)^2 - \frac{n}{2}(\sigma_1 - b_1),$$

$$-\mathcal{B}_1 F = \left(\sigma_3 \frac{\partial}{\partial t_3} + b_0 \frac{\partial}{\partial t_2} - (\sigma_1 - b_1)\frac{\partial}{\partial t_1} + \sum_{i\geq 1} it_i(\frac{\partial}{\partial t_{i+3}} - \frac{\partial}{\partial t_{i+1}})\right.$$
$$\left.+\beta \frac{\partial^2}{\partial t_1 \partial t_2}\right) F + \beta \frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_2},$$

$$-\mathcal{B}_2 F = \left(\sigma_4 \frac{\partial}{\partial t_4} + b_0 \frac{\partial}{\partial t_3} - (\sigma_2 - b_1)\frac{\partial}{\partial t_2} + \sum_{i\geq 1} i t_i \left(\frac{\partial}{\partial t_{i+4}} - \frac{\partial}{\partial t_{i+2}}\right)\right.$$
$$\left. + \frac{\beta}{2}\left(\frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_1 \partial t_3}\right)\right) F + \frac{\beta}{2}\left(\left(\frac{\partial F}{\partial t_2}\right)^2 - \left(\frac{\partial F}{\partial t_1}\right)^2 + 2\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_3}\right).$$

2. Matrix integrals and associated integrable systems

2.1. *Hermitian matrix integrals and the Toda lattice.* Given a weight $\rho(z) = e^{-V(z)}$ defined as in (0.0.1), the inner-product

$$\langle f, g\rangle_t = \int_E f(z)g(z)\rho_t(z)dz, \quad \text{with } \rho_t := e^{\sum_1^\infty t_i z^i}\rho(z), \tag{2.1.1}$$

leads to a moment matrix

$$m_n(t) = (\mu_{ij}(t))_{0\leq i,j<n} = (\langle z^i, z^j\rangle_t)_{0\leq i,j<n}, \tag{2.1.2}$$

which is a *Hänkel matrix*[4], thus symmetric. Hänkel is tantamount to $\Lambda m_\infty = m_\infty \Lambda^\top$. The semi-infinite moment matrix m_∞ evolves in t according to the equations

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j}, \text{ and thus } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \qquad \left(\begin{array}{c}\text{commuting}\\ \text{vector fields}\end{array}\right). \tag{2.1.3}$$

Another important ingredient is the factorization of m_∞ into a lower- times an upper-triangular matrix[5]

$$m_\infty(t) = S(t)^{-1} S(t)^{\top -1},$$

where $S(t)$ is lower-triangular with nonzero diagonal elements.

THEOREM 2.1. *The vector $\tau(t) = (\tau_n(t))_{n\geq 0}$, with*

$$\tau_n(t) := \det m_n(t) = \frac{1}{n!}\int_{E^n} \Delta_n^2(z) \prod_{k=1}^n \rho_t(z_k) dz_k \tag{2.1.4}$$

satisfies:

(i) Virasoro constraints (1.1.4) *for $\beta = 2$,*

$$\left(-\sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i} + \sum_{i\geq 0}\left(a_i\ \mathbb{J}_{k+i}^{(2)} - b_i\ \mathbb{J}_{k+i+1}^{(1)}\right)\right)\tau = 0 \tag{2.1.5}$$

[4]Hänkel means μ_{ij} depends on $i + j$ only.

[5]This factorization is possible for those t's for which $\tau_n(t) := \det m_n(t) \neq 0$ for all $n > 0$.

(ii) *the* KP-hierarchy[6]

$$\left(p_{k+4}(\tilde{\partial}) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right)\tau_n \circ \tau_n = 0,$$

of which the first equation reads:

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right)\log\tau_n + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau_n\right)^2 = 0,$$

$$k = 0, 1, 2, \ldots$$

(iii) *The* standard Toda lattice; *i.e., the tridiagonal matrix*

$$L(t) := S(t)\Lambda S(t)^{-1} = \begin{pmatrix} \frac{\partial}{\partial t_1}\log\frac{\tau_1}{\tau_0} & \left(\frac{\tau_0\tau_2}{\tau_1^2}\right)^{1/2} & 0 & \\ \left(\frac{\tau_0\tau_2}{\tau_1^2}\right)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_2}{\tau_1} & \left(\frac{\tau_1\tau_3}{\tau_2^2}\right)^{1/2} & \\ 0 & \left(\frac{\tau_1\tau_3}{\tau_2^2}\right)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_3}{\tau_2} & \\ & & & \ddots \end{pmatrix} \tag{2.1.6}$$

satisfies the commuting equations[7]

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2}(L^k)_{\mathfrak{s}}, L\right]. \tag{2.1.7}$$

(iv) Orthogonal polynomials: *The* n^{th} *degree polynomials* $p_n(t; z)$ *in* z, *depending on* $t \in \mathbb{C}^\infty$, *orthonormal with respect to the* t*-dependent inner product* (2.1.1)

$$\langle p_k(t; z), p_\ell(t; z)\rangle = \delta_{k\ell}$$

are eigenvectors of L, *i.e.*, $(L(t)p(t; z))_n = zp_n(t; z)$, $n \geq 0$, *and enjoy the following representations*

$$\begin{aligned} p_n(t; z) := (S(t)\chi(z))_n &= \frac{1}{\sqrt{\tau_n(t)\tau_{n+1}(t)}}\det\left(\begin{array}{ccc|c} & m_n & & 1 \\ & & & z \\ & & & \vdots \\ \hline \mu_{n,0} & \ldots & \mu_{n,n-1} & z^n \end{array}\right) \\ &= z^n h_n^{-1/2}\frac{\tau_n(t - [z^{-1}])}{\tau_n(t)}, \qquad h_n := \frac{\tau_{n+1}(t)}{\tau_n(t)}. \end{aligned}$$

---

[6] for the customary Hirota symbol $p(\partial_t)f \circ g := p(\frac{\partial}{\partial y})f(t+y)g(t-y)\Big|_{y=0}$. The p_ℓ's are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} := \sum_{i\geq 0} p_i(t_1, t_2, \ldots)z^i$ and $p_\ell(\tilde{\partial}) := p_\ell(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots)$.

[7] $()_{\mathfrak{s}}$ means: take the skew-symmetric part of () in the decomposition "skew-symmetric" + "lower-triangular."

The functions $q_n(t;z) := z\int_{\mathbb{R}^n} \frac{p_n(t;u)}{z-u}\rho_t(u)du$ *are "dual eigenvectors" of* L, *i.e.*, $(L(t)q(t;z))_n = zq_n(t;z), \quad n \geq 1$, *and have the following* τ*-function representation*: (*see the remark at the end of this section*)

$$
\begin{aligned}
(2.1.8)\quad q_n(t;z) := z\int_{\mathbb{R}^n} \frac{p_n(t;u)}{z-u}\rho_t(u)du &= \Big(S^{\top -1}(t)\chi(z^{-1})\Big)_n \\
&= \Big(S(t)m_\infty(t)\chi(z^{-1})\Big)_n \\
&= z^{-n}h_n^{-1/2}\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}.
\end{aligned}
$$

(v) Bilinear relations: *for all* $n,m \geq 0$, *and* $a,b \in \mathbb{C}^\infty$, *such that* $a-b = t-t'$,

$$
\begin{aligned}
(2.1.9)\qquad &\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_1^\infty a_i z^i} z^{n-m-1}\frac{dz}{2\pi i} \\
&= \oint_{z=0} \tau_{n+1}(t+[z])\tau_m(t'-[z])e^{\sum_1^\infty b_i z^{-i}} z^{n-m-1}\frac{dz}{2\pi i}.
\end{aligned}
$$

In the case $\beta = 2$, the Virasoro expressions take on a particularly elegant form, namely for $n \geq 0$,

$$
\begin{aligned}
\mathbb{J}^{(2)}_{k,n}(t) &= \sum_{i+j=k} :\ \mathbb{J}^{(1)}_{i,n}(t)\ \mathbb{J}^{(1)}_{j,n}(t) :\ = J^{(2)}_k(t) + 2nJ^{(1)}_k(t) + n^2\delta_{0k} \\
\mathbb{J}^{(1)}_{k,n}(t) &= J^{(1)}_k(t) + n\delta_{0k},
\end{aligned}
$$

with[8]

$$
\begin{aligned}
(2.1.10)\qquad J^{(1)}_k &= \frac{\partial}{\partial t_k} + \frac{1}{2}(-k)t_{-k}, \\
J^{(2)}_k &= \sum_{i+j=k}\frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i+j=k} it_i\frac{\partial}{\partial t_j} + \frac{1}{4}\sum_{-i-j=k} it_i jt_j.
\end{aligned}
$$

Statement (i) is already contained in Theorem 1.1, whereas the other statements can be found in [1], [2], and [5]. Notice that the standard Toda lattice is a reduction of the semi-infinite 2-Toda lattice, where $\tau_n(t,s) = \tau_n(t-s)$. The 2-Toda lattice arises in the context of a factorization of a generic semi-infinite matrix $m_\infty(t,s)$, satisfying the simple equations $\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty$, $\frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{\top k}$, whereas the standard Toda lattice is related to the same factorization of $m_\infty(t,s)$, but where $m_\infty(t,s)$ is Hänkel (i.e., $\Lambda m_\infty = m_\infty \Lambda^\top$).

[8]The expression $J^{(1)}_k = 0$ for $k = 0$.

Remark. The vectors p and q are eigenvectors of L. Indeed, remembering $\chi(z) = (1, z, z^2, ...)^\top$, we have

$$\Lambda\chi(z) = z\chi(z) \quad \text{and} \quad \Lambda^\top\chi(z^{-1}) = z\chi(z^{-1}) - ze_1, \quad \text{with } e_1 = (1, 0, 0, ...)^\top.$$

Therefore, $p(z) = S\chi(z)$ and $q(z) = S^{\top -1}\chi(z^{-1})$ are eigenvectors, in the sense

$$\begin{aligned} Lp &= S\Lambda S^{-1}S\chi(z) = zS\chi(z) = zp, \\ L^\top q &= S^{\top -1}\Lambda^\top S^\top S^{\top -1}\chi(z^{-1}) \\ &= zS^{\top -1}\chi(z^{-1}) - zS^{\top -1}e_1 = zq - zS^{\top -1}e_1. \end{aligned}$$

Then, using $L = L^\top$, one is lead to

$$((L - zI)p)_n = 0, \quad \text{for } n \geq 0 \quad \text{and} \quad ((L - zI)q)_n = 0, \quad \text{for} \quad n \geq 1.$$

2.2. *Symmetric/symplectic matrix integrals and the Pfaff lattice.* Consider an inner-product, with a skew-symmetric weight $\rho(y, z)$,

(2.2.1)
$$\langle f, g\rangle_t = \int\int_{\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty t_i(y^i+z^i)}\rho(y, z)dy\,dz, \text{ with } \rho(z, y) = -\rho(y, z).$$

Then, since

$$\langle f, g\rangle_t = -\langle g, f\rangle_t$$

the (semi-infinite) moment matrix, depending on $t = (t_1, t_2, \ldots)$,

$$m_n(t) = (\mu_{ij}(t))_{0\leq i,j\leq n-1} = (\langle y^i, z^j\rangle_t)_{0\leq i,j\leq n-1}$$

is skew-symmetric and the semi-infinite matrix m_∞ evolves in t according to the *commuting vector fields*

$$\frac{\partial\mu_{ij}}{\partial t_k} = \mu_{i+k,j} + \mu_{i,j+k}, \text{ i.e., } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty\Lambda^{\top k}. \tag{2.2.2}$$

It is well known that the determinant of an odd skew-symmetric matrix equals 0, whereas the determinant of an even skew-symmetric matrix is the square of a polynomial in the entries, the Pfaffian, with a sign specified below. So

$$\begin{aligned} \det(m_{2n-1}(t)) &= 0 \\ (\det m_{2n}(t))^{1/2} &= pf(m_{2n}(t)) = \frac{1}{n!}(dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{2n-1})^{-1} \\ &\qquad \left(\sum_{0\leq i<j\leq 2n-1} \mu_{ij}(t)dx_i \wedge dx_j\right)^n. \end{aligned}$$

Define now the *Pfaffian τ-functions*:

$$\tau_{2n}(t) := pf\ m_{2n}(t), \tag{2.2.3}$$

and the semi-infinite skew-symmetric matrix, 0 everywhere, except for the 2×2 blocks, along the diagonal:

$$J := \left(\begin{array}{cccc} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & & \ddots \end{array}\right), \text{ with } J^2 = -I. \tag{2.2.4}$$

Since m_∞ is skew-symmetric, m_∞ does not admit a Borel factorization in the standard sense, but m_∞ admits a unique factorization, with the matrix J inserted (see [6]):

$$m_\infty(t) = Q^{-1}(t) J\, Q^{\top -1}(t),$$

where
(2.2.5)

$$Q(t) = \left(\begin{array}{cccc} \ddots & & & 0 \\ & & 0 & \\ & \boxed{\begin{matrix} Q_{2n,2n} & 0 \\ 0 & Q_{2n,2n} \end{matrix}} & & \\ & * & \boxed{\begin{matrix} Q_{2n+2,2n+2} & 0 \\ 0 & Q_{2n+2,2n+2} \end{matrix}} & \\ & & & \ddots \end{array}\right) \in K.$$

K is the group of lower-triangular invertible matrices of the form above, with Lie algebra $\mathfrak{k}$ of matrices of precisely the same form. In this problem, the Lie algebra splitting of semi-infinite matrices is given by

$$gl(\infty) = \mathfrak{k} \oplus \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (2.2.5)}\} \\ \mathfrak{n} = sp(\infty) = \{a \text{ such that } Ja^\top J = a\}, \end{cases} \tag{2.2.6}$$

with unique decomposition ($a_\pm$ refers to projection onto strictly upper- (strictly lower) triangular matrices, with all 2×2 diagonal blocks equal to zero)

$$\begin{aligned} a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\ &= \left((a_- - J(a_+)^\top J) + \frac{1}{2}(a_0 - J(a_0)^\top J)\right) \\ &\qquad + \left((a_+ + J(a_+)^\top J) + \frac{1}{2}(a_0 + J(a_0)^\top J)\right). \end{aligned} \tag{2.2.7}$$

Considering as a special skew-symmetric weight (2.2.1),

$$\rho(y,z) := 2D^{\alpha}\delta(y-z)\tilde\rho(y)\tilde\rho(z)\ \text{, with } \alpha=\mp 1, \quad \tilde\rho(y)=e^{-\tilde V(y)}, \tag{2.2.8}$$

the inner-product (2.2.1) becomes[9] (see [8])

$$\begin{aligned}\langle f,g\rangle_t &= \int\int_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)}2D^{\alpha}\delta(y-z)\tilde\rho(y)\tilde\rho(z)dy\,dz\\ &= \begin{cases}\iint_{\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty t_i(y^i+z^i)}\varepsilon(y-z)\tilde\rho(y)\tilde\rho(z)dy\,dz, & \text{for } \alpha=-1\\ \int_{\mathbb{R}}\{f,g\}(y)e^{\sum_1^\infty 2t_iy^i}\tilde\rho(y)^2dy, & \text{for } \alpha=+1,\end{cases}\end{aligned}$$

and (see [16], [4])

(2.2.9)

$$pf\left(\langle y^i,z^j\rangle_t\right)_{0\le i,j\le 2n-1} = \begin{cases}\frac{1}{(2n)!}\int_{\mathbb{R}^{2n}}|\Delta_{2n}(z)|\prod_{k=1}^{2n}e^{\sum_1^\infty t_iz_k^i}\tilde\rho(z_k)dz_k\\ \qquad = \frac{1}{(2n)!}\int_{\mathcal{S}_{2n}}e^{\operatorname{Tr}(-\tilde V(X)+\sum t_iX^i)}dX, & \text{for } \alpha=-1,\\ \frac{1}{n!}\int_{\mathbb{R}^n}|\Delta_n(z)|^4\prod_{k=1}^{n}e^{\sum_1^\infty 2t_iz_k^i}\tilde\rho^2(z_k)dz_k\\ \qquad = \frac{1}{n!}\int_{\mathcal{T}_{2n}}e^{\operatorname{Tr}(-2\tilde V(X)+\sum 2t_iX^i)}dX, & \text{for } \alpha=+1.\end{cases}$$

Setting

$$\begin{cases}\tilde\rho(z)=\rho(z)I_E(z) & \text{for } \alpha=-1\\ \tilde\rho(z)=\rho^{1/2}(z)I_E(z),\ t\mapsto t/2 & \text{for } \alpha=+1\end{cases}$$

in the identities (2.2.9), we are led to the identities between integrals and Pfaffians, which are spelled out in Theorem 2.2:

THEOREM 2.2. *The integrals* $I_n(t,c)$,

$$I_n = \int_{E^n}|\Delta_n(z)|^{\beta}\prod_{k=1}^{n}\left(e^{\sum_1^\infty t_iz_k^i}\rho(z_k)dz_k\right)$$

[9] $\varepsilon(y)=sign(y)$, and $\{f,g\}:=f'g-fg'$. Also notice that $\varepsilon'=2\delta(x)$.

$$
= \begin{cases} n!pf\left(\iint_{E^2} y^i z^j \varepsilon(y-z) e^{\sum_1^\infty t_k(y^k+z^k)}\rho(y)\rho(z)dydz\right)_{0\le i,j\le n-1} \\ \qquad\qquad = n!\tau_n(t,c), \qquad\qquad n \textit{ even, for } \beta=1 \\ \\ n!pf\ \left(\int_E \{y^i,y^j\} e^{\sum_1^\infty t_k y^k}\rho(y)dy\right)_{0\le i,j\le 2n-1} = n!\tau_{2n}(t/2,c), \\ \qquad\qquad\qquad\qquad n \textit{ arbitrary, for } \beta=4 \end{cases}
$$

and the $\tau_n(t,c)$*'s above satisfy the following equations*:

(i) *The* Virasoro constraints[10] (1.1.4) *for* $\beta = 1,4$,

$$
\left(-\sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i} + \sum_{i\ge 0}\left(a_i\ {}^\beta\mathbb{J}^{(2)}_{k+i,n} - b_i\ {}^\beta\mathbb{J}^{(1)}_{k+i+1,n}\right)\right) I_n = 0 \tag{2.2.10}
$$

(ii) *The* Pfaff-KP hierarchy: (*see footnote* 6)

$$
\left(p_{k+4}(\tilde\partial) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right)\tau_n \circ \tau_n = p_k(\tilde\partial)\ \tau_{n+2}\circ\tau_{n-2} \tag{2.2.11}
$$

n *even*, $k = 0,1,2,\dots$.

of which the first equation reads

$$
\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1\partial t_3}\right)\log\tau_n + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau_n\right)^2 = 12\frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2},
$$

n *even*.

(iii) *The* Pfaff lattice: *The time-dependent matrix*

$$
L(t) = Q(t)\Lambda Q(t)^{-1} \tag{2.2.12}
$$

satisfies the Hamiltonian commuting equations, *given by the Adler-Kostant-Symes splitting theorem*, *applied to the splitting* $gl(\infty) = \mathfrak{k}\oplus\mathfrak{n}$, *as in* (2.2.6) *and* (2.2.7),

$$
\frac{\partial L}{\partial t_i} = [-(L^i)_{\mathfrak{k}}, L], \qquad \text{(Pfaff lattice)}
$$

(iv) Skew-orthogonal polynomials: *The vector of time-dependent polynomials* $q(t;z) := (q_n(t;z))_{n\ge 0} = Q(t)\chi(z)$ *in* z *satisfy the eigenvalue problem*

$$
L(t)q(t,z) = zq(t,z) \tag{2.2.13}
$$

[10]here the a_i's and b_i's are defined in the usual way, in terms of $\rho(z)$; namely, $-\frac{\rho'}{\rho} = \frac{\sum b_i z^i}{\sum a_i z^i}$.

and enjoy the following representations:

$$q_{2n}(t;z) = z^{2n}h_{2n}^{-1/2}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}, \qquad h_{2n}=\frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}$$

$$q_{2n+1}(t;z) = z^{2n}h_{2n}^{-1/2}\frac{1}{\tau_{2n}(t)}\left(z+\frac{\partial}{\partial t_1}\right)\tau_{2n}(t-[z^{-1}]).$$

They are skew-orthogonal polynomials in z; *i.e.*,

$$\langle q_i(t;z), q_j(t;z)\rangle_t = J_{ij}.$$

(v) The bilinear identities: *For all* $n,m\geq 0$, *the* τ_{2n}*'s satisfy the following bilinear identity*

$$\oint_{z=\infty}\tau_{2n}(t-[z^{-1}])\tau_{2m+2}(t'+[z^{-1}])e^{\sum_1^\infty (t_i-t_i')z^i}z^{2n-2m-2}\frac{dz}{2\pi i} + \oint_{z=0}\tau_{2n+2}(t+[z])\tau_{2m}(t'-[z])e^{\sum_1^\infty (t_i'-t_i)z^{-i}}z^{2n-2m}\frac{dz}{2\pi i}=0. \tag{2.2.14}$$

Note that (2.2.10) is a consequence of Theorem 1.1, while items (ii) to (v) are shown in [4], [6]. (See [8] for the Pfaff lattice, viewed as a reduction of the 2-Toda lattice.) A semi-infinite matrix $m_\infty(t,s)$, satisfying $\frac{\partial m_\infty}{\partial s_k} = \Lambda^k m_\infty$, $\frac{\partial m_\infty}{\partial t_k} = -m_\infty \Lambda^{\top k}$, leads to the semi-infinite 2-Toda lattice. When the initial condition $m_\infty(0,0)$ is skew-symmetric, then $m_\infty(t,-t)$ remains skew-symmetric in time and $\tau_n(t)=(\tau_n(t,-t))^{1/2}=pf\, m_n(t,-t)$ is a Pfaff lattice τ-function.

3. Expressing t-partials in terms of boundary-partials

3.1. *Gaussian and Laguerre ensembles.* Given first-order linear operators $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$ in $c=(c_1,...,c_{2r})\in\mathbb{R}^{2r}$ and a function $F(t,c)$, with $t\in\mathbb{C}^\infty$, satisfying the following partial differential equations in t and c:

$$\mathcal{D}_kF=\frac{\partial F}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}V_j(F)+\gamma_k+\delta_k t_1, \quad k=1,2,3,\ldots, \tag{3.1.1}$$

with $V_j(F)$ nonlinear differential operators in t_i of which the first few are given here:

$$V_j(F)=\sum_{i,i+j\geq 1} it_i\frac{\partial F}{\partial t_{i+j}}+\frac{\beta}{2}\delta_{2,j}\left(\frac{\partial^2 F}{\partial t_1^2}+\left(\frac{\partial F}{\partial t_1}\right)^2\right), \quad -1\leq j\leq 2. \tag{3.1.2}$$

In (3.1.1) and (3.1.2), $\beta>0,\gamma_{kj},\gamma_k,\delta_k$ are arbitrary parameters; also $\delta_{2j}=0$ for $j\neq 2$ and $\delta_{2j}=1$ for $j=2$. The claim is that the equations (3.1.1) enable one to express all partial derivatives,

$$(3.1.3)\qquad \left.\frac{\partial^{i_1+\ldots+i_k}F(t,c)}{\partial t_1^{i_1}\ldots\partial t_k^{i_k}}\right|_{\mathcal{L}},\ \text{along } \mathcal{L}:=\{\text{all } t_i=0,\ c=(c_1,\ldots,c_{2r}) \text{ arbitrary}\},$$

uniquely in terms of polynomials in $\mathcal{D}_{j_1}\ldots\mathcal{D}_{j_r}F(0,c)$. Indeed, the method consists of expressing $\left.\frac{\partial F}{\partial t_k}\right|_{t=0}$ in terms of $\left.\mathcal{D}_k f\right|_{t=0}$, using (3.1.1). Second derivatives are obtained by acting on $\mathcal{D}_k F$ with $\mathcal{D}_\ell$, by noting that $\mathcal{D}_\ell$ commutes with all t-derivatives, by using the equation for $\mathcal{D}_\ell F$, and by setting in the end $t=0$:

$$\begin{aligned}
\mathcal{D}_\ell\mathcal{D}_k F &= \mathcal{D}_\ell\frac{\partial F}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}\mathcal{D}_\ell(V_j(F))\\
&= \left(\frac{\partial}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}V_j\right)\mathcal{D}_\ell(F), \quad \text{provided } V_j(F) \text{ does not}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{contain nonlinear terms}\\
&= \left(\frac{\partial}{\partial t_k}+\sum_{-1\leq j<k}\gamma_{kj}V_j\right)\left(\frac{\partial F}{\partial t_\ell}+\sum_{-1\leq j<\ell}\gamma_{\ell j}V_j(F)+\delta_\ell t_1\right)\\
&= \frac{\partial^2 F}{\partial t_k\partial t_\ell}+\text{lower-weight terms.}
\end{aligned}$$

When the nonlinear term is present, it is taken care of as follows:

$$\begin{aligned}
\mathcal{D}_\ell\left(\frac{\partial F}{\partial t_1}\right)^2 &= 2\frac{\partial F}{\partial t_1}\mathcal{D}_\ell\frac{\partial F}{\partial t_1}\\
&= 2\frac{\partial F}{\partial t_1}\frac{\partial}{\partial t_1}\mathcal{D}_\ell F\\
&= 2\frac{\partial F}{\partial t_1}\frac{\partial}{\partial t_1}\left(\frac{\partial F}{\partial t_\ell}+\sum_{-1\leq j<\ell}\gamma_{\ell j}V_j(F)+\gamma_\ell+\delta_\ell t_1\right);
\end{aligned}$$

higher derivatives are obtained in the same way. Explicit expressions for only a few partials, useful in the next subsection, will be given here:

(3.1.4)

$$\begin{aligned}
\left.\frac{\partial F}{\partial t_1}\right|_{\mathcal{L}} &= \mathcal{D}_1 F-\gamma_1,\\
\left.\frac{\partial^2 F}{\partial t_1^2}\right|_{\mathcal{L}} &= \left(\mathcal{D}_1^2-\gamma_{10}\mathcal{D}_1\right)F+\gamma_{10}\gamma_1-\delta_1,
\end{aligned}$$

$$
\begin{aligned}
\left.\frac{\partial^3 F}{\partial t_1^3}\right|_{\mathcal{L}} &= \left(\mathcal{D}_1^3 - 3\gamma_{10}\mathcal{D}_1^2 + 2\gamma_{10}^2\mathcal{D}_1\right) F + 2\gamma_{10}(\delta_1 - \gamma_1\gamma_{10}),\\
\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{\mathcal{L}} &= \left(\mathcal{D}_1^4 - 6\gamma_{10}\mathcal{D}_1^3 + 11\gamma_{10}^2\mathcal{D}_1^2 - 6\gamma_{10}^3\mathcal{D}_1\right) F - 6\gamma_{10}^2(\delta_1 - \gamma_1\gamma_{10}),\\
\left.\frac{\partial F}{\partial t_2}\right|_{\mathcal{L}} &= \mathcal{D}_2 F - \gamma_2,\\
\left.\frac{\partial^2 F}{\partial t_2^2}\right|_{\mathcal{L}} &= \Big(\mathcal{D}_2^2 - 2\gamma_{20}\mathcal{D}_2 + \beta\gamma_{21}\gamma_{32}\mathcal{D}_1^2\\
&\qquad - ((2\gamma_1+\gamma_{10})\gamma_{21}\gamma_{32}\beta + 2\gamma_{2,-1})\mathcal{D}_1 - 2\gamma_{21}\mathcal{D}_3\Big) F\\
&\quad + \beta\gamma_{21}\gamma_{32}(\mathcal{D}_1 F)^2 + \beta\gamma_{21}\gamma_{32}(\gamma_1^2 + \gamma_{10}\gamma_1 - \delta_1)\\
&\quad + 2(\gamma_{21}\gamma_3 + \gamma_{20}\gamma_2 + \gamma_1\gamma_{2,-1}),\\
\left.\frac{\partial F}{\partial t_3}\right|_{\mathcal{L}} &= \left(\mathcal{D}_3 - \frac{\beta}{2}\gamma_{32}\mathcal{D}_1^2 + \frac{\beta}{2}\gamma_{32}(2\gamma_1+\gamma_{10})\mathcal{D}_1\right) F - \frac{\beta}{2}\gamma_{32}(\mathcal{D}_1 F)^2\\
&\qquad + \frac{\beta}{2}\gamma_{32}(\delta_1 - \gamma_1\gamma_{10} - \gamma_1^2) - \gamma_3,\\
\left.\frac{\partial^2 F}{\partial t_1 \partial t_3}\right|_{\mathcal{L}} &= \Big(\mathcal{D}_1\mathcal{D}_3 - \frac{\beta}{2}\gamma_{32}\mathcal{D}_1^3 + \beta\gamma_{32}(\gamma_1 + 2\gamma_{10})\mathcal{D}_1^2\\
&\qquad - \frac{3\beta}{2}\gamma_{10}\gamma_{32}(2\gamma_1+\gamma_{10})\mathcal{D}_1 - 3\gamma_{1,-1}\mathcal{D}_2 - 3\gamma_{10}\mathcal{D}_3\Big) F\\
&\quad + \frac{3\beta}{2}\gamma_{10}\gamma_{32}(\mathcal{D}_1 F)^2 - \beta\gamma_{32}(\mathcal{D}_1 F)(\mathcal{D}_1^2 F)\\
&\quad + \frac{3}{2}(2\gamma_{10}\gamma_3 + \beta\gamma_{32}\gamma_{10}(\gamma_1^2 + \gamma_{10}\gamma_1 - \delta_1) + 2\gamma_{1,-1}\gamma_2).
\end{aligned}
$$

3.2. *Jacobi ensemble.*

1. From the expressions (1.3.11), upon evaluating $\mathcal{B}_{-1}F|_{t=0}$, $\mathcal{B}_{-1}^2 F|_{t=0}$, $\mathcal{B}_0 F|_{t=0}$, one finds the following equations, both sides of which are evaluated at $t=0$,

$$
\begin{aligned}
-\mathcal{B}_{-1}F &= \sigma_1\frac{\partial}{\partial t_1}F + b_0 n,\\
\frac{1}{\sigma_1}\mathcal{B}_{-1}^2 F &= \left(\sigma_1\frac{\partial^2}{\partial {t_1}^2} + \frac{\partial}{\partial t_2}\right) F - n,
\end{aligned}
$$

$$-\mathcal{B}_0 F = \left(b_0\frac{\partial}{\partial t_1}+\sigma_2\frac{\partial}{\partial t_2}\right)F+\frac{\beta}{2}\left(\left(\frac{\partial}{\partial t_1}\right)^2 F+\left(\frac{\partial F}{\partial t_1}\right)^2\right)-\frac{n}{2}(\sigma_1-b_1).$$

From these expressions, one extracts

$$\left.\frac{\partial F}{\partial t_1}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_1^2}\right|_{t=0},\ \left.\frac{\partial F}{\partial t_2}\right|_{t=0},$$

in terms of $\mathcal{B}_j^i F$.

2. From the expressions for $\mathcal{B}_{-1}^3 F\big|_{t=0}$, $\mathcal{B}_0\mathcal{B}_{-1}F\big|_{t=0}$, $\mathcal{B}_1 F\big|_{t=0}$, namely

$$\mathcal{B}_1 F = \left(-b_0\frac{\partial}{\partial t_2}+(\sigma_1-b_1)\frac{\partial}{\partial t_1}-\sigma_3\frac{\partial}{\partial t_3}\right)F-\beta\left(\frac{\partial^2 F}{\partial t_1\partial t_2}+\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_2}\right),$$

$$\frac{1}{\sigma_1}\mathcal{B}_0\mathcal{B}_{-1}F = \left(\sigma_2\frac{\partial^2}{\partial t_1\partial t_2}+\frac{\partial}{\partial t_3}-\frac{\partial}{\partial t_1}+b_0\frac{\partial^2}{\partial t_1^2}\right)F+\frac{\beta}{2}\left(\frac{\partial^3 F}{\partial t_1^3}+2\frac{\partial F}{\partial t_1}\frac{\partial^2 F}{\partial t_1^2}\right),$$

$$-\frac{1}{\sigma_1}\mathcal{B}_{-1}^3 F = \left(\sigma_1^2\frac{\partial^3}{\partial t_1^3}+3\sigma_1\frac{\partial^2}{\partial t_1\partial t_2}-2\frac{\partial}{\partial t_1}+2\frac{\partial}{\partial t_3}\right)F,$$

one extracts

$$\left.\frac{\partial F}{\partial t_3}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_1^3}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_1\partial t_2}\right|_{t=0}$$

in terms of $\mathcal{B}_j^i F$, using the previous extractions.

3. From the expressions for $\mathcal{B}_2 F\big|_{t=0}$, $\mathcal{B}_1\mathcal{B}_{-1}F\big|_{t=0}$, $\mathcal{B}_0^2 F\big|_{t=0}$, $\mathcal{B}_0\mathcal{B}_{-1}^2 F\big|_{t=0}$, $\mathcal{B}_{-1}^4 F\big|_{t=0}$, namely, (where both sides are evaluated at $t=0$)

(3.2.1)

$$\mathcal{B}_2 F = \left(-\sigma_4\frac{\partial}{\partial t_4}-b_0\frac{\partial}{\partial t_3}+(\sigma_2-b_1)\frac{\partial}{\partial t_2}+\frac{\beta}{2}\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial t_2^2}-2\frac{\partial^2}{\partial t_1\partial t_3}\right)\right)F$$
$$+\beta\left(\left(\frac{\partial F}{\partial t_1}\right)^2-\left(\frac{\partial F}{\partial t_2}\right)^2-\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_3}\right),$$

$$\frac{1}{\sigma_1}\mathcal{B}_1\mathcal{B}_{-1}F = \left(\frac{\partial}{\partial t_4}-\frac{\partial}{\partial t_2}+b_0\frac{\partial^2}{\partial t_1\partial t_2}+\sigma_3\frac{\partial^2}{\partial t_1\partial t_3}-(\sigma_1-b_1)\frac{\partial^2}{\partial t_1^2}+\beta\frac{\partial^3}{\partial t_1^2\partial t_2}\right)F$$
$$+\ \beta\left(\frac{\partial^2 F}{\partial t_1^2}\frac{\partial F}{\partial t_2}+\frac{\partial F}{\partial t_1}\frac{\partial^2 F}{\partial t_1\partial t_2}\right),$$

$$\mathcal{B}_0^2 F = \left(b_0\frac{\partial}{\partial t_1}+\sigma_2\frac{\partial}{\partial t_2}+\frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}+\beta\frac{\partial}{\partial t_1}F\frac{\partial}{\partial t_1}\right)$$
$$\left(b_0\frac{\partial F}{\partial t_1}+\sigma_2\frac{\partial F}{\partial t_2}+\sum_1^2 it_i\left(\frac{\partial F}{\partial t_{i+2}}-\frac{\partial F}{\partial t_i}\right)+\frac{\beta}{2}\left(\frac{\partial^2 F}{\partial t_1^2}+\left(\frac{\partial F}{\partial t_1}\right)^2\right)\right),$$

$$\frac{1}{\sigma_1}\mathcal{B}_0\mathcal{B}_{-1}^2 F = -\frac{\partial}{\partial t_1}\left(\sigma_1\frac{\partial}{\partial t_1}+t_1\frac{\partial}{\partial t_2}\right)\left(b_0\frac{\partial F}{\partial t_1}+\sigma_2\frac{\partial F}{\partial t_2}+\sum_1^2 it_i\left(\frac{\partial F}{\partial t_{i+2}}-\frac{\partial F}{\partial t_i}\right)+\frac{\beta}{2}\left(\frac{\partial^2 F}{\partial t_1^2}+\left(\frac{\partial F}{\partial t_1}\right)^2\right)\right),$$

$$\mathcal{B}_{-1}^4 F = \sigma_1\frac{\partial}{\partial t_1}\left(\sigma_1\frac{\partial}{\partial t_1}+t_1\frac{\partial}{\partial t_2}\right)\left(\sigma_1\frac{\partial}{\partial t_1}+t_1\frac{\partial}{\partial t_2}+2t_2\left(\frac{\partial}{\partial t_3}-\frac{\partial}{\partial t_1}\right)\right)\left(\sigma_1\frac{\partial F}{\partial t_1}+t_1\frac{\partial F}{\partial t_2}+\sum_2^3 it_i\left(\frac{\partial F}{\partial t_{i+1}}-\frac{\partial F}{\partial t_{i-1}}\right)+b_0 n-nt_1\right),$$

one extracts

$$\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{t=0},\ \left.\frac{\partial F}{\partial t_4}\right|_{t=0},\ \left.\frac{\partial^3 F}{\partial t_1^2\partial t_2}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_1\partial t_3}\right|_{t=0},\ \left.\frac{\partial^2 F}{\partial t_2^2}\right|_{t=0}, \tag{3.2.2}$$

again in terms of $\mathcal{B}_j^i F$, using all the previous extractions.

3.3. *Evaluating the matrix integrals on the full range.* The denominators of the probabilities (0.0.4), for $\beta = 1, 4$; namely:

$$I_n^{(\beta)} := \begin{cases} \displaystyle\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-bz_k^2}dz_k \\ \displaystyle\int_{\mathbb{R}_+^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k}dz_k \\ \displaystyle\int_{[-1,1]^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a(1+z_k)^b dz_k \end{cases},$$

can be evaluated, using Selberg's integral (see Mehta [16, p. 340]):

$$I_n^{(\beta)} \begin{cases} = (2\pi)^{n/2}(2b)^{-n(\beta(n-1)+2)/4}\displaystyle\prod_{j=0}^{n-1}\frac{\Gamma((j+1)\beta/2+1)}{\Gamma(\beta/2+1)} \\ = b^{-n(\beta(n-1)+2a+2)/2}\displaystyle\prod_{j=0}^{n-1}\frac{\Gamma(a+1+j\beta/2)\Gamma((j+1)\beta/2+1)}{\Gamma(\beta/2+1)} \\ = 2^{n(2a+2b+\beta(n-1)+2)/2} \\ \qquad\cdot\displaystyle\prod_{j=0}^{n-1}\frac{\Gamma(a+j\beta/2+1)\Gamma(b+j\beta/2+1)\Gamma((j+1)\beta/2+1)}{\Gamma(\beta/2+1)\Gamma(a+b+(n+j-1)\beta/2+2)} \end{cases}.$$

LEMMA 3.1. *For future use, the following expressions*

$$b_n^{(\beta=1)} := \frac{(n!)^2}{(n-2)!(n+2)!}\frac{I_{n-2}^{(1)}I_{n+2}^{(1)}}{(I_n^{(1)})^2} = \begin{cases} \frac{n(n-1)}{16b^2} & \text{(Gauss)} \\ \frac{n(n-1)(n+2a)(n+2a+1)}{16b^4} & \text{(Laguerre)} \\ \frac{Q}{Q_6^\pm} & \text{(Jacobi)} \end{cases}$$

$$b_n^{(\beta=4)} \quad := \quad \frac{(n!)^2}{(n-1)!(n+1)!}\frac{I_{n-1}^{(4)}I_{n+1}^{(4)}}{(I_n^{(4)})^2} = \begin{cases} \frac{2n(2n+1)}{4b^2} & \text{(Gauss)} \\ \frac{2n(2n+1)(2n+a)(2n+a-1)}{b^4} & \text{(Laguerre)} \\ \frac{Q}{Q_6^{\pm}} & \text{(Jacobi)} \end{cases}$$

satisfy the following functional dependence:

$$b_n^{(4)}(n,a,b) = b_n^{(1)}\left(-2n, -\frac{a}{2}, -\frac{b}{2}\right).$$

In the expressions above, Q (*already appearing in* (0.3.1)), *and a new expression* $Q_6^{\pm}$ *are expressible in terms of the variables* q, r, s *introduced in* (0.3.1):

$$\begin{aligned} Q \quad &:= \begin{cases} 48(n-1)n(2a+n)(2a+n+1)(2b+n)(2b+n+1) \\ \quad (2b+2a+n+1)(2b+2a+n+2), & \textit{for } (\beta=1) \\ 96n(2n+1)(a+2n-1)(a+2n)(b+2n-1) \\ \quad (b+2n)(b+a+2n-2)(b+a+2n-1), & \textit{for } (\beta=4) \end{cases} \\ &= \frac{3}{16}\left((s^2-qr+q^2)^2 - 4(rs^2-4qs^2-4s^2+q^2r)\right) \end{aligned}$$

and[11]

$$\begin{aligned} Q_6^{\pm} \quad &= \begin{cases} = 48\,(b+a+n)\,(b+a+n+1)^2\,(b+a+n+2)\,(2b+2a+2n-1) \\ \quad (2b+2a+2n+1)^2\,(2b+2a+2n+3), & \textit{for } \beta=1 \\ = 3(b+a+4n-4)(b+a+4n-3)(b+a+4n-2)^2 \\ \quad (b+a+4n-1)^2(b+a+4n)(b+a+4n+1), & \textit{for } \beta=4 \end{cases} \\ &= 3q\,(q+1)\,(q-3)\left(q+4\pm 4\sqrt{q+1}\right) \qquad \begin{cases} + & \textit{for} \quad \beta=1 \\ - & \textit{for} \quad \beta=4 \end{cases}. \end{aligned}$$

Proof. For instance, in the Jacobi case, one computes

$$\begin{aligned} \frac{I_{n+2}^{(1)}}{I_n^{(1)}} \quad &= \quad 2^{2n+2a+2b+3}\ \frac{\Gamma(\frac{n+3}{2}+a+b)\Gamma(\frac{n+4}{2}+a+b)\Gamma(\frac{n+2}{2}+a)\Gamma(\frac{n+2}{2}+b)}{\Gamma(n+a+b+\frac{3}{2})\Gamma(n+a+b+2)} \\ &\quad \cdot \frac{\Gamma(\frac{n+3}{2}+a)\Gamma(\frac{n+3}{2}+b)\Gamma(\frac{n+3}{2})\Gamma(\frac{n+4}{2})}{\Gamma(n+a+b+\frac{5}{2})\Gamma(n+a+b+3)} \\ \frac{I_{n+1}^{(4)}}{I_n^{(4)}} \quad &= \quad 2^{4n+a+b}\ \frac{\Gamma(2n+a+b)\Gamma(2n+a+1)\Gamma(2n+b+1)\Gamma(2n+3)}{\Gamma(4n+a+b)\Gamma(4n+a+b+2)} \end{aligned}$$

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[11] $\sqrt{q+1} = 2n+2b+2a+1$ for $\beta=1$ and $\sqrt{q+1} = 4n+b+a-1$ for $\beta=4$

and so,

$$\left.\frac{(n!)^2}{(n-2)!(n+2)!}\frac{I^{(\beta)}_{n-2}I^{(\beta)}_{n+2}}{(I^{(\beta)}_n)^2}\right|_{\beta=1} = \left.\frac{Q}{Q^{\pm}_6}\right|_{\beta=1}$$

$$\left.\frac{(n!)^2}{(n-1)!(n+1)!}\frac{I^{(\beta)}_{n-1}I^{(\beta)}_{n+1}}{(I^{(\beta)}_n)^2}\right|_{\beta=4} = \left.\frac{Q}{Q^{\pm}_6}\right|_{\beta=4}. \qquad \square$$

4. Proof of Theorems 0.1, 0.2, 0.3

From Theorems 2.1 and 2.2, the integrals $I_n(t,c)$, depending on $\beta = 2,1,4$, on $t=(t_1,t_2,...)$ and on the boundary points $c=(c_1,...,c_{2r})$ of E, relate to τ-functions, as follows:

$$I_n(t,c) = \int_{E^n}|\Delta_n(z)|^{\beta}\prod_{k=1}^{n}\left(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)dz_k\right) \tag{4.0.1}$$

$$= \begin{cases} n!\tau_n(t,c), & n \text{ arbitrary}, \quad \beta=2 \\ n!\tau_n(t,c), & n \text{ even}, \quad \beta=1 \\ n!\tau_{2n}(t/2,c), & n \text{ arbitrary}, \quad \beta=4. \end{cases}$$

$I_n(t)$ refers to the integral (4.0.1) over the full range. It also follows that $\tau_n(t,c)$ satisfies the KP-like equation[12]

(4.0.2)

$$12\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\delta^{\beta}_{1,4} = (\mathrm{KP})_t\log\tau_n(t,c), \quad \begin{cases} n \text{ arbitrary for } \beta=2 \\ n \text{ even for } \beta=1,4\end{cases}$$

where

$$(\mathrm{KP})_t F := \left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1\partial t_3}\right)F + 6\left(\frac{\partial^2}{\partial t_1^2}F\right)^2.$$

4.1. $\beta = 2,1$. Evaluating the left-hand side of (4.0.2)(for $\beta=1$) yields, taking into account $P_n := P_n(E) = I_n(0,c)/I_n(0)$:

$$\left.12\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\right|_{t=0} = \left.12\frac{(n!)^2}{(n-2)!(n+2)!}\frac{I_{n-2}(t,c)I_{n+2}(t,c)}{I_n(t,c)^2}\right|_{t=0}$$

$$= 12\frac{n(n-1)}{(n+1)(n+2)}\frac{I_{n-2}(0)I_{n+2}(0)}{I_n(0)^2}\frac{P_{n-2}P_{n+2}}{P_n^2}$$

$$= 12b_n^{(1)}\frac{P_{n-2}(E)P_{n+2}(E)}{P_n^2(E)},$$

[12]Remember $\delta^{\beta}_{1,4}=1$ for $\beta=1,4$, and $=0$ for $\beta=2$.

with $b_n^{(1)}$ given by Lemma 3.1. Concerning the right-hand side of (4.0.2), it follows from Section 2.1 that $F_n(t;c) = \log I_n(t;c)$, as in (4.0.1), satisfies Virasoro constraints, corresponding precisely to the situation of Sections 3.1 and 3.2 for Gauss, Laguerre and Jacobi. As explained in (3.1.4), (3.2.1) and (3.2.2), we express

$$\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_2^2}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1 \partial t_3}\right|_{t=0}, \left.\frac{\partial^2 F}{\partial t_1^2}\right|_{t=0}, \qquad F = \log I_n(t,c),$$

in terms of $\mathcal{D}_k$ and $\mathcal{B}_k$, which when substituted in the right-hand side of (4.0.2), i.e., in the KP-expressions, leads to (upon comparing the expressions (1.3.4) and (1.3.8) with (3.1.1) for Gauss and Laguerre and using (3.2.1) directly for Jacobi):

• Gauss with $\begin{cases} \gamma_{1,-1} = -\frac{1}{2}, \gamma_{1,0} = \gamma_1 = 0, \delta_1 = -\frac{n}{2} \\ \gamma_{2,-1} = 0, \gamma_{2,0} = -1/2, \gamma_{2,1} = 0, \gamma_2 = -\frac{n}{4}\sigma_1, \delta_2 = 0 \\ \gamma_{3,-1} = -\frac{1}{4}\sigma_1, \gamma_{3,0} = 0, \gamma_{3,1} = -\frac{1}{2}, \gamma_{3,2} = \gamma_3 = 0, \delta_3 = -\frac{n}{4}\sigma_1. \end{cases}$

$$\begin{aligned} &(\mathrm{KP})_t \log \tau_n(t,c)\big|_{t=0} \\ &= (\mathcal{D}_1^4 + 6n\mathcal{D}_1^2 + 3\mathcal{D}_2^2 - 3\mathcal{D}_2 - 4\mathcal{D}_1\mathcal{D}_3)F + 6(\mathcal{D}_1^2 F)^2 + \frac{3}{4}(2-\beta)n(n-1) \\ &= \frac{1}{16}\Big((\mathcal{B}_{-1}^4 + 8(n + (2-\beta)(n-1))\mathcal{B}_{-1}^2 + 12\mathcal{B}_0^2 + 24\mathcal{B}_0 - 16\mathcal{B}_{-1}\mathcal{B}_1)F \\ &\qquad + 6(\mathcal{B}_{-1}^2 F)^2 + 12(2-\beta)n(n-1)\Big) \end{aligned}$$

• Laguerre with $\begin{cases} \gamma_{1,-1} = 0, \gamma_{1,0} = -1, \gamma_1 = -\frac{n}{2}(\sigma_1 + a), \\ \gamma_{2,-1} = 0, \gamma_{2,0} = -\sigma_1, \gamma_{2,1} = -1, \gamma_2 = -\frac{n}{2}\sigma_1(\sigma_1 + a), \\ \gamma_{3,-1} = 0, \gamma_{3,0} = -\sigma_1\sigma_2, \gamma_{3,1} = -\sigma_2, \\ \gamma_{3,2} = -1, \gamma_3 = -\frac{n}{2}\sigma_1\sigma_2(\sigma_1 + a). \end{cases}$

$$\begin{aligned} &(\mathrm{KP})_t \log \tau_n(t,c)\big|_{t=0} \\ &= \Big(\mathcal{D}_1^4 - 2(\beta-3)\mathcal{D}_1^3 \\ &\quad - \Big(2n(n-1)(\beta-2)(\beta-1) + (\beta-2)(4an+4n+5) - 4n^2 - 4an - 1\Big)\mathcal{D}_1^2 \\ &\quad - 3(\beta-2)\Big(\beta n^2 - \beta n + 2an + 2n + 1\Big)\mathcal{D}_1 \\ &\quad + 3\mathcal{D}_2^2 + 6(\beta(n-1) + a + 2)\mathcal{D}_2 - 6\mathcal{D}_3 - 4\mathcal{D}_1\mathcal{D}_3 \Big) F_n \end{aligned}$$

$$-3(\beta-2)(\mathcal{D}_1F_n)^2+6(\mathcal{D}_1^2\log\tau_N)^2-4(\beta-3)(\mathcal{D}_1F_n)(\mathcal{D}_1^2F_n)$$

$$-\frac{3}{4}(\beta-2)n(n-1)(\beta n-2\beta+2a+2)(\beta n-\beta+2a+2)$$

$$\begin{aligned} = \ & \Big(\mathcal{B}_{-1}^4+2(\beta-3)\mathcal{B}_{-1}^3 \\ & -\Big((\beta-2)\Big(3(\beta-1)(n-1)^2+3n^2+6an-4a+2\Big)+\Big(a^2-1\Big)\Big)\mathcal{B}_{-1}^2 \\ & +3(\beta-2)\Big((\beta-1)(n-1)^2+n^2+2an-a\Big)\mathcal{B}_{-1}-4\mathcal{B}_1\mathcal{B}_{-1}-2\mathcal{B}_1 \\ & +2(\beta n+a)\mathcal{B}_0\mathcal{B}_{-1}+3\mathcal{B}_0^2-(\beta n+a)\mathcal{B}_0\Big)F \\ & +6(\mathcal{B}_{-1}^2F)^2+4(\beta-3)(\mathcal{B}_{-1}F)(\mathcal{B}_{-1}^2F)+3(2-\beta)(\mathcal{B}_{-1}F)^2 \\ & -\frac{3}{4}(\beta-2)n(n-1)(\beta n-2\beta+2a+2)(\beta n-\beta+2a+2). \end{aligned}$$

• Jacobi[13]

for $\beta=2$,

$$\frac{1}{8}q(q^2-4)\ (\mathrm{KP})_t\log\tau_n(t,c)\big|_{t=0}$$

$$\begin{aligned} = \ & \Big(2\mathcal{B}_{-1}^4+(q-r+4)\mathcal{B}_{-1}^2-(4\mathcal{B}_{-1}F-s)\mathcal{B}_{-1}+3q\mathcal{B}_0^2-2q\mathcal{B}_0+8\mathcal{B}_0\mathcal{B}_{-1}^2 \\ & -4(q-1)\mathcal{B}_1\mathcal{B}_{-1}+(4\mathcal{B}_{-1}F-s)\mathcal{B}_1+2(4\mathcal{B}_{-1}F-s)\mathcal{B}_0\mathcal{B}_{-1}+2q\mathcal{B}_2\Big)F \\ & +4\mathcal{B}_{-1}^2F\Big(2\mathcal{B}_0F+3\mathcal{B}_{-1}^2F\Big) \end{aligned}$$

for $\beta=1$,

$$Q_6^{\pm}\ (\mathrm{KP})_t\log\tau_n(t,c)\big|_{t=0}$$

$$\begin{aligned} = \ & (q+1)\Big(4q\mathcal{B}_{-1}^4+12(4\mathcal{B}_{-1}F-s)\mathcal{B}_{-1}^3+2(q+12)(4\mathcal{B}_{-1}F-s)\mathcal{B}_0\mathcal{B}_{-1} \\ & +3q^2\mathcal{B}_0^2-4(q-4)q\mathcal{B}_1\mathcal{B}_{-1}+q(4\mathcal{B}_{-1}F-s)\mathcal{B}_1+20q\mathcal{B}_0\mathcal{B}_{-1}^2+2q^2\mathcal{B}_2\Big)F \\ & +\Big(Q_2\mathcal{B}_{-1}^2-sQ_1\mathcal{B}_{-1}+Q_3\mathcal{B}_0\Big)F+48(\mathcal{B}_{-1}F)^4 \\ & -48s(\mathcal{B}_{-1}F)^3+2Q_4(\mathcal{B}_{-1}F)^2 \\ & +12q^2(\mathcal{B}_0F)^2+16q(2q-1)\mathcal{B}_{-1}^2F\mathcal{B}_0F+24(q-1)q(\mathcal{B}_{-1}^2F)^2 \\ & +24\Big(2\mathcal{B}_{-1}F-s\Big)\mathcal{B}_{-1}F\Big((q+2)\mathcal{B}_0F+(q+3)\mathcal{B}_{-1}^2F\Big)+Q, \end{aligned}$$

[13] In the Jacobi $\beta=2$ case, we have $b_0=a-b,\ b_1=a+b$; thus $r=2(b_0^2+b_1^2), q_n=2(2n+a+b)^2$ and $q(q^2-4)=16(2n+\gamma+\delta)^2(2n+\gamma+\delta-1)(2n+\gamma+\delta+1)$.

where the Q_1, Q_2, Q_3, Q_4, Q are given by (0.3.1) and where the auxiliary $Q_6^{\pm}$ happens to be exactly the one of Lemma 3.1. This establishes Theorems 0.1, 0.2 and 0.3 for $\beta = 2, 1$, at least when $b = 1$ in the exponent of the Gaussian and Laguerre ensembles, upon noting that $\mathcal{B}_k^j \log P_n(E) = \mathcal{B}_k^j \log I_n(0,c)/I_n(0) = \mathcal{B}_k^j \log \tau_n(0,c)$.

Finally, a simple argument captures the case $b \neq 1$. Indeed, setting $\alpha E := \bigcup_1^{2r} [\alpha\ c_{2i-1}, \alpha\ c_{2i}] \subset F$, for $\alpha > 0$, the elementary identities

$$\begin{aligned} I_n(t,c) &= \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-bz_k^2} dz_k = C \int_{(\sqrt{b}\ E)^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-z_k^2} dz_k \\ I_n(t,c) &= \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k} dz_k = C \int_{(b\ E)^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-z_k} dz_k, \end{aligned}$$

where $C(a,b,n,\beta)$ is a constant independent of E, lead to the same Virasoro constraints as in Examples 1 and 2 (§1.3), but with the following mapping for the differential operators

$$(4.1.1) \qquad (\mathcal{B}_{-1}, \mathcal{B}_0, \mathcal{B}_1) \quad \to \quad \left(\frac{\mathcal{B}_{-1}}{\sqrt{b}}, \mathcal{B}_0, \mathcal{B}_1 \sqrt{b} \right) \quad \text{(Gauss)}$$

$$(4.1.2) \qquad \to \quad \left(\mathcal{B}_{-1}, b\mathcal{B}_0, b^2 \mathcal{B}_1 \right) \quad \text{(Laguerre).}$$

Therefore, the equations (0.1.2) and (0.2.2) for the probabilities (0.1.1) and (0.2.1) are obtained by making the substitutions (4.1.1) and (4.1.2) in the PDEs $(0.1.2)|_{b=1}$ and $(0.2.2)|_{b=1}$; this process yields the precise equations (0.1.2) and (0.2.2), with $b \neq 1$. This ends the proof of Theorems 0.1, 0.2 and 0.3 for the cases $\beta = 1, 2$.

4.2. $\beta = 4$, *using duality.* From (4.0.1), the integral for $\beta = 4$ is expressible in terms of a τ-function, in which t is replaced by $t/2$. Hence (4.0.2) becomes:

$$(4.2.1) \qquad 12 \frac{\tau_{2n-2}(t/2,c)\tau_{2n+2}(t/2,c)}{\tau_{2n}(t/2,c)^2} = (\mathrm{KP})_{t/2} (\log \tau_{2n})(t/2,c).$$

So, the left-hand side of (4.2.1) equals ($P_n := P_n(E) = I_n(0,c)/I_n(0)$)

$$\begin{aligned} 12 \frac{\tau_{2n-2}(t/2,c)\tau_{2n+2}(t/2,c)}{\tau_{2n}(t/2,c)^2} \Big|_{t=0} &= 12 \frac{(n!)^2}{(n-1)!(n+1)!} \ \frac{I_{n-1}(t,c) I_{n+1}(t,c)}{I_n(t,c)^2} \Big|_{t=0} \\ &= 12 \frac{n}{(n+1)} \frac{I_{n-1}(0) I_{n+1}(0)}{I_n(0)^2} \frac{P_{n-1} P_{n+1}}{P_n^2} \\ &= 12 b_n^{(4)} \frac{P_{n-1}(E) P_{n+1}(E)}{P_n^2(E)}, \end{aligned}$$

where $b_n^{(4)} = b_n^{(4)}(n,a,b)$ is given by Lemma 3.1 and satisfies

$$b_n^{(4)}(n,a,b) = b_n^{(1)} \left(-2n, -\frac{a}{2}, -\frac{b}{2} \right).$$

Recall from Theorem 1.1 (1.1.4) that $I_n^{(\beta)}(t,c;a_i,b_i)$ and $I_n^{(4/\beta)}(t,c;a_i,b_i)$ (where we indicate the explicit dependence on the coefficients a_i and b_i of ρ'/ρ) satisfy the same equations, with altered parameters:

$$\left(\mathcal{B}_k-\mathcal{V}_{k,n}^{(\beta)}(t;n,a_i,b_i)\right)I_n^{(\beta)}(t,c;a_i,b_i)=0,$$

$$\left(\mathcal{B}_k-\mathcal{V}_{k,n}^{(\beta)}(-\frac{\beta}{2}t;-\frac{2}{\beta}n,a_i,-\frac{\beta}{2}b_i)\right)I_n^{(4/\beta)}(t,c;a_i,b_i)=0.$$

Setting $\beta=1$ in the equations above, extracting t-partials in terms of $\mathcal{B}_k$'s, and using the procedure explained in this section, we have that

$$\begin{aligned}(\mathrm{KP})_t(\log I_n^{(1)}(t,c;a_i,b_i))\Big|_{t=0} &= R(\mathcal{B};n,a_i,b_i)\log I_n^{(1)}(0,c;a_i,b_i)\\ &= R(\mathcal{B};n,a_i,b_i)\log P_n^{(1)}(E),\end{aligned}$$

$$\begin{aligned}(\mathrm{KP})_{t/2}(\log I_n^{(4)}(t,c;a_i,b_i))\Big|_{t=0} &= (\mathrm{KP})_{-t/2}(\log I_n^{(4)}(t,c;a_i,b_i))\Big|_{t=0}\\ &= R(\mathcal{B};-2n,a_i,-b_i/2)\log I_n^{(4)}(0,c;a_i,b_i)\\ &= R(\mathcal{B};-2n,a_i,-b_i/2)\log P_n^{(4)}(E),\end{aligned}$$

where $R(\mathcal{B};a_i,b_i,n)$ denotes the right-hand side of the equations (0.1.2), (0.2.2) and (0.3.4) for $\beta=1$. The coefficients a_i and b_i of the rational function $-\rho'/\rho$ are as follows: the a_i and b_i all vanish, except for

Hermite	$a_0=1$	$a_1=0$	$a_2=0$	$b_0=0$	$b_1=2b$
Laguerre	$a_0=0$	$a_1=1$	$a_2=0$	$b_0=-a$	$b_1=b$
Jacobi	$a_0=1$	$a_1=0$	$a_2=-1$	$b_0=a-b$	$b_1=a+b$;

thus the map

$$(n,a_i,b_i)\longrightarrow(-2n,a_i,-b_i/2)$$

translates into the map

$$(n,a,b)\longrightarrow(-2n,-a/2,-b/2),\tag{4.2.2}$$

which shows that the PDEs (0.1.2), (0.2.2) and (0.3.4) for the case $\beta=4$ are obtained by means of the map (4.2.2) from the same PDEs for $\beta=1$. But according to (0.0.5), this is the precise way the coefficients $q,s,Q_{-1},Q_0,Q_1,Q_2,Q_3,Q_4,Q$, evaluated at $\beta=4$, are obtained from the same coefficients at $\beta=1$. This ends the proof of Theorem 0.3. □

4.3. *Reduction to Chazy and Painlevé equations* ($\beta=2$). Setting $E=[-\infty,x],E=[0,x],E=[-1,x]$ in the PDEs (0.1.2), (0.2.2) and (0.3.4) respectively, leads to the equations (0.4.1), (0.4.2) and (0.4.3) respectively, as announced in Section 0.4. Furthermore setting $\beta=2$, the inductive terms on the left-hand side of (0.4.1) and (0.4.2) vanish and one obtains the ODEs:

- Gauss: $P_n(\max_i \lambda_i \leq x) = \exp(-\int_x^\infty f(u)du)$, where f satisfies:

$$f''' + 6\ f'^2 + 4b(2n - bx^2)f' + 4b^2 x\ f = 0.$$

- Laguerre: $P_n(\max_i \lambda_i \leq x) = \exp\left(-\int_x^\infty \frac{f(u)}{u}du\right)$, where f satisfies

$$x^2 f''' + xf'' + 6xf'^2 - 4ff' - ((a - bx)^2 - 4nbx)f' - b(2n + a - bx)f = 0.$$

- Jacobi: $P_n(\max_i \lambda_i \leq x) = \exp\left(-\int_x^1 \frac{f(u)}{1-u^2}du\right)$, where f satisfies:

$$2(x^2-1)^2 f''' + 4(x^2-1)\left(xf'' - 3f'^2\right) + \left(16xf - q_n(x^2-1) - 2sx - r\right) f' \\ -f\,(4f - q_n x - s) = 0,$$

where r, s, q_n are defined in (0.3.1).

These three equations are of the form

$$f''' + \frac{P'}{P}f'' + \frac{6}{P}f'^2 - \frac{4P'}{P^2}ff' + \frac{P''}{P^2}f^2 + \frac{4Q}{P^2}f' - \frac{2Q'}{P^2}f + \frac{2R}{P^2} = 0, \tag{4.3.1}$$

with the following coefficients P, Q, R:

Gauss	$P(x) = 1$	$4Q(x) = -4b^2x^2 + 8bn$	$R = 0$
Laguerre	$P(x) = x$	$4Q(x) = -(bx - a)^2 + 4bnx$	$R = 0$
Jacobi	$P(x) = 1 - x^2$	$4Q(x) = -\frac{1}{2}(q_n(x^2-1) + 2sx + r)$	$R = 0.$

The general Chazy class of differential equations are equations of the form $f''' = F(z, f, f', f'')$, where F is rational in f, f', f'' and locally analytic in z, subjected to the requirement that the general solution be free of movable branch points; the latter is a branch point whose location depends on the integration constants. In his classification, Chazy found thirteen cases, the first of which is given by (4.3.1), with arbitrary polynomials $P(z), Q(z), R(z)$ of degree $3, 2, 1$ respectively.

Cosgrove ([11], [12]), (A.3), shows this third-order equation has a first integral, which is second-order in f and quadratic in f'',

$$f''^2 + \frac{4}{P^2}\Big((Pf'^2 + Qf' + R)f' - (P'f'^2 + Q'f' + R')f \\ + \frac{1}{2}(P''f' + Q'')f^2 - \frac{1}{6}P'''f^3 + c\Big) = 0, \tag{4.3.2}$$

with an integration constant c. In the three cases, discussed above, $c = 0$. Notice equations of the general form

$$f''^2 = G(x, f, f')$$

are invariant under the map

$$x \mapsto \frac{a_1 z + a_2}{a_3 z + a_4} \quad \text{and} \quad f \mapsto \frac{a_5 f + a_6 z + a_7}{a_3 z + a_4}.$$

Using this map, the polynomial $P(z)$ can be normalized to

$$P(z) = z(z-1), \ z, \ \text{or } 1.$$

Equation (4.3.2) is a master Painlevé equation, containing the six Painlevé equations. If $f(x)$ satisfies the first three equations above, then the new function $g(z)$, defined below,

$$\begin{array}{ll} \text{Gauss} & g(z) = b^{-1/2} f(zb^{-1/2}) + \frac{2}{3} nz \\ \text{Laguerre} & g(z) = f(z) + \frac{b}{4}(2n+a)z + \frac{a^2}{4} \\ \text{Jacobi} & g(z) := -\frac{1}{2} f(x)|_{x=2z-1} - \frac{q}{8} z + \frac{q+s}{16} \end{array}$$

satisfies the following canonical equations of Cosgrove and Scoufis ([11], [12]):

- $g''^2 = -4g'^3 + 4(zg' - g)^2 + A_1 g' + A_2,$ (Painlevé IV)
- $(zg'')^2 = (zg' - g)\Big(-4g'^2 + A_1(zg' - g) + A_2\Big) + A_3 g' + A_4,$ (Painlevé V)
- $(z(z-1)g'')^2 = (zg' - g)\Big(4g'^2 - 4g'(zg' - g) + A_2\Big) + A_1 g'^2 + A_3 g' + A_4$ (Painlevé VI)

with respective coefficients

- $A_1 = 3\left(\frac{4n}{3}\right)^2, \ A_2 = -\left(\frac{4n}{3}\right)^3,$
- $A_1 = b^2, \ A_2 = b^2((n + \frac{a}{2})^2 + \frac{a^2}{2}), \ A_3 = -a^2 b(n + \frac{a}{2}), \ A_4 = \frac{(ab)^2}{2} .((n + \frac{a}{2})^2 + \frac{a^2}{8}),$
- $A_1 = \frac{2q+r}{8}, \ A_2 = \frac{qs}{16}, \ A_3 = \frac{(q-s)^2 + 2qr}{64}, \ A_4 = \frac{q}{512}(2s^2 + qr).$

Each of the equations above can be transformed into the standard Painlevé equations.

5. Appendix. Self-similarity proof of the Virasoro constraints (Theorem 1.1)

Given the data (0.0.1) to (0.0.3), namely $\rho = e^{-V}$ and $-\rho'/\rho = V' = g/f = \sum_0^\infty b_i z^i / \sum_0^\infty a_i z^i$ and $E = \bigcup_1^r \, [c_{2i-1}, c_{2i}] \subseteq F \subseteq \mathbb{R}$, we show that the multiple integral

$$I_n(t,c;\beta) := \int_{E^n} |\Delta_n(x)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i x_k^i} \rho(x_k) dx_k \right), \quad \text{for } n > 0 \tag{5.0.1}$$

satisfies the Virasoro constraints of Theorem 1.1, using a (much less conceptual!) self-similarity argument. Setting

$$dI_n(x) := |\Delta_n(x)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i x_k^i} \rho(x_k) dx_k \right),$$

we state the following lemma:

Lemma 5.1. *The following variational formula holds*:

$$\frac{d}{d\varepsilon} dI_n(x_i \mapsto x_i + \varepsilon f(x_i) x_i^{k+1}) \Big|_{\varepsilon=0} = \sum_{\ell=0}^\infty \left(a_\ell \; {}^\beta\mathbb{J}^{(2)}_{k+\ell,n} - b_\ell \; {}^\beta\mathbb{J}^{(1)}_{k+\ell+1,n} \right) dI_n. \tag{5.0.2}$$

Proof. Upon setting

$$\begin{aligned} E(x,t) &:= \prod_1^n e^{\sum_{i=1}^\infty t_i x_k^i} \rho(x_k) \\ &= \prod_1^n e^{-V(x_k,t)}, \quad \text{where } V(x,t) := V(x) - \sum_1^\infty t_i x^i, \end{aligned} \tag{5.0.3}$$

the following two relations hold:

$$\begin{aligned} \left(\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{n}{2}\delta_{k,0} \right) E &= \left(\sum_{\substack{1\le a<\beta\le n \\ i,j>0 \\ i+j=k}} x_\alpha^i x_\beta^j + \frac{k-1}{2} \sum_{1\le\alpha\le n} x_\alpha^k \right) E, \\ \left(\frac{\partial}{\partial t_k} + n\delta_{k,0} \right) E &= \left(\sum_{1\le\alpha\le n} x_\alpha^k \right) E, \quad \text{for all } k \ge 0. \end{aligned} \tag{5.0.4}$$

So, the point now is to compute the ε-derivative

$$\frac{d}{d\varepsilon} \left(|\Delta_n(x)|^\beta e^{\sum_{k=1}^n (-V(x_k) + \sum_{i=1}^\infty t_i x_k^i)} dx_1 \ldots dx_n \right)_{x_i \mapsto x_i + \varepsilon f(x_i) x_i^{k+1}} \Bigg|_{\varepsilon=0}, \tag{5.0.5}$$

which consists of three contributions:

Contribution 1:

(5.0.6)

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\left|\Delta(x+\varepsilon f(x)x^{k+1})\right|^{\beta}\Big|_{\varepsilon=0}\\
&= \beta|\Delta(x)|^{\beta}\sum_{1\le\alpha<\gamma\le n}\frac{\partial}{\partial\varepsilon}\log\left(|x_\alpha-x_\gamma+\varepsilon(f(x_\alpha)x_\alpha^{k+1}-f(x_\gamma)x_\gamma^{k+1})|\right)\Big|_{\varepsilon=0}\\
&= \beta|\Delta(x)|^{\beta}\sum_{1\le\alpha<\gamma\le n}\frac{f(x_\alpha)x_\alpha^{k+1}-f(x_\gamma)x_\gamma^{k+1}}{x_\alpha-x_\gamma}\\
&= \beta|\Delta(x)|^{\beta}\sum_{\ell=0}^{\infty}a_\ell\sum_{1\le\alpha<\gamma\le n}\frac{x_\alpha^{k+\ell+1}-x_\gamma^{k+\ell+1}}{x_\alpha-x_\gamma}\\
&= \beta|\Delta(x)|^{\beta}\sum_{\ell=0}^{\infty}a_\ell\left(\sum_{\substack{i+j=\ell+k\\ i,j>0\\ 1\le\alpha<\gamma\le n}}x_\alpha^i x_\gamma^j+(n-1)\sum_{1\le\alpha\le n}x_\alpha^{\ell+k}-\frac{n(n-1)}{2}\delta_{\ell+k,0}\right)\\
&= \beta E^{-1}|\Delta(x)|^{\beta}\sum_{\ell=0}^{\infty}a_\ell\Bigg(\frac12\sum_{\substack{i+j=k+\ell\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}-\frac{n}{2}\delta_{k+\ell,0}\\
&\quad+\left(n-\frac{k+\ell+1}{2}\right)\left(\frac{\partial}{\partial t_{k+\ell}}+n\delta_{k+\ell,0}\right)-\frac{n(n-1)}{2}\delta_{k+\ell,0}\Bigg)E\\
&= \beta E^{-1}|\Delta(x)|^{\beta}\sum_{\ell=0}^{\infty}a_\ell\\
&\quad\left(\frac12\sum_{\substack{i+j=k+\ell\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}+\left(n-\frac{k+\ell+1}{2}\right)\frac{\partial}{\partial t_{k+\ell}}+\frac{n(n-1)}{2}\delta_{k+\ell,0}\right)E.
\end{aligned}$$

Contribution 2:

(5.0.7)
$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\prod_1^n d(x_\alpha+\varepsilon f(x_\alpha)x_\alpha^{k+1})\Big|_{\varepsilon=0}\\
&\quad=\sum_1^n\left(f'(x_\alpha)x_\alpha^{k+1}+(k+1)f(x_\alpha)x_\alpha^k\right)\prod_1^n dx_i\\
&\quad=\sum_{\ell=0}^{\infty}(\ell+k+1)a_\ell\sum_{\alpha=1}^n x_\alpha^{k+\ell}\prod_1^n dx_i\\
&\quad=E^{-1}\sum_{\ell=0}^{\infty}(\ell+k+1)a_\ell\left(\frac{\partial}{\partial t_{k+\ell}}+n\delta_{k+\ell,0}\right)E\prod_1^n\ dx_i,
\end{aligned}$$

Contribution 3:

$$
(5.0.8)\quad \frac{\partial}{\partial\varepsilon}\prod_{\alpha=1}^{n}\exp\left(-V\left(x_\alpha+\varepsilon f(x_\alpha)x^{k+1}\right)\right.
$$

$$
\left.+\sum_{i=1}^{\infty}t_i\sum_{\alpha=1}^{n}\left(x_\alpha+\varepsilon f(x_\alpha)x_\alpha^{k+1}\right)^i\right)\Bigg|_{\varepsilon=0}
$$

$$
=\left(-\sum_{\alpha=1}^{n}V'(x_\alpha)f(x_\alpha)x_\alpha^{k+1}+\sum_{i=1}^{\infty}it_i\sum_{\alpha=1}^{n}f(x_\alpha)x_\alpha^{i+k}\right)E
$$

$$
=\left(-\sum_{\ell=0}^{\infty}b_\ell\sum_{\alpha=1}^{n}x_\alpha^{k+\ell+1}+\sum_{\substack{\ell\geq 0\\ i\geq 1}}a_\ell it_i\sum_{\alpha=1}^{n}x_\alpha^{i+k+\ell}\right)E
$$

$$
=\left(-\sum_{\ell=0}^{\infty}b_\ell\left(\frac{\partial}{\partial t_{k+\ell+1}}+n\delta_{k+\ell+1,0}\right)\right.
$$

$$
\left.+\sum_{\ell=0}^{\infty}a_\ell\sum_{i=1}^{\infty}it_i\left(\frac{\partial}{\partial t_{i+k+\ell}}+n\delta_{i+k+\ell,0}\right)\right)E.
$$

As mentioned, to conclude (5.0.2), we must add up the three contributions (5.0.6), (5.0.7) and (5.0.8), resulting in:

(5.0.9)

$$
\frac{\partial}{\partial\varepsilon}dI_n(x_i\mapsto x_i+\varepsilon f(x_i)x_i^{k+1})\Big|_{\varepsilon=0}
$$

$$
=\left(\sum_{\ell=0}^{\infty}a_\ell\left(\frac{\beta}{2}J_{k+\ell}^{(2)}+(n\beta+(\ell+k+1)(1-\frac{\beta}{2}))J_{k+\ell}^{(1)}\right.\right.
$$

$$
\left.\left.+n((n-1)\frac{\beta}{2}+1)\delta_{k+\ell,0}\right)-\sum_{\ell=0}^{\infty}b_\ell\left(J_{k+\ell+1}^{(1)}+n\delta_{k+\ell+1,0}\right)\right)dI_n(x).
$$

where $J_k^{(i)} := {}^{\beta}J_k^{(i)}$, as in (1.1.8). Thus we use (1.1.8) to end the proof of Lemma 5.1. □

Proof of Theorem 1.1. The change of integration variable $x_i\mapsto x_i+\varepsilon f(x_i)x_i^{k+1}$ in the integral (5.0.1) leaves the integral invariant, but it induces a change of limits of integration, given by the inverse of the map above; namely the c_i's in $E=\bigcup_1^r[c_{2i-1},c_{2i}]$, get mapped as follows:

$$
c_i\mapsto c_i-\varepsilon f(c_i)c_i^{k+1}+O(\varepsilon^2).
$$

Therefore, setting

$$
E^\varepsilon=\bigcup_1^r[c_{2i-1}-\varepsilon f(c_{2i-1})c_{2i-1}^{k+1}+O(\varepsilon^2),c_{2i}-\varepsilon f(c_{2i})c_{2i}^{k+1}+O(\varepsilon^2)],
$$

we find, using Lemma 5.1 and the fundamental theorem of calculus,

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon}\int_{(E^{\varepsilon})^{2n}} |\Delta_{2n}(x+\varepsilon f(x)x^{k+1})| \prod_{i=1}^{2n} e^{-V(x_i+\varepsilon f(x_i)x_i^{k+1},t)} d(x_i+\varepsilon f(x_i)x_i^{k+1}) \\ &= \left(-\sum_{i=1}^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i} + \sum_{\ell=0}^{\infty}\left(a_\ell\ {}^{\beta}\mathbb{J}^{(2)}_{k+\ell,n} - b_\ell\ {}^{\beta}\mathbb{J}^{(1)}_{k+\ell+1,n}\right)\right) I_n(t,c,\beta). \end{aligned}$$

This ends the alternative proof of Theorem 1.1. □

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References

[1] M. Adler and P. van Moerbeke, Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials, *Duke Math. J.* **80** (1995), 863–911.

[2] ———, The spectrum of coupled random matrices, *Ann. of Math.* **149** (1999), 921–976.

[3] ———, Integrals over classical groups, random permutations, Toda and Toeplitz lattices, *Comm. Pure Appl. Math.* **54** (2001), 153–205 (math.CO9912143).

[4] ———, The Pfaff lattice, matrix integrals and a map from Toda to Pfaff, *Duke Math. J.* (2001), to appear (solv-int/9912008).

[5] ———, Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems, *Comm. Math. Phys.* **207** (1999), 589–620.

[6] M. Adler, E. Horozov, and P. van Moerbeke, The Pfaff lattice and skew-orthogonal polynomials, *Internat. Math. Res. Not.* **11** (1999), 569–588.

[7] M. Adler, T. Shiota, and P. van Moerbeke, Random matrices, vertex operators and the Virasoro algebra, *Phys. Lett.* A **208** (1995), 67–78.

[8] ———, Pfaff τ-functions, *Math. Ann.* (2001), to appear (solv-int/9909010).

[9] K. Aomoto, Jacobi polynomials associated with Selberg integrals, *SIAM J. Math. Anal.* **18** (1987), 545–549.

[10] H. Awata, Y. Matsuo, S. Odake, and J. Shiraishi, Collective field theory, Calogero-Sutherland model and generalized matrix models, *Phys. Lett.* B **347** (1995), 49–55.

[11] C. M. Cosgrove, Chazy classes IX-XII of third-order differential equations, *Stud. Appl. Math.* **104** (2000), 171–228.

[12] C. M. Cosgrove and G. Scoufis, Painlevé classification of a class of differential equations of the second order and second degree, *Stud. Appl. Math.* **88** (1993), 25–87.

[13] L. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific Publ. Co., River Edge, NJ (1991).

[14] L. Haine and J. P. Semengue, The Jacobi polynomial ensemble and the Painlevé VI equation, *J. Math. Phys.* **40** (1999), 2117–2134.

[15] G. Mahoux and M. L. Mehta, A method of integration over matrix variables: IV, *J. Physique* I **1** (1991), 1093–1108.

[16] M. L. Mehta, *Random Matrices*, 2nd ed. Academic Press, Boston, MA, 1991.

[17] ———, A nonlinear differential equation and a Fredholm determinant, *J. Physique* I, (1992), 1721–1729.

[18] M. L. Mehta and G. Mahoux, Level spacing functions and nonlinear differential equations, *J. Physique* I **3** (1993), 697–715.

[19] G. Moore, Matrix models of 2D gravity and isomonodromic deformations, *Progr. Theor. Phys. Suppl.*, No. 102 (1990), 255–285.

[20] C. A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models, *Comm. Math. Phys.* **163** (1994), 33–72.

[21] ———, On orthogonal and symplectic matrix ensembles, *Comm. Math. Phys.* **177** (1996), 727–754.

[22] K. Ueno and K. Takasaki, Toda lattice hierarchy, *Adv. Stud. Pure Math.* **4** (1984), 1–95.

[23] P. van Moerbeke, The spectrum of random matrices and integrable systems, Group 21, in *Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras*, Vol. II, 835–852 (H.-D. Doebner, W. Scherer, and C. Schulte, eds.), World Scientific Publ. Co., Singapore, 1997.

[24] ———, Integrable lattices: random matrices and random permutations, in *Random Matrices and their Applications*, MSRI publications **40** (2001), 321–406.

[25] E. P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, *Proc. Cambr. Philos. Soc.* **47** (1951), 790–798.

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