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Triangulations and Lattice Points

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Abstract. We study triangulations of the plane, in which for each triangle a special point of the triangle is a lattice point. Depending on the selection of this special point, we prove sharp upper bounds on the number of lattice points which occur as special points of triangles. The problems are partially motivated by an application in numerical integration.

1. Introduction

Let S denote a function which selects from each triangle Δ an interior point $S(\Delta)$. In the following we study edge-to-edge triangulations T of the plane, in which for each triangle $\Delta \in T$ the point $S(\Delta)$ is a point of the integer lattice $\mathbb{Z}^2$. We prove for the special choices $S =$ centroid and $S =$ circumcenter upper bounds for the number of lattice points which occur as selected points $S(\Delta)$ for some $\Delta \in T$.

In the following "triangulation" is always understood to mean edge-to-edge triangulation of the plane, i.e. we do not allow a vertex of one triangle to be an interior point of an edge of another triangle.

2. Centroid

Subdivisions of sets in subsets with prescribed centroids are of interest in the construction of lower bounds for the numerical integration of convex functions. For if f is a convex function on some set $M \subset \mathbb{R}^d$ with centroid p, then

$$\begin{aligned}\int_M f(x)\,dx \geq \int_M \Big(a * x + b\Big)\,dx &= \mathrm{Vol}(M)\left(b + a * \left(\frac{1}{\mathrm{Vol}(M)}\int_M x\,dx\right)\right)\\ &= \mathrm{Vol}(M)\Big(b + a * p\Big) = f(p)\mathrm{Vol}(M)\end{aligned}$$

holds, where $a * x + b$ denotes the hyperplane of support of f in p. So if we have a subdivision $D = M_1 \cup \cdots \cup M_k$ with $f(p_i)$ known for the centroid p_i of each set M_i, then we have the lower bound $\int_D f(x)\,dx \geq \sum_i f(p_i)\mathrm{Vol}(M_i)$. This is known as the centroid

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method [1], [3]. Theorem 1 shows that if the function was observed at lattice points, we cannot find a subdivision in triangles which uses all information.

Since the centroid is affine-invariant, the theorem holds for any lattice.

Theorem 1. *For any triangulation, in which the centroid of each triangle is a point of a lattice* Γ*, at most* $\frac{2}{3}$ *of the points of* Γ *occur as centroids.*

Figure 1.

This constant cannot be improved, as the triangulation in Figure 1 shows. To prove the theorem, we need the following lemma.

Lemma 1. *If* T *is a triangulation in which the centroid of each triangle is a point of the lattice* Γ*, and* $\overline{ab}$ *is an edge of* T*, then each triangle of* T *has one vertex in each of the three sets* $a+3\Gamma$*,* $b+3\Gamma$ *and* $-a-b+3\Gamma$*.*

Proof. Let c denote the third vertex of a triangle $\Delta \in T$ with edge $\overline{ab}$. By the definition of the centroid of a triangle we have $\frac{1}{3}(a+b+c) \in \Gamma$, so $a+b+c \in 3\Gamma$ and $c \in -a-b+3\Gamma$. So the lemma holds for the triangle Δ. For any other triangle Δ^* there is a sequence of triangles $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_r = \Delta^*$ in T such that each triangle shares a common edge with the preceding triangle. We now use induction on the length of this sequence and assume that the lemma holds already for Δ_{r-1}. Since $\Delta_r = \Delta^*$ shares two vertices with Δ_{r-1}, already two vertices of Δ^* belong to two of the three sets $a+3\Gamma$, $b+3\Gamma$ and $-a-b+3\Gamma$. By the assumption, the centroid of Δ^* is a point of Γ, so the sum of the three vertices is in 3Γ, and the third vertex of Δ^* must belong to the third set. $\square$

Proof of Theorem 1. By Lemma 1, the vertices of triangles of the triangulation T are contained in the union of three sets, each of which has density $\frac{1}{9}\Big(\det(\Gamma)\Big)^{-1}$ in the plane [2]. So in a big square of area A there are at most $\frac{1}{3}A\Big(\det(\Gamma)\Big)^{-1} + o(A)$ possible vertices of triangles. Each of these vertices is surrounded by triangles whose sum of interior angles in this vertex is 2π, so the total sum of interior angles of triangles in this square is at most $\frac{2}{3}\pi A\Big(\det(\Gamma)\Big)^{-1} + o(A)$. Each triangle contributes a sum of interior angles of π, so there are at most $\frac{2}{3}A\Big(\det(\Gamma)\Big)^{-1} + o(A)$ triangles in this square. As the density of Γ is $\Big(\det(\Gamma)\Big)^{-1}$, at most $\frac{2}{3}$ of the points of Γ can occur as centroid of a triangle. $\square$

3. Circumcenter

Theorem 2. *For any triangulation, in which the circumcenters of the triangles always lie in the interior of the triangle and belong to the integer lattice, at most $\frac{1}{3}$ of the lattice points occur as circumcenter.*

The constant is again best possible, as the triangulation in Figure 2a shows. If we drop the condition that the circumcenter lies in the interior of the triangle, and allow the triangulation to be not edge-to-edge, we can have each lattice point as circumcenter (Figure 2b).

Figure 2a.

Figure 2b.

Proof. We define a dual graph of the triangulation on the set of lattice points occuring as $S(\Delta)$ for some Δ by $\{S(\Delta_1), S(\Delta_2)\} \in E$ iff Δ_1 and Δ_2 have a common edge. This dual graph can be defined for any triangulation and any function S, and is always a 3-regular plane graph. Furthermore each face of this plane graph corresponds to a vertex point of the triangulation (Figure 3).

In the triangulations considered in Theorem 2, the circumcenter of each triangle is assumed to be in the interior of the triangle. This happens iff the interior angles of each triangle are all smaller than $\frac{\pi}{2}$. Since the edges of the dual graph are parts of the mid-perpendiculars of the edges of the triangulation, it follows by elementary geometry that the interior angles of each face of the dual graph are greater than $\frac{\pi}{2}$ and smaller than π. So the faces of the dual graph are convex lattice polygons.

We call a vertex of a face of the dual graph extremal, if it has maximal or minimal x_1- or x_2-coordinate among the vertices of that face. If there are two vertices with the same maximal or minimal coordinate, they are connected by an edge of the face, and we select

Figure 3

only the first of them in clockwise direction as extremal vertex. Since the interior angle in each vertex is greater than $\frac{\pi}{2}$, no vertex is extremal for both coordinates. So each face has exactly four extremal vertices. We call a pair of extremal vertices antipodal, if they are maximal and minimal with respect to the same coordinate.

For the triangulations considered in Theorem 2 we have the following lemma, which is proved below.

Lemma 2. *Each vertex of the dual graph is extremal for exactly two of its three incident faces.*

We take a big square Q which contains n vertices of the dual graph and m lattice points which are not vertices of the dual graph, i. e. which do not occur as circumcenters.

Let f_i denote the number of faces among those with nonempty intersection with Q that have i of their extremal vertices contained in Q $(i = 1, \dots, 4)$. By Lemma 2 we have $\sum_{i=1}^{4} i\, f_i \;= 2n$. To get a lower bound on m we apply the following lemma.

Lemma 3. *If X, Y is an antipodal pair of extremal vertices of a face F, then there are four lattice points X_1, X_2, Y_1, Y_2 in the interior of that face with distance at most $\sqrt{2}$ to X and Y, respectively.*

Let $\tilde{Q}$ denote the set of points with distance at most $\sqrt{2}$ to Q. If Q contains at least three or at least one of the extremal vertices of the face F, then at least four or at least two interior lattice points of F lie in $\tilde{Q}$, respectively. Therefore at least $4f_4 + 4f_3 + 2f_2 + 2f_1$ interior points of faces of the dual graph are contained in $\tilde{Q}$. There are only $O(\sqrt{m+n})$ lattice points in $\tilde{Q} \setminus Q$, so we get

$$\begin{aligned} m &\geq 4f_4 + 4f_3 + 2f_2 + 2f_1 - O(\sqrt{m+n}) \geq 4f_4 + 3f_3 + 2f_2 + f_1 - O(\sqrt{m+n}) \\ &\geq 2n - O(\sqrt{m+n}) \end{aligned}$$

and the theorem follows. □

Proof of Lemma 2. Let V denote a vertex of the dual graph, incident to the faces F_1, F_2 and F_3. Since there are four quadrants around V, but only three edges with endpoint V separating these faces, one of the quadrants must be contained in a face. Without loss of generality we can assume that F_3 contains the lower right quadrant (Figure 4a). Since the interior angles of F_1 and F_2 in V are both greater than $\frac{\pi}{2}$, the edge that separates F_1 and F_2 must lie in the upper left quadrant (Figure 4b), so V is extremal for F_1 and F_2, but not F_3. □

Figure 4a. Figure 4b.

Proof of Lemma 3. Without loss of generality we can assume that $X = (r, s)$ is the leftmost vertex of F. We will show that the two lattice points $(r+1, s)$ and $(r+1, s+1)$ or $(r+1, s-1)$ lie in the interior of F.

For if there is a second leftmost vertex $(r, \hat{s})$ of F, then $\overline{(r,s)\,(r,\hat{s})}$ is an edge of F. Since the interior angles in these two vertices are greater than $\frac{\pi}{2}$, the preceding and following edges of F cut the line $(r+1, \cdot\,)$ above and below $(r+1, \hat{s})$ and $(r+1, s)$, respectively (Figure 5a). By convexity, the lattice points $(r+1, s), \ldots, (r+1, \hat{s})$ are in the interior of F.

Figure 5a. Figure 5b. Figure 5c.

If (r, s) is the unique leftmost vertex, then the preceding and following edges of F lie on the right of the line $(r, \cdot\,)$. Since the interior angle in this vertex is greater than $\frac{\pi}{2}$, one of the edges must lie above and one below the line $(\cdot\,, s)$. So the point $(r+1, s)$ lies in the interior of F (Figure 5b). Furthermore, the points $(r+1, s+1)$ and $(r+1, s-1)$ enclose an angle of $\frac{\pi}{2}$ at (r, s), so one of them must also lie in the interior of F.

Repeating the argument for the rightmost vertex of F, we find four interior lattice points of F, two of which have distance at most $\sqrt{2}$ to X and Y, respectively.

If these four points are not distinct, the rightmost vertex must lie on the line $(r+2, \cdot\,)$. Let $(r+1, b)$ denote the lowest vertex of F, and (r, a), $(r+2, c)$ the preceding and following vertices on the boundary of the face. Since the interior angles in these vertices are greater than $\frac{\pi}{2}$, we have $a, c > b$. But if $a, c \geq b + 1$, an angle of at most $\frac{\pi}{2}$ in $(r + 1, b)$ results (Figure 5c). So the rightmost vertex is at least three units right of the leftmost vertex, and the four interior lattice points are distinct. □

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