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A New Algebraic Criterion for Shellability

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Abstract. In this paper, various types of filtrations of modules are explored, with emphasis on what are called "clean" filtrations. If M is a module over a commutative ring R, a filtration by submodules $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ is said to be "clean" if $M_i/M_{i-1} \cong R/P_i$ with $P_i \in \min(\mathrm{Spec}(R/\mathrm{Ann}(M)))$ for $i = 1, \ldots, n$. A module is said to be "clean" if it has a clean filtration. It will be shown that a simplicial complex is shellable if and only if its face ring is clean (as a module over itself).

Key words and phrases: Prime decomposition, "clean" filtrations of modules, prime ideals associated with a module, simplicial complexes, shelling, face ring, Cohen-Macaulay property.

§1 Introduction

Ever since the relevance of shellability was discovered in the context of convex polytopes (cf. [BM], [M1], [M2]) this intriguing concept has been the subject of a considerable number of geometric and algebraic investigations (cf. for instance [St] and the references in there). In particular, the discovery that the face ring of a pure simplicial complex is a Cohen-Macaulay ring whenever the complex is shellable (cf. [H1], [H2], [R], and [KK]) proved to be a crucial insight in this field.

In this note I want to introduce an apparently new algebraic concept, called *cleanness*, which — as will be shown — can actually be used to characterize shellable complexes via their face ring. The concept of cleanness was first considered in [DDM] where it was used to study the dimension of various multivariate spline spaces. It is, though of course related to the Cohen-Macaulay property, a much simpler concept; so it seems rather surprising that its simple relationship with shellability appears to have been overlooked so far.

§2 Prime decomposition of modules

To motivate the definition of cleanness let us first summarize what is known about *prime decompositions* of modules over commutative rings. To this end recall the following standard notations:

For a subset T of a commutative ring R and subsets X, Y of a (left) R-module M we define the product $T \cdot X \subseteq M$, the sum $X + Y \subseteq M$, and the quotient $(X : Y) \subseteq R$ as usual by

$$T \cdot X := \{t \cdot x \mid t \in T, x \in X\},$$
$$X + Y := \{x + y \mid x \in X, y \in Y\},$$

and

$$(Y : X) := \{r \in R \mid r \cdot x \in Y \text{ for all } x \in X\}.$$

Note, that $(Y : X)$ is an ideal in R whenever Y is an R-submodule of M. Moreover, for $X, Y, Z \subseteq M$ one has

$$(Z : Y) \cdot (Y : X) \subseteq (Z : X) \tag{1}$$

and for $r \in R$ one has

$$(Y : r \cdot X) = ((Y : X) : r) \tag{2}$$

(where I have written r instead of $\{r\}$, — a convention I will use from now on without any further mentioning).

In particular, an ideal $I \subsetneqq R$ is a prime ideal in R if and only if $(I : r) \in \{I, R\}$ for all $r \in R$ or, equivalently, if and only if $(I : r) = I$ for all $r \in R - I$, hence $(Y : X)$ is a prime ideal for some $X, Y \subseteq M$ if and only if $R \cdot X \not\subseteq Y$ and $(Y : rX) = (Y : X)$ for all $r \in R - (Y : X)$ or, again equivalently, if and only if $R \cdot X \not\subseteq Y$ and $(Y : rX) \in \{(Y : X), R\}$ for all $r \in R$.

Next we define a *filtration* F of an R-module M to consist of a finite set of R-submodules of M which is linearly ordered by inclusion and contains M and $\{0\}$. In other words, a finite collection F of R-submodules of M is a filtration of M if and only if the modules in F can be labelled (uniquely!) as $M_0 = M_0(F) = \{0\}, M_1 = M_1(F), \ldots, M_i = M_i(F), \ldots, M_k = M_k(F) = M$ (with $k + 1 = \#F$) so that

$$M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_i \subsetneqq M_{i+1} \subsetneqq \ldots \subsetneqq M_k = M,$$

in which case we say that F is a filtration of length $k =: l(F)$ or, just, a k-filtration.

Such a filtration $F = \{M_0 = \{0\}, M_1, \ldots, M_k = M\}$ of M will be called a *weak prime decomposition* of M if for every $i \in \{1, \ldots, k\}$ there exists a prime ideal $P = P_i = P_i(F) \in \operatorname{Spec}(R)$ with $P = (M_{i-1} : x)$ for all $x \in M_i - M_{i-1}$ and we define F to be a *prime decomposition* or, more precisely, a *proper prime decomposition* of M if in addition the factor modules $F_i := M_i/M_{i-1}$ are generated by one element, that is, if $F_i \cong R/P_i$ for all $i = 1, \ldots, k$.

Obviously, any filtration F of M induces a filtration

$$F' = F \cap M' := \{N \cap M' \mid N \in F\}$$

on every R-submodule M' of M, which is a weak prime decomposition of M' whenever F was a weak prime decomposition of M; similarly, if M', M'' are two submodules of M with $M' \cap M'' = \{0\}$ and if

$$F' = \{M'/M', M_1'/M', \dots, M/M'\}$$

and

$$F'' = \{M''/M'', M_1''/M'', \dots, M/M''\}$$

are weak prime decompositions of M/M' and M/M'', respectively, then the collection

$$F = F' \circ F'' := \{\{0\} = M' \cap M'', M_1' \cap M'', \dots, M \cap M'' = M'', M_1'', M_2'', \dots, M\}$$

of submodules of M constitutes a weak prime decomposition of M.

In particular, a ring R without nilpotent elements, considered as an R-module, admits a weak prime decomposition provided R contains finitely many minimal prime ideals, only: indeed, if we denote the minimal prime ideals in R by $P_1, P_2, \dots, P_k$, say, with $P_i \neq P_j$ for $1 \leq i < j \leq k$, then

$$F = F(P_1, \dots, P_k) := \{\{0\} = \bigcap_{j=1}^{k} P_j, \bigcap_{j=2}^{k} P_j, \dots, \bigcap_{j=i}^{k} P_j, \dots, P_k, R\}$$

is a weak prime decomposition of R of length k with $P_i = P_i(F)$ for $i = 1, \dots, k$.

The fact that one cannot replace "weak" by "proper" in the above assertions is, of course, one of the reasons to consider weak prime decompositions simultaneously with proper prime decompositions.

Concerning weak and proper prime decompositions of modules, one is interested not only in existence and constructions but also, of course, in the finite set $\mathcal{P}(F) = \{P_i(F) \mid i = 1, \dots, l(F)\}$ of prime ideals "occuring" in a weak or a proper prime decomposition F and in the multiplicity $l_P(F) := \#\{i \in \{1, \dots, l(F)\} \mid P_i(F) = P\}$ with which a given prime ideal $P \in \mathcal{P}(F)$ occurs in F.

Concerning the existence of prime decompositions it is obvious that any module M which admits a proper prime decomposition must be finitely generated, and it is well known (cf. [B], Chapter IV) that — vice versa — for every noetherian R-module there exists a proper prime decomposition (and so, a fortiori, there exists a weak prime decomposition, too, — see (PD7) below for more details concerning the construction of particularly nice weak prime decompositions for noetherian R-modules).

In addition, the following facts are well known and easily established:

(**PD1**) *If $F = \{M_0 = \{0\}, M_1, \dots, M_k = M\}$ is a weak prime decomposition of length k of a given R-module M, and if $P_i = P_i(F)$ for $i = 1, \dots, k$, then*

$$\begin{aligned} \prod_{P \in \mathcal{P}(F)} P^{l_P(F)} &= P_k \cdot P_{k-1} \cdot \ldots \cdot P_1 = \\ (3) \qquad & (0 : M_1) \cdot (M_1 : M_2) \cdot \ldots \cdot (M_{k-1} : M) \subseteq \\ & (0 : M) \subseteq \bigcap_{i=1}^{k} (M_{i-1} : M_i) = \bigcap_{i=1}^{k} P_i. \end{aligned}$$

(**PD2**) *The set* $\mathcal{P}_{\min}(F) := \{P \in \mathcal{P}(F) \mid P' \in \mathcal{P}(F)$ *and* $P' \subseteq P$ *implies* $P' = P\}$ *of minimal prime ideals occuring in a weak prime decompsition* F *of an* R*-module* M *coincides with the set* $\mathcal{P}_{\min}(M)$ *of all minimal prime ideals containing the annulator* $\mathrm{Ann}_R(M) := (0 : M)$ *of* M*, that is, those prime ideals* P *which contain* $\mathrm{Ann}_R(M)$ *and for which*

$$\{r^k \cdot s \mid k \in \mathbb{N}, s \in R - P\} \cap \mathrm{Ann}_R(M) \neq \emptyset$$

holds for all $r \in P$.

In particular, $\mathcal{P}_{\min}(M)$ must be finite, whenever M admits a weak prime decomposition. So a ring R without nilpotent elements as considered above admits a weak prime decomposition if and only if R contains finitely many minimal prime ideals, only.

This follows easily from (PD1).

(**PD3**) *For every (proper) prime decomposition* F *of an* R*-module* M *and for every* $P \in \mathcal{P}_{\min}(F) = \mathcal{P}_{\min}(M)$ *the multiplicity* $l_P(F)$ *coincides with the length* $l_P(M)$ *of the "localized" module* $M_P := R_P \otimes_R M$*, considered as a module over the local ring* R_P*, while for a weak prime decomposition* F *of* M *we have at least* $1 \leq l_P(F) \leq l_P(M)$ *for all* $P \in \mathcal{P}_{\min}(F) = \mathcal{P}_{\min}(M)$ *and therefore* $l_P(F) = l_P(M) = 1$ *for all* $P \in \mathcal{P}_{\min}(F) = \mathcal{P}_{\min}(M)$ *with* $l_P(M) = 1$.

This follows easily from the exactness of localization.

(**PD4**) *Any prime ideal contained in the set*

$$\mathrm{Ass}(M) := \{P \in \mathrm{Spec}(R) \mid P = (0 : x) \text{ for some } x \in M\}$$

of prime ideals associated with an R*-module* M *occurs in every weak prime decomposition* F *of* M*. In particular,* $\mathrm{Ass}(M)$ *must be finite, if* M *admits a weak prime decomposition.*

Indeed, if F is a weak prime decomposition of M, if $i \in \{1, \ldots, l(F)\}$ is minimal with $P \in \mathrm{Ass}(M_i(F))$, and if $m \in M_i(F)$ satisfies $(0 : m) = P$, then

$$R \cdot m \cap M_{i-1}(F) = \{0\},$$

since any non-zero multiple $r \cdot m$ of m satisfies $(0 : r \cdot m) = P$, too, and therefore

$$P = (0 : m) = (M_{i-1}(F) : m) = P_i \in \mathcal{P}(F).$$

Vice versa, we have

(**PD5**) *If* M *admits a weak prime decomposition* F*, then* $\mathcal{P}_{\min}(M) \subseteq \mathrm{Ass}(M)$*.*

To verify this assertion consider for a weak prime decomposition F of M of length k and a prime ideal $P \in \mathcal{P}_{\min}(M) = \mathcal{P}_{\min}(F)$ the smallest index $i \in \{1, \ldots, k\}$ with $P_i(F) = P$. Choose $x \in M_i(F) - M_{i-1}(F)$, choose $r_1 \in P_1 - P, \ldots, r_{i-1} \in P_{i-1} - P$, and put $y := r_1 \cdot r_2 \cdot \ldots \cdot r_{i-1} \cdot x$. Obviously $y \in M_i(F) - M_{i-1}(F)$ and $r \cdot y = 0$ for all $r \in P$, hence $P \subseteq (0 : y) \subseteq (M_{i-1}(F) : y) = P$, that is, $P = (0 : y) \in \mathrm{Ass}(M)$.

It follows that $\mathcal{P}_{\min}(M) \subseteq \mathrm{Ass}(M)$ is a further necessary condition for the existence of a weak prime decomposition of a given R-module M.

(**PD6**) *If F is a weak prime decomposition of an R-module M, then*

$$I(F) := \bigcap_{P \in \mathcal{P}_{\min}(F)} P^{l_P(F)} \cdot \prod_{P \in \mathcal{P}(F) - \mathcal{P}_{\min}(F)} P^{l_P(F)}$$

is contained in $\mathrm{Ann}_R(M)$*; in particular*

$$\mathrm{Ann}_R(M) = \bigcap_{P \in \mathcal{P}_{\min}(M)} P,$$

if $l_P(M) = 1$ for all $P \in \mathcal{P}_{\min}(M)$ and there exists a weak prime decomposition F of M with $\mathcal{P}(F) = \mathcal{P}_{\min}(M)$ or, more generally, if there exists a weak prime decomposition F of M with $\mathcal{P}(F) = \mathcal{P}_{\min}(F)$ and $l_P(F) = 1$ for all $P \in \mathcal{P}(F)$.

Proof: By induction on $k := l(F)$ we may assume that $P_1 = P_1(F)$ is contained in $\mathcal{P}_{\min}(F)$ and that for $r \in I(F)$ and $m \in M$ we have $r \cdot m \in M_1(F)$. For $P \in \mathcal{P}_{\min}(F) - \{P_1\}$ choose $s_P \in P - P_1$ and put $s := \prod_{P \in \mathcal{P}_{\min}(F) - \{P_1\}} s_P^{l_P(F)}$. Then $s \notin P_1$, but $r \cdot s \in \prod_{P \in \mathcal{P}(F)} P^{l_P(F)} \subseteq \mathrm{Ann}_R(M)$ by (PD1) and therefore

$$s \in (0 : r \cdot m) \in \{(0 : m_1) \mid m_1 \in M_1(F)\} = \{P_1, R\},$$

which implies $(0 : r \cdot m) = R$, that is $r \cdot m = 0$. Hence $I(F) \subseteq \mathrm{Ann}_R(M)$, as claimed. □

The last fact we want to mention appears to be new. It clarifies the rôle of $\mathrm{Ass}(M)$ in this context and offers another good reason to consider weak as well as proper prime decompositions:

(**PD7**) *If F is a weak prime decomposition of an R-submodule M of a given R-module N and if $\mathrm{Ass}(N/M) \neq \emptyset$ (so, in particular, $M \subsetneq N$), then there exists an R-module M' with $M \subsetneq M'$ and a weak prime decomposition F' of M' with $F' \cap M = F$ and $\mathcal{P}(F') \subseteq \mathcal{P}(F) \cup \mathrm{Ass}(M')$.*
In particular, any noetherian R-module M admits a weak prime decomposition F with $\mathcal{P}(F) = \mathrm{Ass}(M)$.

Proof: Assume $P \in \mathrm{Ass}(N/M)$. If $P \in \mathcal{P}(F)$ choose $x \in N - M$ with $P = (M : x)$ and put $M' := R \cdot x + M$ as well as $F' := F \cup \{M'\}$. Otherwise assume $i \in \{0, 1, \ldots, l(F)\}$ to be the smallest index with $P \in \mathrm{Ass}(N/M_i(F))$ and choose $x \in N$ with $(M_i(F) : x) = P$. Obviously, $R \cdot x \cap M = R \cdot x \cap M_i(F) = P \cdot x$, since otherwise

$$P \in \mathrm{Ass}(M/M_i(F)) \subseteq \{P_{i+1}(F), \dots, P_{l(F)}(F)\} \subseteq \mathcal{P}(F).$$

If $i = 0$, put $M' := R \cdot x + M$ and $F' := F \cup \{M'\}$ as above, which is legitimate this time because $P = (0 : x) \in \mathrm{Ass}(M')$. Otherwise put $M' := Rx + M$, again as above, but now put

$$F' := \{M_0(F), M_1(F), \dots, M_{i-1}(F), R \cdot x + M_i(F), \dots, R \cdot x + M = M'\}.$$

One has to show that $I := (M_{i-1}(F) : y)$ coincides with P_i for all elements y in $(R \cdot x + M_i(F)) - M_{i-1}(F)$. This holds by definition if $y \in M_i(F)$. Otherwise we have $(M_i(F) : y) = P$, so in view of the minimality of i we must have

$$I = (M_{i-1}(F) : y) \subsetneqq (M_i(F) : y) = P$$

as well as

$$(I : r) = (M_{i-1}(F) : r \cdot y) = \begin{cases} \subsetneqq P & \text{if } r \in R - P \qquad (4.1) \\ P_i & \text{if } r \in P - I \qquad (4.2) \\ R & \text{if } r \in I \qquad (4.3) \end{cases} .$$

Together with the fact that $P_i \neq P$ we will use these alternatives in each of the next arguments which will finally show that also in this case we have $I = P_i$, too: Indeed, since $I \subseteq (I : r)$ for all $r \in R$, (4.2) implies $I \subseteq P_i$. It implies also $P_i \cdot (P - I) \subseteq I$ and therefore $P_i \cdot P \subseteq I$. Hence $P \subseteq (I : r)$ for $r \in P_i$ and consequently $r \notin R - P$ by (4.1) for all $r \in P_i$, that is $P_i \subseteq P$. We also get $(I : r) \neq P_i$ for $r \in P_i$, — otherwise $P \subseteq (I : r) = P_i \subseteq P$ in contradiction to $P \neq P_i$. So for all $r \in P_i$ only the last alternative (4.3), that is, $(I : r) = R$ can hold, therefore we have necessarily $P_i \subseteq I$ as well as $I \subseteq P_i$ and therefore $P_i = I$, as claimed. □

§3 Clean R-modules

As above, let R denote a commutative ring. Using the notations and concepts introduced and discussed in §2, one can now define a particularly well-behaved class of R-modules:

One defines an R-module M to be *clean* if there exists a proper prime decomposition F of M with $\mathcal{P}(F) = \mathcal{P}_{\min}(M)$ or, equivalently, with $\mathcal{P}(F) = \mathcal{P}_{\min}(F)$ and one defines M to be *almost clean* if there exists at least a weak prime decomposition F of M with $\mathcal{P}(F) = \mathcal{P}_{\min}(M)$. Correspondingly, one defines a filtration F of M to be clean or, more precisely, to be a clean prime decomposition of M, if F is a proper prime decomposition of M, satisfying $\mathcal{P}(F) = \mathcal{P}_{\min}(M)$, and one defines F to be almost clean or, more precisely, to be an almost clean prime decomposition of M if F is a weak prime decomposition of M, satisfying $\mathcal{P}(F) = \mathcal{P}_{\min}(M)$.

Of course, the ring R itself will be called clean or almost clean, if R — considered as an R-module — is clean or almost clean, respectively.

The following facts follow easily from §2:

(C1) *For every almost clean R-module M one has*

$$\mathrm{Ass}(M) = \mathcal{P}_{\min}(M);$$

vice versa, if $\mathrm{Ass}(M) = \mathcal{P}_{\min}(M)$ *and M is noetherian, then M is almost clean.*

(C2) *If F is an almost clean prime decomposition of an R-module M and if M' is an R-submodule of M, then $F' := F \cap M' = \{N \cap M' \mid N \in F\}$ is an almost clean prime decomposition of M'; in particular, every submodule M' of an almost clean R-module M is almost clean, too, and satisfies*

$$\mathcal{P}_{\min}(M') \subseteq \mathcal{P}_{\min}(M).$$

Indeed, $\mathcal{P}_{\min}(M') = \mathcal{P}_{\min}(F') \subseteq \mathcal{P}(F') \subseteq \mathcal{P}(F) = \mathcal{P}_{\min}(F) = \mathcal{P}_{\min}(M)$ implies $\mathcal{P}_{\min}(M') \subseteq \mathcal{P}_{\min}(M)$ as well as $\mathcal{P}_{\min}(F') = \mathcal{P}(F')$.

(C3) *For every ideal I in R the factor ring R/I is (almost) clean if and only if R/I is (almost) clean as an R-module.*

(C4) *Any R-module of finite length is clean.*

(C5) *Every clean R-module is finitely generated.*

(C6) *For every almost clean prime decomposition F of an R-module M one has*

$$\bigcap_{P \in \mathcal{P}_{\min}(M)} P^{l_P(F)} \subseteq \mathrm{Ann}_R(M)$$

and, hence, one has also

$$\bigcap_{P \in \mathcal{P}_{\min}(M)} P^{l_P(M)} \subseteq \mathrm{Ann}_R(M)$$

for every almost clean R-module M (with $P^{l_P(M)} := P^l$ for some large enough l if $l_P(M) = \infty$);
in particular

$$\mathrm{Ann}_R(M) = \bigcap_{P \in \mathcal{P}_{\min}(M)} P$$

for every almost clean R-module M with $l_P(M) = 1$ for all $P \in \mathcal{P}_{\min}(M)$.

(**C7**) *If for an R-module M we define its clean length $l_c(M)$ as the sum*

$$l_c(M) := \sum_{P \in \mathcal{P}_{\min}(M)} l_P(M)$$

then it is easy to see that

- *for any prime decomposition F of M one has*

$$l(F) \geq l_c(M),$$

- *a prime decomposition F of M is clean if and only if $l(F)$ equals $l_c(M)$, and*
- *M is clean if and only if ($l_c(M) < \infty$ and) there exists a prime decomposition of M of length $l_c(M)$.*

To consider a particular example, let R be a commutative ring without nilpotent elements and, hence, with $l_P(R) = 1$ for all $P \in \mathcal{P}_{\min}(R)$: We have seen in §2 that R admits a weak prime decompostion if and only if $\mathcal{P}_{\min}(R)$ is finite, and one can verify easily that in this case the filtrations $F = F(P_1, P_2, \ldots, P_k) = \{\bigcap_{j=i}^{k} P_j \mid i = 1, \ldots, k\} \cup \{R\}$, considered in §2, which are associated to a given labelling $P_1, P_2, \ldots, P_k$ of the prime ideals in $\mathcal{P}_{\min}(R)$ (with $k = \#\mathcal{P}_{\min}(R)$, that is, $P_i \neq P_j$ for $1 \leq i < j \leq k$, of course), are almost clean. Vice versa, it follows from (C6) that any almost clean prime decomposition F of R is necessarily of such a form, that is, for any such F one has $l(F) = k := \#\mathcal{P}_{\min}(R), \mathcal{P}_{\min}(R) = \{P_1(F), \ldots, P_k(F)\}$, and $F = F(P_1(F), P_2(F), \ldots, P_k(F))$. If follows that a ring R without nilpotent elements is clean if and only if $\mathcal{P}_{\min}(R)$ is finite and we can label its elements as, say, $P_1, P_2, \ldots, P_k$ ($k = \#\mathcal{P}_{\min}(R)$) in such a way that for every $i = 1, \ldots, k-1$ there exists some $r_i \in \bigcap_{j=i+1}^{k} P_j$ with

$$\bigcap_{j=i+1}^{k} P_j = R \cdot r_i + \bigcap_{j=i}^{k} P_j.$$

§4 Cleanness and shellability

Now let K denote a (commutative) field, let X denote a finite set, and let $R = K[X]$ denote the polynomial ring over K, generated by X as its set of indeterminates. For any subset $A \subseteq X$ let P_A denote the prime ideal in $R = K[X]$, generated by all indeterminates $x \in X - A$, so that R/P_A is canonically isomorphic to $K[A]$ and let $m_A := \prod_{x \in A} x$ denote the monomial in R which is the product of all $x \in A$, so $m_{A'} \notin P_A$ if and only if $A' \subseteq A$ for all $A, A' \subseteq X$.

For a given nonempty collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X consider the simplicial complex

$$\Delta_{\mathcal{B}} := \{A \subseteq X \mid A \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

Recall that $\mathcal{B}$ (or $\Delta_{\mathcal{B}}$) is defined to be shellable if one can label the maximal subsets in $\mathcal{B}$ as $B_1, B_2, \ldots, B_k$ in such a way that for all $1 \leq j < i \leq k$ there exists some $x \in B_i - B_j$ and some $j' \in \{1, 2, \ldots, i-1\}$ with $B_i - B_{j'} = \{x\}$.

Next, for any such $\mathcal{B} \subseteq \mathcal{P}(X)$ we define the ideal

$$I(\mathcal{B}) := \bigcap_{B \in \mathcal{B}} P_B = \bigcap_{A \in \Delta_{\mathcal{B}}} P_A.$$

Note that $I(\mathcal{B})$ is generated by all monomials of the form m_C with $C - B \neq \emptyset$ for all $B \in \mathcal{B}$. The factor ring

$$K[\Delta_{\mathcal{B}}] := R/I(\mathcal{B})$$

has often been called the face ring of $\Delta_{\mathcal{B}}$ and it is well known that in the particular case $\mathcal{B} \subseteq \binom{X}{d} := \{B \subseteq X \mid \#B = d\}$ this face ring is a Cohen-Macaulay ring whenever $\mathcal{B}$ is shellable while the converse is not necessarily true.

In contrast to this "one-sided" relationship between shellability and the Cohen-Macaulay property it will be shown in this note that — as indicated in the introduction — shellability of $\mathcal{B}$ is, in fact, equivalent to cleanness of the face ring $R/I(\mathcal{B}) = K[\Delta_{\mathcal{B}}]$, even if not all maximal subsets in $\mathcal{B}$ have the same cardinality. In other words, I want to prove the following

Theorem. *For a finite set X and for a given non-empty collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X, the associated face ring $K[\Delta_{\mathcal{B}}]$ is clean if and only if $\mathcal{B}$ is shellable.*

Proof: In view of the last remarks in §3 it is enough to establish the following

Lemma. *Given K, X, and $\mathcal{B} \subseteq \mathcal{P}(X)$ as above as well as another subset A of X with $A \notin \Delta_{\mathcal{B}}$ there exists some $r \in R$ with*

$$I(\mathcal{B}) = R \cdot r + I(\mathcal{B} \cup \{A\})$$

or, equivalently, there exists some $r \in I(\mathcal{B})$ with $I(\mathcal{B}) \subseteq R \cdot r + I(\mathcal{B} \cup \{A\})$, if and only if for every $B \in \mathcal{B}$ there exists some $x \in A - B$ and some $B' \in \mathcal{B}$ with $A - B' = \{x\}$, in which case r can be chosen to be the product of all such $x \in A$ with $\{x\} = A - B$ for some $B \in \mathcal{B}$.

Proof: First of all note that $A \notin \Delta_{\mathcal{B}}$ implies $I(\mathcal{B}) \not\subseteq P_A$, e.g. because $A \notin \Delta_{\mathcal{B}}$ is equivalent with $A - B \neq \emptyset$ for all $B \in \mathcal{B}$, that is, with $m_A \in I(\mathcal{B})$, while $m_A \notin P_A$ holds by definition.

Next note that for $\mathcal{B} \subseteq \mathcal{P}(X)$ and $A \subseteq X$ with $A - B \neq \emptyset$ for all $B \in \mathcal{B}$ a subset C of A is contained in *all* subsets $C' \subseteq A$ with $C' - B \neq \emptyset$ for all $B \in \mathcal{B}$ if and only if C is contained in

$$C_0 := \{x \in A \mid A - B = \{x\} \text{ for some } B \in \mathcal{B}\} :$$

indeed, if $x \in C_0$, say $\{x\} = A - B_0$ for some $B_0 \in \mathcal{B}$, then

$$\{x\} = A - B_0 \supseteq C' - B_0 \neq \emptyset$$

and therefore $x \in C'$ for all $C' \subseteq A$ with $C' - B \neq 0$ for all $B \in \mathcal{B}$, while $y \in A - C_0$ implies $A - B \neq \{y\}$ for all $B \in \mathcal{B}$ and therefore, in view of $A - B \neq \emptyset$ for $B \in \mathcal{B}$, it implies $(A - \{y\}) - B \neq \emptyset$ for all $B \in \mathcal{B}$; hence, with

$$\mathcal{C} = \mathcal{C}(A, \mathcal{B}) := \{C' \subseteq A \mid C' - B \neq \emptyset \text{ for all } B \in \mathcal{B}\}$$

one has

$$C_0 \subseteq \bigcap_{C' \in \mathcal{C}} C'$$

as well as

$$\bigcap_{C' \in \mathcal{C}} C' \subseteq \bigcap_{y \in A - C_0} (A - \{y\}) = C_0,$$

that is, one has

$$\bigcap_{C' \in \mathcal{C}} C' = C_0.$$

Finally note that since both, $I(\mathcal{B})$ and $I(\mathcal{B} \cup \{A\})$, are generated by monomials of the form m_C for some $C \subseteq X$, there exists an $r \in I(\mathcal{B})$ with

$$I(\mathcal{B}) \subseteq R \cdot r + I(\mathcal{B} \cup \{A\})$$

if and only if there exists a monomial m_C, C an appropriate subset of X, with $m_C \in I(\mathcal{B})$ and

$$I(\mathcal{B}) \subseteq R \cdot m_C + I(\mathcal{B} \cup \{A\}),$$

while from the definition of $I(\mathcal{B})$ and of $I(\mathcal{B} \cup \{A\})$ it follows that for a subset C of X one has $m_C \in I(\mathcal{B})$ and

$$I(\mathcal{B}) \subseteq R \cdot m_C + I(\mathcal{B} \cup \{A\})$$

if and only if $C - B \neq \emptyset$ for all $B \in \mathcal{B}$ and $m_{C'} \in R \cdot m_C$ for every subset $C' \subseteq X$ with

$$m_{C'} \in I(\mathcal{B}) - I(\mathcal{B} \cup \{A\}),$$

that is, C is contained in every subset $C' \in \mathcal{C}(A, \mathcal{B})$. Hence, there exists some $r \in R$ with

$$I(\mathcal{B}) = R \cdot r + I(\mathcal{B} \cup \{A\})$$

if and only if

$$C_0 = \{x \in A \mid A - B = \{x\} \text{ for some } B \in \mathcal{B}\}$$

satisfies $C_{0-B} \neq \emptyset$ for all $B \in \mathcal{B}$, that is, if and only if for every $B \in \mathcal{B}$ there exists some $x \in A - B$ with $\{x\} = A - B'$ for some $B' \in \mathcal{B}$, as claimed. $\square$

Acknowledgement: While working on [DDM] I observed that a face ring $K[\Delta_{\mathcal{B}}]$ is clean if and only if one can order the minimal subsets in $\{X - B \mid B \in \mathcal{B}\}$ as $C_1, C_2, \ldots, C_k$ in such a way that for all $1 \leq j < i \leq k$ there exists some $x \in C_j - C_i$ and some $j' \in \{1, 2, \ldots, i-1\}$ with $C_{j'} - C_i = \{x\}$. I am grateful to Dr. R. Simon for pointing out to me that this just means shellability of $\Delta_{\mathcal{B}}$ (or $\mathcal{B}$). His own elaborations of this observation will appear soon in the J. of Algebra under the titel: Combinatorial Properties of "Cleanness".

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