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Tile-Transitive Partial Tilings of the Plane

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Abstract. There are 3 different kinds of periodic tile-transitive partial tilings of the plane; those having none, one or an infinite number of unbounded components. In each case all possible equivariant types can be enumerated. These results are obtained by using a generalization of the method of Delaney-Dress symbols.

1. Introduction

It is well-known that there exist precisely 93 equivariant types of tile-transitive tilings of the euclidean plane. From [11] follows that the number of equivariant types of "(discrete) periodic closed disk patterns" is 52.

Both of these results are special cases of the problem of classifying all possible equivariant types of periodic tile-transitive *partial* tilings of the plane, i.e. of such tilings that do not necessarily completely cover the plane.

In [7] and [9] Andreas Dress introduces the theory of *Delaney symbols* (or -more appropriately- *Delaney-Dress symbols*), which proves to be quite a successful approach to classification problems concerning periodic tilings. For example, using algorithms described in [6] and [15], all tile-k-transitive tilings of the plane, the sphere or the hyperbolic plane can be systematically enumerated. In this paper we discuss how to adapt the theory of Delaney-Dress symbols to *partial* tilings of the plane.

In Section 2 we define periodic tile-transitive partial tilings and discuss some of their properties. In Section 3 we prove that there exist three different families of periodic tile-transitive partial tilings, which we call *patchy*, *stripey* and *net-like*.

We investigate the relationship between such tilings and Delaney-Dress symbols in Section 4. This is followed by a detailed inspection of the *net-like*, *patchy*, and finally, the *stripey* tile-transitive partial tilings, in Sections 5, 6 and 7, respectively.

The main result of this paper is that there exist three different families of periodic tile-transitive partial tilings; those having none, one or an infinite number of unbounded components. Furthermore, there exist precisely 152+52 types of such *patchy*, 248 types of *stripey* and 1028+93 types of *net-like* periodic tile-transitive partial tilings of the plane, respectively.

This paper is is based on my Ph. D. thesis [13]. I would like to thank my Ph. D. advisor, Andreas Dress.

2. Periodic Tile-Transitive Partial Tilings

A system $\mathcal{T} = \{t_1, t_2, \ldots\}$ of subsets $t_i \subseteq \mathbb{E}^2$ of the euclidean plane, called *tiles*, is called a *tiling* (of the plane) if it satisfies:

(T1) Every tile is a topological disk.

(T2) The intersection of any two different tiles is either empty, a point or an arc.

(T3) The tiles are uniformly bounded, i.e. there exist two positive numbers r, R such that each tile contains a disk of radius r and is contained in a disk of radius R.

(T4) The tiling covers the plane, i.e. $\bigcup_{t \in \mathcal{T}} t = \mathbb{E}^2$.

By properties (T2) and (T3), all tilings considered in this paper are so-called *normal* tilings.

For a tiling $\mathcal{T}$, the intersection of a finite number of distinct tiles is either empty, a point called a *vertex* or an arc called an *edge*. The boundary of any tile $t \in \mathcal{T}$ is divided into a number of parts (at least two) by vertices, each such arc being an edge of the tiling.

We called any system $\mathcal{T} = \{t_1, t_2, \ldots\}$ of subsets of the plane satisfying conditions (T1)—(T3), but not necessarily (T4), a *partial* tiling. For any such partial tiling $\mathcal{T}$ let T denote the support of $\mathcal{T}$, i.e. $T := \bigcup_{t \in \mathcal{T}} t$ and let K denote the closure of the complement of T, i.e. $K := \overline{\mathbb{E}^2 \setminus T} = \bigcup_{H \in \mathcal{K}} H$, where $\mathcal{K} := \{\overline{H} \subset \mathbb{E}^2 \mid H \in \pi_0(\mathbb{E}^2 \setminus K)\}$ is the set of all (closures of) connected components of the complement of T. Finally, let $\mathcal{K}_\infty$ denote the set of all *unbounded* elements in $\mathcal{K}$. See Figure 1.

Figure 1a: A net-like partial tiling	Figure 1b: A stripey partial tiling	Figure 1c: A patchy partial tiling

Let $\mathcal{T}$ be a partial tiling. If $t \in \mathcal{T}$ doesn't intersect any other tile in $\mathcal{T}$, then the boundary ∂t of t is said to contain neither vertices nor edges. Otherwise, call any point in ∂t that lies in the intersection of at least three tiles, or at least two tiles and K, a *vertex*. The edges (in ∂t) are defined as the closures of the connected components of the set that one obtains from ∂t by removing all the contained vertices. Following V. Klee, a vertex or an edge is called *exposed* if it lies in K and *protected* otherwise.

The *degree* of a vertex v is the number of edges incident to v (where "loops" are counted twice) and the *degree* of a tile t is the number of edges contained in the boundary of t. Note that an edge always lies in either exactly two tiles or exactly one tile and K. Furthermore, an exposed edge may be a *loop*, i.e. a simple closed curve that is incident to precisely one vertex.

Let $\mathcal{T}$ be a partial tiling and let Γ be a discrete group of isometries of the euclidean plane. If $\mathcal{T}$ is Γ-invariant, i.e. if $\mathcal{T} = \gamma\mathcal{T} := \{\gamma t \mid t \in \mathcal{T}\}$ for all $\gamma \in \Gamma$, then we call the system $(\mathcal{T}, \Gamma)$ an *equivariant partial tiling*. Such a system is called *tile-transitive*, if for any two tiles $t, t' \in \mathcal{T}$ there exists a *symmetry* $\gamma \in \Gamma$ such that $\gamma t = t'$. If Γ is a crystallographic group, i.e. if Γ contains translations in two linearly independent directions, then $(\mathcal{T}, \Gamma)$ is called *periodic*.

Let $(\mathcal{T}, \Gamma)$ be a periodic equivariant partial tiling. We will make use of the following notation:

$$\begin{aligned} \bar{\Gamma} &:= \{\gamma \in \Gamma \mid \gamma \text{ is a translation } \} \leq \Gamma \\ \Gamma_H &:= \{\gamma \in \Gamma \mid \gamma H = H\} \\ \bar{\Gamma}_H &:= \bar{\Gamma} \cap \Gamma_H \end{aligned}$$

for any set $H \in \mathcal{T} \cup \mathcal{K}$.

Two equivariant partial tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are called *equivariantly equivalent*, if there exists a homeomorphism $\phi : \mathbb{E}^2 \to \mathbb{E}^2$ with $\phi\mathcal{T} = \mathcal{T}'$ and $\Gamma' = \phi\Gamma\phi^{-1}$. We intend to enumerate all possible equivariant types of periodic tile-transitive partial tilings of the plane.

From now on we will refer to equivariant partial tilings simply as partial tilings. Remember, however, that an equivariant tiling is not plainly a family of tiles, but rather a system consisting of a set of tiles *and* a prescribed symmetry group.

3. Patchy, Stripey and Net-Like Tilings

In this Section we investigate the basic properties of a periodic tile-transitive partial tiling $(\mathcal{T}, \Gamma)$. Let us first look at the bounded elements of $\mathcal{K}$.

Theorem 3.1 *Let $(\mathcal{T}, \Gamma)$ be a periodic tile-transitive partial tiling. Every bounded set $L \in \mathcal{K}$ is a topological disk.*

Proof. For a compact set $X \subseteq \mathbb{E}^2$, let $D(X)$ denote its *diameter*, i.e. the maximal distance between any two points in X. For a Jordan curve J, let $int(J)$ denote its interior region.

Let $L \in \mathcal{K} \setminus \mathcal{K}_\infty$. It suffices to show that for any point $p \in \partial L$, the set $\partial L \setminus \{p\}$ is connected. Assume that $\partial L \setminus \{p\}$ is disconnected for some $p \in \partial L$. Because L is bounded, there exists a Jordan curve $J \subseteq \partial L$ with $L \subseteq J \cup int(J)$. Let $J' \subseteq \partial L$ be a Jordan curve with $J \cap J' \subseteq \{p\}$ that has maximal diameter. Let $t \in \mathcal{T}$, or $t' \in \mathcal{T}$, be a tile that intersects J, or J', respectively, in at least two points. Note that $t \neq t'$. For any Jordan curve $C \subseteq \partial T$ that intersects t' in at least two points we have

$$D(C) \leq D(J') < D(J)$$

because $C, J' \subseteq int(J) \cup \{p\}$. However, the existence of an isometry $\gamma \in \Gamma$ with $\gamma t = t'$ implies that there exists a Jordan curve γJ in ∂T of diameter $D(\gamma J) = D(J)$ that intersects t' in at least two points, a contradiction. □

Let $(\mathcal{T}, \Gamma)$ be a periodic tile-transitive partial tiling. If $H \subseteq T$ is a connected component of $\mathcal{T}$, then we call $(\mathcal{T}_H, \Gamma_H)$ a component of $(\mathcal{T}, \Gamma)$, with $\mathcal{T}_H := \{t \in \mathcal{T} \mid t \cap H \neq \emptyset\} = \{t \in \mathcal{T} \mid t \subseteq H\}$.

Theorem 3.2 *Let $(\mathcal{T}, \Gamma)$ be a periodic tile-transitive partial tiling. Exactly one of the following cases holds (see Figure 1):*
(1) *The tiling is net-like: It possesses exactly one unbounded connected component. All components of the complement are (bounded) topological disks ("holes").*
(2) *The tiling is stripey: It consists of an infinite number of unbounded components ("stripes"), all invariant under translations in exactly one same direction. The complement of the tiling possesses an infinite number of unbounded components, which are all invariant under precisely the same translations as the components of the tiling.*
(3) *The tiling is patchy: It consists of an infinite number of bounded components ("patches"), all surrounded by the same (and only) unbounded component of the complement.*

To prove this result, let us first show two Lemmas on "crystal-structures". A system (C, Γ) consisting of a connected set $C \subseteq \mathbb{E}^2$ and a group Γ of symmetries of C is called a *connected (2-dimensional) crystal-structure* if Γ is a discrete group of translations in two linearly independent directions, i.e. a crystallographical group of type **p1**. If Γ is a discrete group of translations in one direction, i.e. a frieze group of type **p111**, then (C, Γ) is called a *connected rod-structure* (compare [1]).

Lemma 3.3 *Let (C, Γ) be a connected crystal-structure. Every connected component of $\mathbb{E}^2 \setminus C$ is bounded.*

Proof. Let $\gamma \in \Gamma$ be a non-trivial translation and $p_0 \in C$. Define $p_i := \gamma^i p_0$ and $\alpha_i := \gamma^i \alpha_0$, where $\alpha_0 \subset C$ denotes an arc joining p_0 and p_1. Let D denote the maximal distance between two points in α_0. The curve $\alpha := \bigcup_{i \in \mathbb{Z}} \alpha_i$ lies in an infinite strip of width D and partitions the plane into two unbounded regions. Let $\gamma' \in \Gamma$ be a translation not parallel to γ. The set $A := \bigcup_{j \in \mathbb{Z}} (\gamma')^j \alpha$ partitions the plane into an infinite number of unbounded regions, each contained in an infinite strip of width $d + D$ (where d is the distance between p_0 and $\gamma' p_0$). Note that the strips are all parallel.

The same construction, but starting with γ' and then applying γ, yields a second such set A'. It now follows from $\gamma \not\parallel \gamma'$ that the intersection of any two components of $\mathbb{E}^2 \setminus A$ and $\mathbb{E}^2 \setminus A'$ is bounded, and hence, any component of $\mathbb{E}^2 \setminus C \subset (\mathbb{E}^2 \setminus A) \cap (\mathbb{E}^2 \setminus A')$ is also bounded. □

This result and similar arguments imply:

Lemma 3.4 *The intersection of any two connected crystal-structures, or a connected crystal-structure and a connected rod-structure, is non-empty. The intersection of any two connected rod-structures is either non-empty, or the translational directions of the two are parallel.*

Proof of Theorem 3.2. Let H be a connected component of T. Consider the (induced) stabilizer group Γ_H. There are precisely three possible cases:

(1) The group Γ_H contains translations in two linearly independent directions. In this case the system $(H, \bar{\Gamma}_H)$ is a connected crystal-structure. The Lemmas imply that all connected components of the complement of the tiling are (bounded) topological disks. Furthermore, tile-transitivity implies $H = T$.

(2) The group Γ_H contains translations in precisely one direction. Note that then *all* translations (in Γ) in that direction must lie in Γ_H. Let $\gamma \in \bar{\Gamma}$ be a translation not contained in Γ_H, and therefore, not parallel to any translation in Γ_H.

Then, the system $(\gamma^i H, \gamma^i \bar{\Gamma}_H (\gamma^i)^{-1})$ is a rod-structure for every $i \in \mathbb{Z}$ and one has $\gamma^i H \cap \gamma^j H = \emptyset$ whenever $i \neq j$. If we now assume that γ was chosen as a (non-trivial) translation of minimal displacement, then any two sets $\gamma^i H$ and $\gamma^{i+1} H$ are separated by an unbounded connected set $L \in \mathcal{K}_\infty$.

We claim that $(L, \bar{\Gamma}_L)$ is a rod-structure. To see why this is true, let $\mathcal{E}_L$ denote the set of all edges of $\mathcal{T}$ that lie in ∂L. The tiling $\mathcal{T}$ is uniformly bounded, so $\#\mathcal{E}_L = \infty$. On the other hand, as the boundary of a tile contains only a finite number of edges and Γ operates transitively on $\mathcal{T}$, it follows that the number of transitivity classes of edges is finite. The only elements of Γ that have infinite orbits are translations (and glide-reflections). So there must exist a non-trivial translation $\gamma \in \bar{\Gamma}$ with $\gamma(\mathcal{E}_L) = \mathcal{E}_L$. But then $\gamma L = L$ and so $\bar{\Gamma}_L \neq \emptyset$.

(3) The group Γ_H does not contain translations. In this case every component of $\mathcal{T}$ must be bounded and the complement of $\mathcal{T}$ must contain at least one unbounded component $L \in \mathcal{K}_\infty$. Again, ∂L contains an infinite set of edges partitioned into a finite number of Γ-orbits. In this case Γ_L must contain translations in two linearly independent directions, and this implies $\mathcal{K}_\infty = \{L\}$. □

4. Delaney-Dress Symbols of Partial Tilings

As we will see, the Delaney-Dress symbol of a periodic 2-dimensional tiling is a finite, edge-colored graph, together with two functions defined on the vertices of the graph, that encode the tiling up to equivariant equivalence. The relationship between such tilings and their symbols has been studied in great detail, see the References. In this Section we discuss how one can define the symbol of a periodic tile-transitive *partial* tiling. First we recall the definition of the Delaney-Dress symbol for a 2-dimensional tiling.

Let $(\mathcal{T}, \Gamma)$ be an equivariant tiling that covers the euclidean plane, the sphere or the hyperbolic plane. Let us assume that the tiling is *tile-k-transitive*, i.e. that the tiling possesses precisely $k \in \mathbb{N}$ transitivity classes of tiles with respect to the symmetry group Γ. We define the *chamber system* $\mathcal{C}_\mathcal{T}$ of $\mathcal{T}$ via formal barycentric subdivision in the usual way: For every vertex, edge and tile choose an interior point, called a 0-, 1- or 2-*center*, respectively. Then, for every tile $A \in \mathcal{T}$, join its 2-center to each 0- and 1-center contained in the boundary of the tile, by pairwise non-intersecting edges. These "triangles" are called *chambers* and they make up the set $\mathcal{C}_\mathcal{T}$. Each chamber has exactly three vertices, namely a 0-, a 1- and a 2-center. The edge opposite the i-center of a chamber is called an *i-edge*.

Note that for every chamber $C \in \mathcal{C}_\mathcal{T}$ and any $i \in \{0, 1, 2\}$ there exists exactly one chamber $C' \in \mathcal{C}_\mathcal{T}$ (with $C' \neq C$) such that C and C' have the same i-edge. We say that C and C' are *i-adjacent* or *i-neighbors* and define a permutation $\sigma_i : \mathcal{C}_\mathcal{T} \to \mathcal{C}_\mathcal{T}$ that maps every $C \in \mathcal{C}_\mathcal{T}$ onto its i-neighbor $C' =: C\sigma_i$ (so $C' = C\sigma_i$ implies $C = C'\sigma_i$).

Note that the chamber system $\mathcal{C}_\mathcal{T}$ can always be constructed in a way compatible with the action of Γ on $\mathcal{T}$. We call two chambers $C, C' \in \mathcal{C}_\mathcal{T}$ *equivalent* whenever there exists a symmetry $\gamma \in \Gamma$ with $\gamma \cdot C = C'$. The assumption that $(\mathcal{T}, \Gamma)$ is tile-k-transitive implies that the set of all equivalence classes (or Γ-orbits) in $\mathcal{C}_\mathcal{T}$ is finite. This set is denoted by $\mathcal{D} := \mathcal{D}_{(\mathcal{T},\Gamma)} = \Gamma \setminus \mathcal{C}_\mathcal{T}$.

Consider $\mathcal{C} := \mathcal{C}_\mathcal{T}$. Any symmetry $\gamma \in \Gamma$ necessarily maps i-neighbors onto i-neighbors, hence the operation of σ_i commutes with Γ on $\mathcal{C}$, for any $i \in \{0,1,2\}$. Thus we can introduce the concept of i-adjacency of Γ-orbits. The permutation $\sigma_i : \mathcal{D} \to \mathcal{D}$ maps every $D \in \mathcal{D}$ onto its *i-neighbor* $D' =: D\sigma_i$, with $i \in \{0,1,2\}$.

The set $\mathcal{D}$ and the maps $\sigma_0, \sigma_1, \sigma_2$ define a finite, connected 3-colored graph with node set $\mathcal{D}$ and set of colored edges $\mathcal{E} := \big\{(\{D, D'\}, i) \mid D, D' \in \mathcal{D} \text{ and } D\sigma_i = D'\big\}$, where the component $i \in \{0,1,2\}$ of an edge $(\{D, D'\}, i)$ is called its *color*. (Note that $D = D\sigma_i$ is possible; the arising edge is a *loop*. Note also that D and D' can be connected by more than one edge.) This *Delaney-Dress graph* $(\mathcal{D}, \mathcal{E})$ makes up the first part of the Delaney-Dress symbol corresponding to $(\mathcal{T}, \Gamma)$. To complete the definition, let us introduce functions $m_{ij} : \mathcal{D} \to \mathbb{N}$ for $0 \le i < j \le 2$ with

$$m_{ij}(D) := \min\{m \in \mathbb{N} \mid C(\sigma_i\sigma_j)^m = C \text{ for any } C \in D\}.$$

It is well worth mentioning that these functions have the following geometric interpretations: The value $m_{01}(D)$ equals the number of edges of any tile that contains a chamber $C \in D \in \mathcal{D}$. Similarly, the value $m_{12}(D)$ equals the number of edges incident to any vertex that is adjacent to a chamber $C \in D \in \mathcal{D}$. Finally, $m_{02}(D) = 2$, because this value equals the number of faces incident to any edge adjacent to $C \in D \in \mathcal{D}$.

We now come to the formal definition of the symbol of a periodic 2-dimensional tiling.

A system $(\mathcal{D}; m) := \big((\mathcal{D}, \mathcal{E}); m_{01}, m_{12}, m_{02}\big)$ consisting of a finite, connected Delaney-Dress graph $\mathcal{D} := (\mathcal{D}, \mathcal{E})$ and functions $m_{01}, m_{12}, m_{02} : \mathcal{D} \to \mathbb{N}$, is called a (2-dimensional) *Delaney-Dress symbol* if and only if for all $D \in \mathcal{D}$ and $0 \le i < j \le 2$ the following hold:
(1) $m_{ij}(D) = m_{ij}(D\sigma_i) = m_{ij}(D\sigma_j)$,
(2) $D(\sigma_i\sigma_j)^{m_{ij}(D)} = D(\sigma_j\sigma_i)^{m_{ij}(D)} = D$,
(3) $m_{02}(D) = 2$ and $m_{12}(D) \ge 3$.

Two Delaney-Dress symbols $(\mathcal{D}; m)$ and $(\mathcal{D}'; m')$ are called *isomorphic* if and only if there exists a bijection $\pi : \mathcal{D} \to \mathcal{D}'$ with $(\pi D)\sigma_k = \pi(D\sigma_k)$ and $m'_{ij}(\pi D) = m_{ij}(D)$ for all $D \in \mathcal{D}$, $0 \le k \le 2$ and $0 \le i < j \le 2$. It is not too difficult to prove the following ([8]):

Lemma 4.1 *Two tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are equivariantly equivalent if and only if the corresponding Delaney-Dress symbols $(\mathcal{D}; m)$ and $(\mathcal{D}'; m')$ are isomorphic.*

Proof. It is obvious that isomorphic symbols give rise to equivalent tilings. Now assume that $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are two periodic tile-transitive partial tilings with Delaney-Dress symbols $(\mathcal{D}; m)$ and $(\mathcal{D}'; m')$, equivariantly equivalent via an homeomorphism $\phi : \mathcal{T} \to \mathcal{T}'$. It follows that the corresponding chamber systems $\mathcal{C}$ and $\mathcal{C}'$, being barycentric subdivisions, are topologically equivalent. Furthermore, ϕ induces a one-to-one correspondence between Γ-orbits in $\mathcal{C}$ and Γ'-orbits in $\mathcal{C}'$, which commutes with σ_i ($i \in \{0,1,2\}$), because $\Gamma' = \phi\Gamma\phi^{-1}$. Hence both arising symbols are isomorphic. □

Consider a Delaney-Dress graph $(\mathcal{D}, \mathcal{E})$. If we delete all edges of color k in the graph, then we obtain a two-colored graph $\mathcal{D}_{ij}$, which we call the *$\{i,j\}$-subgraph* of $\mathcal{D}$, with $\{i,j,k\} = \{0,1,2\}$. The connected components of $\mathcal{D}_{ij}$ are called the *$\{i,j\}$-orbits* of $\mathcal{D}$. An

$\{i, j\}$-orbit $\mathcal{O}$ is called a *cycle*, if $D\sigma_i \neq D$ and $D\sigma_j \neq D$ for all $D \in \mathcal{O}$, and a *chain*, otherwise. Any $\{i, j\}$-orbit $\mathcal{O}$ with $0 \leq i < j \leq 2$ is also called a *2-color-orbit*. It is a nice fact that the different equivalence classes of vertices, edges and tiles of a periodic tiling correspond directly to the different $\{1,2\}$-, $\{0,2\}$- and $\{0,1\}$-orbits of the associated Delaney-Dress symbol, respectively.

How does one adjust the above definition of a Delaney-Dress symbol to periodic tile-transitive *partial* tilings? As we have seen, periodicity and tile-transitivity imply that such a tiling is made up of (one or infinitely many) equivalent components. We intend to define the symbol for any such *component*. The important question is how to treat the components of the complement of such a periodic tile-transitive partial tiling?

Let $(\mathcal{T}, \Gamma)$ be a periodic tile-transitive partial tiling and let $\mathcal{T}_H \subseteq \mathcal{T}$ be the set of all tiles that make up a given connected component H of T. Let $\mathcal{K}_H = \{L \in \mathcal{K} \setminus \mathcal{K}_\infty \mid L \cap H \neq \emptyset\}$ be the set of all bounded connected components of the complement of $\mathcal{T}$ that are adjacent to tiles in $\mathcal{T}_H$.

Note that if $L \in \mathcal{K}_H$, then $\partial L \cap \partial H \neq \emptyset$ and therefore $\partial L \subseteq \partial H$, since ∂L is connected. Hence, all elements of $\mathcal{K}_H$ can be considered as additional tiles, which we call *holes*. To define the chamber system $\mathcal{C} := \mathcal{C}_{(\mathcal{T}_H, \Gamma_H)}$ of the component $(\mathcal{T}_H, \Gamma_H)$ of a periodic tile-transitive partial tiling we begin as discussed above, by first subdividing all tiles and all holes. Let $\mathcal{C}_{tiles}$ and $\mathcal{C}_{holes}$ denote the sets of the two kind of chambers that arise. We can define the 0-, 1-, and 2-neighbor relationships for all chambers in $\mathcal{C}_{tiles} \cup \mathcal{C}_{holes}$ as above, expect for those chambers that are adjacent to some unbounded set $L \in \mathcal{K}_\infty$. The latter *unprotected* chambers all lie in $\mathcal{C}_{tiles}$ and for these we must now discuss how to define σ_2.

With each unprotected chamber $C \in \mathcal{C}_{tiles}$ associate a “formal” chamber C' and set $C\sigma_2 := C'$. Let $\mathcal{C}_\infty$ be the set of all such formal chambers. The chamber system of $(\mathcal{T}_H, \Gamma_H)$ is the set $\mathcal{C} = \mathcal{C}_{tiles} \dot\cup \mathcal{C}_{holes} \dot\cup \mathcal{C}_\infty$. Now, for every formal chamber $C' \in \mathcal{C}_\infty$, we set $C'\sigma_0 := C'\sigma_2\sigma_0\sigma_2 \in \mathcal{C}_\infty$ and $C'\sigma_1 := C\sigma_2 \in \mathcal{C}_\infty$, where $C \in \mathcal{C}_{tiles}$ is the first chamber that one encounters when going around a small circle around the 0-vertex of $C'\sigma_2 \in \mathcal{C}_{tiles}$ in the orientation defined by leaving the chamber $C'\sigma_2$ across its 2-edge.

Now that we have defined the chamber system $\mathcal{C}$ and the i-neighbor relationships we obtain a finite Delaney-Dress graph $(\mathcal{D}, \mathcal{E})$ by considering Γ_H-orbits on $\mathcal{C}$. The partitioning of $\mathcal{C}$ carries over to $\mathcal{D}$ as follows:

$$\mathcal{D} = \mathcal{D}_{tiles} \dot\cup \mathcal{D}_{holes} \dot\cup \mathcal{D}_\infty.$$

The functions m_{ij} are defined as above, except that there are two special cases: If the tiling is stripey, then every connected component of the boundary of any set $L \in \mathcal{K}_\infty$ is unbounded. In this case one must set $m_{01}(D) = \infty$ for all $D \in \mathcal{D}_\infty$. If the tiling is patchy, then every connected component of the boundary of the one set $L \in \mathcal{K}_\infty$ is bounded, and we must define $m_{01}(D) = N$ for all $D \in \mathcal{D}_\infty$, where N denotes the number of edges of the tiling that lie in the considered component of the boundary of L.

We now state the formal definition of a “PTTP Delaney-Dress symbol”, where PTTP stands for “periodic tile-transitive partial”. A system $(\mathcal{D}; m) := \big((\mathcal{D}, \mathcal{E}); m_{01}, m_{12}, m_{02}\big)$ consisting of a finite, connected Delaney-Dress graph $\mathcal{D} := (\mathcal{D}, \mathcal{E})$, with a partitioning $D = \mathcal{D}_{tiles} \dot\cup \mathcal{D}_{holes} \dot\cup \mathcal{D}_\infty$, and functions $m_{01}, m_{12}, m_{02} : \mathcal{D} \to \mathbb{N} \cup \{\infty\}$, is called a *PTTP Delaney-Dress symbol* if for all $D \in \mathcal{D}$ and $0 \leq i < j \leq 2$ the following hold:

(DS1) $m_{ij}(D) = m_{ij}(D\sigma_i) = m_{ij}(D\sigma_j)$,
(DS2) $D(\sigma_i\sigma_j)^{m_{ij}(D)} = D(\sigma_j\sigma_i)^{m_{ij}(D)} = D \quad$ if $m_{ij}(D) < \infty$,
(DS3) $m_{02}(D) = 2 \quad$ and $\quad m_{12}(D) \geq 3$,
(DS4) $(\mathcal{D}_{holes} \cup \mathcal{D}_\infty)\sigma_2 \subseteq \mathcal{D}_{tiles}$,
(DS5) $\mathcal{D}_{tiles}\sigma_k = \mathcal{D}_{tiles}$, $\mathcal{D}_{holes}\sigma_k = \mathcal{D}_{holes}$ and $\mathcal{D}_\infty\sigma_k = \mathcal{D}_\infty$ for $k \in \{0,1\}$,
(DS6) $\mathcal{D}_{tiles}$ consists of precisely one $\{0,1\}$-orbit.

Note that $m_{ij}(D) < \infty$ if $D \notin \mathcal{D}_\infty$ or $(i,j) \neq (0,1)$. For an example, see Figure 2.

Figure 2: The Delaney-Dress graph of a component of a partial tiling $(\mathcal{T}_H, \Gamma_H)$. The component consists of three tiles, which are shaded grey. The stabilizer group Γ_H is a rotation group of type C3. The Delaney-Dress graph has 12 vertices, drawn here as circles labeled $A,B,\ldots,L$. Note that $\mathcal{D}_{tiles}=\{A,\ldots,H\}$, $\mathcal{D}_{holes}=\{I,J\}$ and $\mathcal{D}_\infty=\{K,L\}$. The edges of the graph are depicted as heavy lines, each labeled 0, 1, or 2, indicating the edge's color. In this example we have $m_{01}(D)=4$ for all $D\in\mathcal{D}_{tiles}$ and $m_{01}(D)=3$ for all $D\in\mathcal{D}_{holes}\cup\mathcal{D}_\infty$. Furthermore, $m_{12}(D)=3$ for all $D\in\mathcal{D}$.

Two PTTP Delaney-Dress symbols $(\mathcal{D};m)$ and $(\mathcal{D}';m')$ are called *isomorphic* if and only if there exists a bijection $\pi : \mathcal{D} \to \mathcal{D}'$ with

$$\pi\mathcal{D}_{tiles} = \mathcal{D}_{tiles}, \quad \pi\mathcal{D}_{holes} = \mathcal{D}_{holes}, \quad \pi\mathcal{D}_\infty = \mathcal{D}_\infty,$$

$$(\pi D)\sigma_k = \pi(D\sigma_k) \quad \text{and} \quad m'_{ij}(\pi D) = m_{ij}(D)$$

for all $D \in \mathcal{D}$, $0 \leq k \leq 2$ and $0 \leq i < j \leq 2$.

Similar to Lemma 4.1, we have:

Lemma 4.2 *Two components $(\mathcal{T}_H, \Gamma_H)$ and $(\mathcal{T}'_{H'}, \Gamma'_{H'})$ of periodic tile-transitive regular tilings are equivariantly equivalent if and only if the two associated PTTP Delaney-Dress Symbols $(\mathcal{D}; m)$ and $(\mathcal{D}'; m')$ are isomorphic.*

Finally, we need to introduce a number of different basic concepts associated with Delaney-Dress symbols in general (not just PTTP symbols). Let $(\mathcal{D}; m)$ be a (PTTP or not) Delaney-Dress symbol. Define

$$r_{ij}(D) := \min\{n \in \mathbb{N} \mid D(\sigma_i\sigma_j)^n = D\}$$

for all $D \in \mathcal{D}$ and $0 \leq i < 2 \leq 2$. Note that (DS2) implies $r_{ij}(D) \mid m_{ij}(D)$ and it is also clear that r_{ij} is constant on $\{i,j\}$-orbits. We call the map $v_{ij}(D) := \frac{m_{ij}(D)}{r_{ij}(D)}$ the *branching number map*. It turns out that the *branching number* $v_{ij}(D \in \mathcal{O})$ of an $\{i,j\}$-orbit $\mathcal{O}$ equals the rotational order of the stabilizer group of the corresponding vertex, edge or tile of the associated tiling (depending on whether $\{i,j\} = \{1,2\}$, $\{0,2\}$ or $\{0,1\}$, respectively). An $\{i,j\}$-orbit $\mathcal{O}$ is said to have a *trivial* branching number, if $v_{ij}(D \in \mathcal{O}) = 1$.

For any Delaney-Dress symbol $(\mathcal{D}; m)$ we call

$$\mathbf{K}_{(\mathcal{D};m)} := \sum_{D \in \mathcal{D}} \left(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} + \frac{1}{m_{02}(D)} - 1\right)$$

the *curvature* of the symbol and

$$\chi(\mathcal{D}) := \sum_{D \in \mathcal{D}} \left(\frac{1}{r_{01}(D)} + \frac{1}{r_{12}(D)} + \frac{1}{r_{02}(D)} - 1\right)$$

is called the *Euler characteristic* of the symbol. A rather long but straight forward calculation shows that this number indeed coincides with the usual Euler characteristic of the so-called topological realization of the given Delaney-Dress graph.

A Delaney-Dress symbol $(\mathcal{D}; m)$ is called *orientable*, if one can define a map $\phi : \mathcal{D} \to \{-1, +1\}$ such that $\phi(D\sigma_i) = -\phi(D)$ for all $D \in \mathcal{D}$ and $i \in \{0,1,2\}$. Now assume $(\mathcal{D}; m)$ encodes a tiling $(\mathcal{T}, \Gamma)$. It can be shown that $(\mathcal{D}; m)$ is orientable if and only if Γ consists of orientation-preserving symmetries only. If Γ contains orientation-reversing symmetries, then the following construction yields the Delaney-Dress symbol of the equivariant tiling $(\mathcal{T}, \Gamma')$ that one obtains from $(\mathcal{T}, \Gamma)$ by removing the orientation reversing symmetries from Γ: Set $\mathcal{D}' := D \times \{-1, +1\}$ and define $(D, \epsilon)\sigma_i := (D\sigma_i, -\epsilon)$ for all $D \in \mathcal{D}$, $\epsilon \in \{-1, +1\}$ and $i \in \{0,1,2\}$. Set $m'_{ij}((D, \epsilon)) := m_{ij}(D)$ for all $D \in \mathcal{D}$, $\epsilon \in \{-1, +1\}$ and $0 \leq i < j \leq 2$. We call $(\mathcal{D}'; m')$ the *oriented covering* of $(\mathcal{D}; m)$.

A Delaney-Dress symbol is called *fixed-point-free*, if $D\sigma_i \neq D$ for all $D \in \mathcal{D}$ and $i \in \{0,1,2\}$.

The following Lemma is easy, but tedious to prove:

Lemma 4.3 *Let $(\mathcal{D};m)$ be a non-orientable Delaney-Dress symbol and $(\mathcal{D}';m')$ its oriented covering. Then $(\mathcal{D}';m')$ fulfills conditions (DS1)–(DS5), but not necessarily (DS6). Furthermore,*

$$\mathbf{K}_{(\mathcal{D}';m')} = 2\mathbf{K}_{(\mathcal{D};m)} \quad \textit{and} \quad \chi(\mathcal{D}') = 2\chi(\mathcal{D}).$$

In the following we will not distinguish between Delaney-Dress symbols and PTTP Delaney-Dress symbols. This should not lead to confusion.

5. Net-Like Partial Tilings

Through out this Section we will assume that all consider partial tilings $(\mathcal{T},\Gamma)$ are periodic tile-transitive partial tilings with $\mathcal{K}_\infty = \emptyset$. We will refer to them as net-like tilings.

First let us state an important result that holds for tile-k-transitive tilings that cover the plane. This theorem is due to A. Dress, see [7] and [8]. By our definition in Section 1, the tiles of a tiling are assumed to be topological disks. For the following result, however, we need to allow the sightly more general *non-disk tiles* which arise from disk-tiles when the boundary of a tile is permitted to self-intersect in vertices and edges, see [3]. For an algorithm that can decide whether a given Delaney-Dress symbol encodes non-disk tiles, see [15].

Theorem 5.1 *A (2-dimensional) Delaney-Dress symbol $(\mathcal{D};m)$ corresponds to a tiling of the euclidean plane if and only if $\mathbf{K}_{(\mathcal{D};m)} = 0$. (The tiling possibly contains non-disk tiles.)*

Proof. The original (unpublished) proof due to A. Dress involves considering 17 different cases, one for each possible type of crystallographic symmetry group. Recently it has become clear that the Theorem also follows from the fact that every 2-dimensional Delaney-Dress symbol $(\mathcal{D};m)$ with curvature $\mathcal{K}_{(\mathcal{D};m)}$ is essentially a 2-dimensional *orbifold* with *Euler characteristic* $\mathcal{K}_{(\mathcal{D};m)}$, as defined by W. Thurston. According to W. Thurston, *every "good", connected, closed 2-orbifold with Euler characteristic > 0, 0 or < 0, is spherical, Euclidean, or hyperbolic, respectively*, see [16], pg. 91. There are exactly four families of 2-orbifolds that are not "good", i.e. that are not globally the orbit space of some symmetry-group, see [16], pg. 88. These are precisely the cases characterized by the integerity condition mentioned in Section 6. □

Let $(\mathcal{T},\Gamma)$ be a net-like tiling. Theorem 3.1 states that every set $L \in \mathcal{K}$ is a topological disk. So $(\mathcal{T} \cup \mathcal{K},\Gamma)$ is obviously a tile-k-transitive tiling that covers the plane, with $k \in \mathbb{N}$, having the following properties:

(1) The symmetry group Γ operates transitively on $\mathcal{T}$.
(2) The symmetry group Γ partitions $\mathcal{K}$ into precisely $k-1 \geq 0$ different Γ-transitivity classes.

We characterize the Delaney-Dress symbols associated with such tilings in the following result, see [4].

Theorem 5.2 *Let $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ be a Delaney-Dress symbol with $\mathcal{D}_{tiles} \neq \emptyset$. There exists a net-like tiling $(\mathcal{T}, \Gamma)$ that is associated with $(\mathcal{D}; m)$ if and only if the following conditions hold:*

(N1) $\mathcal{D}_\infty = \emptyset$,
(N2) $\infty > m_{01}(D) \geq 3$ *if* $D \in \mathcal{D}_{tiles}$ *and* $\infty > m_{01}(D) \geq 2$ if $D \in \mathcal{D}_{holes}$,
(N3) $\mathbf{K}_{(\mathcal{D};m)} = 0$.

Proof. Condition (N1) reflects $\mathcal{K}_\infty = \emptyset$. Conditions (N2) corresponds to the fact that tiles in $\mathcal{T}$ (or $\mathcal{K}$) must contain at least two (or three, respectively) edges in their boundary. The condition (N3) ensures that the associated tiling can be realized in the euclidean plane. □

The following result was established in [4]:

Lemma 5.3 *There exist only a finite number of Delaney-Dress symbols $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ that satisfy conditions (N1)—(N3). For these we have $\#\mathcal{D} \leq 36$.*

This high bound for the maximal size of a Delaney-Dress symbol of a net-like tiling makes a brute-force approach to the classification of all possible symbols impracticable. Hence in this paper we discuss a more direct approach. The idea is to use geometric algorithms to "construct" rather than "find" all feasible symbols. See [15] for a very similar discussion.

Let $(\mathcal{T}, \Gamma)$ be a net-like partial tiling. If $\mathcal{K} = \emptyset$, then $\mathcal{T}$ covers the plane and $(\mathcal{T}, \Gamma)$ belongs to one of the well-known 93 equivariant types of tile-transitive tilings of the plane. We call such net-like tilings *simple*. The Delaney-Dress symbols of all such tilings were classified in [10].

In the following we will show that all net-like tilings can be obtained from the 93 types of simple ones by systematic application of two geometric operations, called *vertex truncation* and *edge-contraction*. We will discuss the operations geometrically in terms of tilings and tiles. For a formulation of both operations in terms of Delaney-Dress symbols, see [14]. The latter discussion leads to algorithms that have been implemented on a computer and that have been used to obtain the complete classification of the net-like tilings.

Algorithm 5.4 (Vertex-Truncation) Let $(\mathcal{T}, \Gamma)$ be a net-like tiling and let v be a *protected* vertex of $\mathcal{T}$, i.e. with $v \notin K$. Let $U \subset \mathbb{E}^{\not{k}}$ be a small neighborhood around v that intersects no other vertex and only such edges and tiles that are incident to v. The intersection with any of the latter is assumed to be connected. Furthermore, assume that $\gamma U = U$ or $\gamma U \cap U = \emptyset$ for every $\gamma \in \Gamma$. Then the system $\mathcal{T}' := \{t \setminus \bigcup_{\gamma \in \Gamma} \gamma U \mid t \in \mathcal{T}\}$ is a Γ-invariant net-like tiling with $\mathcal{K}' = \mathcal{K} \cup \{\gamma U \mid \gamma \in \Gamma\}$. We say that $(\mathcal{T}', \Gamma)$ can be derived from $(\mathcal{T}, \Gamma)$ by *(equivariant) vertex-truncation*. See Figure 3.

Algorithm 5.5 (Edge-Contraction) Let $(\mathcal{T}, \Gamma)$ be a net-like tiling and let e be a *protected* edge of $\mathcal{T}$. Furthermore, assume that at least one of the two vertices incident to e is unprotected, with degree greater than 3. Call this vertex v and let w be the other vertex incident to e. Note that $v \neq w$. Assume that the number of edges incident to v, say, that are equivalent to e, is one, then $\Gamma_v \leq \Gamma_e$. If $\Gamma_e \leq \Gamma_w$, then we can "move" v along e to w (bending other edges incident to v), thus contracting e to the point w. If $\Gamma_w \leq \Gamma_e$,

Figure 3: (Equivariant) vertex-truncation of the vertex v
in the left-hand tiling yields the right-hand tiling

$\Gamma_w \neq \Gamma_e$, then e has a unique Γ_e-fixed-point x, and we can move both v and w along e to x (again bending incident edges), thus contracting e to x. The conditions fulfilled by v insure that this can be done to every edge $e' \in \Gamma e$. We thus obtain a new net-like partial tiling $(\mathcal{T}', \Gamma)$ and say that it can be derived from $(\mathcal{T}, \Gamma)$ by *(equivariant) edge-contraction*. See Figure 4.

Figure 4: (Equivariant) edge-contraction of the edge e
in the left-hand tiling yields the right-hand tiling

We can now state an important observation.

Theorem 5.6 *If $(\mathcal{T}', \Gamma)$ is a net-like tiling then there exists a finite sequence of net-like tilings*

$$(\mathcal{T}, \Gamma) = (\mathcal{T}_0, \Gamma), (\mathcal{T}_1, \gamma), \ldots, (\mathcal{T}_p, \Gamma), \ldots, (\mathcal{T}_{p+q}, \Gamma) = (\mathcal{T}', \Gamma)$$

with the properties

(1) $(\mathcal{T}, \Gamma)$ *is a simple net-like tiling,*
(2) $(\mathcal{T}_{i+1}, \Gamma)$ *can be derived from* $(\mathcal{T}_i, \Gamma)$ *by vertex-truncation* $(0 \leq i < p)$,
(3) $(\mathcal{T}_{i+1}, \Gamma)$ *can be derived from* $(\mathcal{T}_i, \Gamma)$ *by edge-contraction* $(0 \leq i < q)$.

Proof. Given a net-like tiling $(\mathcal{T}', \Gamma)$. Let v be a unprotected vertex with degree 3. Let $k \in \mathcal{K}'$ be a hole incident to v. Let e and e' be the two adjacent edges of k that meet at v. By definition of a hole, both edges lie between k and some tile. Hence, starting at v we can "zip-up" the edges e and e' along a short distance and thus push v out of the boundary of k, creating a new vertex in the boundary of k, which has degree 3. This operation is precisely the reversion of *edge-contraction.* Applying it precisely q times, where q is the number of non-equivalent unprotected vertices of degree greater than 3, yields a net-like tiling for which every unprotected vertex has degree 3.

Now, given a net-like tiling $(\mathcal{T}', \Gamma)$ with the property that every unprotected vertex has degree 3. We can obtain a simple net-like tiling $(\mathcal{T}, \Gamma)$ by "dividing" each hole $k \in \mathcal{K}'$ among the tiles of $\mathcal{T}'$ that share an edge with k, in a way compatible with the operation of Γ. One way to do this is to consider a Γ-invariant barycentric subdivision of $\mathcal{T}'$. We "glue" each pair of chambers in k that are incident to an edge e of k to the tile that lies on the other side of e. The fact that all unprotected vertices have degree 3 implies that vertices of degree 2 cannot arise. This operation is precisely the reversion of *vertex-truncation.* Applying it precisely p times, where p is the number of non-equivalent holes, yields a simple net-like tiling. □

One can obtain a complete classification of all possible equivariant types of net-like tilings in the following three steps:
(1) Produce the complete list of 93 Delaney-Dress symbols that encode simple net-like tilings (or isohedral tilings that cover the plane), as in [10] or [15].
(2) For every symbol listed in (1), apply vertex-truncation in all possible ways. Discard isomorphic symbols.
(3) For every symbol produced in (2), apply edge-contraction in all possible ways. Discard isomorphic symbols.

We have implemented this approach on a computer, see [14]. Using three programs ISOHEDRAL, TRUNCATE-VERTEX and CONTRACT-EDGE we obtain the following result:

Lemma 5.7 *There exist exactly 1028+93 Delaney-Dress symbols* $(\mathcal{D}; m)$ *satisfying (N1)—(N3), up to isomorphism. Exactly 237 of the 1028 non-simple symbols are derived from the simple symbols by equivariant vertex-truncation, whereas the other 791 are obtained by equivariant vertex-truncation, followed by equivariant edge-contraction.*

By virtue of Lemma 4.2 and Theorem 5.2 there exists a one-to-one correspondence between the above mentioned 1028+93 isomorphism classes of Delaney-Dress symbols and the equivariant equivalence classes of net-like tilings. We have the following

Theorem 5.8 *There exist exactly 1028+93 equivariant types of periodic tile-transitive partial tilings* $(\mathcal{T}, \Gamma)$ *that are net-like, i.e. for which* $\mathcal{K}_\infty = \emptyset$ *holds.*

All enumerated net-like tilings are available on a disk encoded as Delaney-Dress symbols from the author, together with the computer program REPTILES, developed in cooperation with Olaf Delgado Friedrichs, which translates Delaney-Dress symbols into tilings and visualizes them, see [5].

6. Patchy Partial Tilings

A periodic tile-transitive partial tiling $(\mathcal{T}, \Gamma)$ is called a *patchy* tiling, if $\#\mathcal{K}_\infty = 1$. A patchy tiling is *discrete* if no two of its tiles touch.

Comparing definitions shows that there exists a natural one-to-one correspondence between the equivariant types of discrete patchy tilings $(\mathcal{T}, \Gamma)$ and the 52 different "homeomeric types" PP1—PP47, PP48A, PP48B, PP49—PP51 of "(discrete periodic) closed disk patterns" enumerated by B. Grünbaum and G. C. Shephard in [12], Table 5.2.3, pg. 228pp and pg. 345. Their results can be stated as

Theorem 6.1 *There are exactly 52 equivariant types of discrete patchy tilings $(\mathcal{T}, \Gamma)$.*

Let us now focus our attention on the non-discrete patchy tilings. First we will discuss how to enumerate the possible *components*.

Lemma 6.2 *Let $(\mathcal{T}_H, \Gamma_H)$ be a component of a patchy tiling $(\mathcal{T}, \Gamma)$. The degree of any tile $t \in \mathcal{T}_H$ cannot exceed* 4.

Proof. Let $p \in \mathbb{E}^{\not\models}$ be a fix-point of Γ_H. Such a point always exists, because Γ_H is a finite group of isometries. Consider a tile $t \in \mathcal{T}_H$. Let $x_{min} \in t$ be a point of minimal distance d_{min} from p. Let $x_{max} \in t$ be a point of maximal distance d_{max} from p. Tile-transitivity implies that all tiles in $\mathcal{T}_H$ lie between the two circles of radius d_{min} and d_{max} centered at p. Furthermore, each tile t contains a path $\alpha(t)$ which connects a point on the inner circle with a point on the outer circle. Clearly, in this situation a tile t can only have at most four edges: one adjacent to the hole that lies within the inner circle (if this circle has positive radius), one adjacent to the unbounded component of the complement that lies outside the outer circle, at most one adjacent to the tile which lies to the one side of the path in $\alpha(t)$ and at most one to the otherside of $\alpha(t)$. $\square$

Lemma 6.3 *Let $(\mathcal{T}_H, \Gamma_H)$ be a component of a non-discrete patchy tiling $(\mathcal{T}, \Gamma)$, let $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ be the associated Delaney-Dress symbol. Then $\#\mathcal{D} \leq 12$.*

Proof. If $t \in \mathcal{T}_H$ has three or less edges, then $\#\mathcal{D}_{tiles} \leq 6$ and by property (DS4), $\#\mathcal{D} \leq \#\mathcal{D}_{tiles} + \#(\mathcal{D}_{tiles}\sigma_2 \setminus \mathcal{D}_{tiles}) \leq 6 + 6 = 12$. If $t \in \mathcal{T}_H$ has four edges, then two are shared by other tiles, and again, $\#\mathcal{D} \leq \#\mathcal{D}_{tiles} + \#(\mathcal{D}_{tiles}\sigma_2 \setminus \mathcal{D}_{tiles}) \leq 8 + (8 - 4) = 12$. $\square$

Let $(\mathcal{T}_H, \Gamma_H)$ be a *component* of a patchy tiling $(\mathcal{T}, \Gamma)$. Let $\mathcal{K}_H$ denote the set of all *bounded* sets in $\mathcal{K}$ and let $L := \mathbb{E}^{\not\models} \setminus \bigcup_{\approx \in \mathcal{T}_{\mathbb{H}} \cup \mathcal{K}_{\mathbb{H}}} \approx$ be the complement of the tiles and holes of the patch. Let $x \in \mathbb{E}^{\not\models}$ be a point such that $\gamma p = p$ for all $\gamma \in \Gamma_H$. Now, let $\tilde{\cdot} : \mathbb{E}^{\not\models} \to \mathbb{S}^{\not\models}$ be the stereographic projection of the plane onto the sphere, based at point p.

It is obvious that the image of $(\mathcal{T}_H, \Gamma_H)$ under this projection: $(\tilde{\mathcal{T}} := \tilde{\mathcal{T}}_H \cup \tilde{\mathcal{K}}_H \cup \tilde{L}, \tilde{\Gamma})$ can be considered as an equivariant tile-k-transitive tiling of the sphere with $k > 1$, having the following properties:
(1) $\tilde{\Gamma}_H$ operates transitively on $\tilde{\mathcal{T}}$,
(2) $\tilde{\Gamma}_H$ operates $(k-2)$-transitively on $\tilde{\mathcal{K}}_H$, with $k \geq 2$,
(3) $\tilde{\Gamma}_H$ stabilizes $\tilde{L}$, i.e. $\gamma\tilde{L} = \tilde{L}$ for all $\gamma \in \tilde{\Gamma}$.
(Note that $\tilde{L}$ is possibly a non-disk tile).

Hence, the problem of classifying all possible *components* of patchy partial tilings can be translated into the problem of classifying appropriate equivariant tilings of the sphere. Our next step is to characterize the Delaney-Dress symbols of these spherical tilings.

First let us state a result that holds for tile-k-transitive tilings that cover the sphere, which is closely related to Theorem 5.1. This result is due to A. Dress, see [7] and [8].

Theorem 6.4 *A (2-dimensional) Delaney-Dress symbol $(\mathcal{D}; m)$ corresponds to a tiling of the sphere if and only if $\mathbf{K}_{(\mathcal{D};m)} > 0$ and the so-called integerity condition*

$$\frac{\chi(\mathcal{D})}{v_{ij}(D)} \in \frac{\mathbf{K}_{(\mathcal{D};m)}}{2}\mathbb{N}$$

holds for all $D \in \mathcal{D}$. (The tiling possibly contains non-disk tiles.)

Proof. See proof of Theorem 5.1 □

For a discussion of the following Lemma, due to A. Dress, see [15].

Lemma 6.5 *Let $(\mathcal{D}; m)$ be the Delaney-Dress symbol corresponding to a spherical tiling $(\mathcal{T}, \Gamma)$. Then $\#\Gamma = \frac{4}{\mathbf{K}_{(\mathcal{D};m)}}$.*

Now we can completely characterize those Delaney-Dress symbols that encode a component of a non-discrete patchy tilings:

Theorem 6.6 *Let $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ be a Delaney-Dress symbol with $\#\mathcal{D} \leq 12$. There exists the component $(\mathcal{T}_H, \Gamma_H)$ of a non-discrete patchy tiling associated with $(\mathcal{D}; m)$ if and only if the following conditions hold:*

(P1) *$\mathcal{D}_\infty \neq \emptyset$ consists of exactly one $\{0,1\}$-orbit and $\mathbf{K}_{(\mathcal{D};m)} = 2\frac{\#\mathcal{D}_\infty}{m_{01}(D)}$ for all $D \in \mathcal{D}_\infty$.*

(P2) $$\infty > m_{01}(D) \geq \begin{cases} 1 & \textit{if } D \in \mathcal{D}_{tiles} \\ 2 & \textit{if } D \in \mathcal{D}_\infty \textit{ and } m_{01}(D') = 1 \textit{ for } D' \in \mathcal{D}_{tiles} \\ 3 & \textit{otherwise.} \end{cases}$$

(P3) *$\mathbf{K}_{(\mathcal{D};m)} > 0$ and the integerity condition holds.*

Proof. Property (P1) ensures that the encoded component is a bounded patch and that Γ stabilizes the complement of the patch. The latter follows from $\frac{2m_{01}(D)}{\#\mathcal{D}_\infty} = \frac{\#(\Gamma \cdot \mathcal{D}_\infty)}{\#\mathcal{D}_\infty} =$

$\#\Gamma = \frac{4}{\mathbf{K}_{(\mathcal{D};m)}}$ for $D \in \mathcal{D}_\infty$, by Lemma 6.5. Property (P2): If $\mathcal{T}_H$ consists of exactly two tiles, then the intersection of K_∞ and T contains exactly two edges and otherwise, it contains at least three. Finally, (P3) ensures that the encoded tiling can indeed be interpreted as a tiling of the sphere. □

One can produce a complete classification of all possible equivariant types of components of patchy tilings by generating all possible Delaney-Dress symbols statisfying (P1)—(P3). Using a computer program PATCHES we obtain the following result:

Theorem 6.7 *There exist exactly 54 equivariant equivalence classes of components* ($\mathcal{T}_H$, Γ_H) *of non-discrete patchy tilings, see Figure 5 at the end of this paper.*

Now that we have classified all possible components of patchy tilings, let us discuss how to classify the patchy tilings themselves.

Let $(\mathcal{T}_H, \Gamma_H)$ be the component of a non-discrete patchy tiling with Delaney-Dress symbol $(\mathcal{D}; m)$. The group Γ_H is a finite cyclic or dihedral group, which can can easily be determined from $(\mathcal{D}; m)$: If Γ_H is a cyclic group of type Cn with $n \in \{2, 3, 4, 6\}$, then $\mathcal{D}$ must be orientable with $\chi(\mathcal{D}) = 2$ and all 2-color-orbits have branching number 1, except exactly two, which have branching number n. If Γ_H is a dihedral group of type Dn with $n \in \{1, 2, 3, 4, 6\}$, then $\mathcal{D}$ must be non-orientable with $\chi(\mathcal{D}) = 1$ and all 2-color-orbits have branching number 1, unless $n > 1$, in which case exactly two chains have branching number n.

In [12], Grünbaum and Shephard classify all "(discrete periodic) closed disk patterns" into 52 different "homeomeric types" PP1—PP47, PP48A, PP48B, PP49—PP51. Comparing definitions makes it obvious how to associate to each non-discrete patchy tiling $(\mathcal{T}, \Gamma)$ a homeomeric type PP1—PP51, i.e. by "engulfing" each component of the tiling in an appropriate "(marked) disk", in such a way that the following result holds.

Theorem 6.8 *Let* $(\mathcal{T}, \Gamma)$ *and* $(\mathcal{T}', \Gamma')$ *be two non-discrete patchy tilings with components* $(\mathcal{T}_H, \Gamma_H)$ *and* $(\mathcal{T}'_{H'}, \Gamma'_{H'})$*, respectively. Then* $(\mathcal{T}, \Gamma)$ *and* $(\mathcal{T}', \Gamma')$ *are equivariantly equivalent if and only if*
(1) $(\mathcal{T}_H, \Gamma_H)$ and $(\mathcal{T}'_{H'}, \Gamma'_{H'})$ *are equivariantly equivalent,*
(2) $(\mathcal{T}, \Gamma)$ *and* $(\mathcal{T}', \Gamma')$ *give rise to the same "homeomeric type" PP1—PP51.*

So now we can produce an enumeration of the non-discrete patchy tilings. Let $(\mathcal{T}_H, \Gamma_H)$ be a representative of one of the 54 classes of components of non-discrete patchy tilings. Consider Table 5.2.3 in [12]. If for the "homeomeric type" PPi (where $i \in \{1, \ldots, 47, 48A, 48B, 49, \ldots, 51\}$) the "induced tile group" in column (3) of the table and the group type of Γ_H coincide, then there exists a patchy tiling $(\mathcal{T}, \Gamma)$ with component $(\mathcal{T}_H, \Gamma_H)$ that is associated with the "homeomeric type" PPi. In this way, the Table 5.2.3 and the above statements lead to the main result of this section:

Theorem 6.9 *There exist exactly 204 equivariant equivalence classes of patchy tilings, 52 of which are classes of discrete patchy tilings* [see Table 1].

Table 1: For each of the possible stabilizer groups we list the number of types of discrete patchy components, the names and number of appropriate patterns types and the resulting number of discrete patchy tilings.

7. Stripey Partial Tilings

A periodic tile-transitive partial tiling $(\mathcal{T}, \Gamma)$ is called a *stripey* tiling, if $\#\mathcal{K}_\infty = \infty$. We will first discuss how to enumerate all possible types of *components* of stripey tilings. Such a classification can be based on the following

Theorem 7.1 *Let* $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ *be a Delaney-Dress symbol with* $\mathcal{D}_{tiles} \neq \emptyset$. *There exists a component* $(\mathcal{T}_H, \Gamma_H)$ *of a stripey tiling associated with* $(\mathcal{D}; m)$ *if and only if the following conditions hold:*

(S1) $\mathcal{D}_\infty \neq \emptyset$ *and* $m_{01}(D) = \infty$ *for all* $D \in \mathcal{D}_\infty$.

(S2) $\infty > m_{01}(D) \geq 3$ *if* $D \in \mathcal{D}_{tiles}$ *and* $\infty > m_{01}(D) \geq 2$ *if* $D \in \mathcal{D}_{holes}$.

(S3) $\mathrm{K}_{(\mathcal{D};m)} = 0$.

Proof. Condition (S1) reflects that $\mathcal{K}_\infty \neq \emptyset$ and that the connected components of the boundary of any set $L \in \mathcal{K}_\infty$ are unbounded. Condition (S2) has the same effect as (N2) above. The condition (S3), together with (S1), ensures that the associated component $(\mathcal{T}_H, \Gamma_H)$ can be realized in the Euclidean plane. The proof of this fact involves studying the orbifolds associated with each of the seven types of frieze groups. This we do in the next Lemma. □

The proof of the following result is analog to the discussion in [10] and [4]. We show that any Delaney-Dress symbol satisfying (S1)—(S3) leads to one of the seven orbifolds (see [2]) associated with the seven frieze groups (see [12], pg. 39).

Lemma 7.2 *Let $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ be a Delaney-Dress symbol with properties (S1)—(S2). Then (S3) holds if and only if one of the following conditions hold:*

- *$\mathcal{D}$ is orientable and $\chi(\mathcal{D}) = 2$. There exist precisely two 2-color-orbits with non-trivial branching numbers. These are the two $\{0,1\}$-cycles that make up $\mathcal{D}_\infty$, and both have branching number ∞. This gives rise to the orbifold $\infty\infty$. The frieze group is* **p111**.
- *$\mathcal{D}$ is orientable and $\chi(\mathcal{D}) = 2$. There exist precisely three 2-color-orbits with non-trivial branching numbers; the $\{0,1\}$-cycle $\mathcal{D}_\infty$ has branching number ∞, whereas two other 2-color-orbits have branching number 2. This gives rise to the orbifold 22∞. The frieze group is* **p112**.
- *$\mathcal{D}$ is non-orientable, but fixed-point-free, and $\chi(\mathcal{D}) = 1$. There exists only one 2-color-orbit with a non-trivial branching number, namely the $\{0,1\}$-cycle $\mathcal{D}_\infty$, with branching number ∞. This gives rise to the orbifold $\infty\times$. The frieze group is* **p1a1**.
- *$\mathcal{D}$ has fixed-points and $\chi(\mathcal{D}) = 1$. There exists only one 2-color-orbit with non-trivial branching number, namely the $\{0,1\}$-cycle $\mathcal{D}_\infty$, with branching number ∞. This gives rise to the orbifold $\infty*$. The frieze group is* **p1m1**.
- *$\mathcal{D}$ has fixed-points and $\chi(\mathcal{D}) = 1$. Furthermore, $\mathcal{D}_\infty$ consists of precisely two $\{0,1\}$-chains, and these are the only 2-color-orbits having non-trivial branching numbers, both ∞. This gives rise to the orbifold $*\infty\infty$. The frieze group is* **pm11**.
- *$\mathcal{D}$ has fixed-points and $\chi(\mathcal{D}) = 1$. Furthermore, $\mathcal{D}$ contains a $\{i,j\}$-cycle with branching number 2 and $\mathcal{D}_\infty$ is a single $\{0,1\}$-chain with branching number ∞. This gives rise to the orbifold $2*\infty$. The frieze group is* **pma2**.
- *$\mathcal{D}$ has fixed-points and $\chi(\mathcal{D}) = 1$. There exist three 2-color-orbits with non-trivial branching numbers, $\infty, 2, 2$, and these are all chains. This gives rise to the orbifold $*22\infty$. The frieze group is* **pmm2**.

Proof. Let us first assume that $\mathcal{D}$ is orientable. In this case the *topological realization* of $\mathcal{D}$ is a compact orientable 2-manifold and thus, $\chi(\mathcal{D}) \in \{2, 0, -2, \ldots\}$. We have

$$2\chi(\mathcal{D}) + \#\mathcal{D} = \sum_{0\le i<j\le 2} \sum_{D\in\mathcal{D}} \frac{1}{r_{ij}(D)}.$$

Thus, $\mathbf{K}_{(\mathcal{D};m)} = 0$ if and only if

$$\#\mathcal{D} = \sum_{0\le i<j\le 2} \sum_{D\in\mathcal{D}} \frac{1}{m_{ij}(D)},$$

where $\frac{1}{m_{ij}(D)} := 0$ whenever $m_{ij}(D) = \infty$. So (S3) holds if and only if

$$2\chi(\mathcal{D}) + \sum_{D\in\mathcal{D}} \sum_{0\le i<j\le 2} \frac{1}{m_{ij}(D)} = \sum_{D\in\mathcal{D}} \sum_{0\le i<j\le 2} \frac{1}{r_{ij}(D)}$$

or

$$\begin{aligned}
2\chi(\mathcal{D}) &= \sum_{D\in\mathcal{D}} \sum_{0\le i<j\le 2} \Big(\frac{1}{r_{ij}(D)} - \frac{1}{m_{ij}(D)}\Big) \\
&= \sum_{0\le i<j\le 2} \; \sum_{\mathcal{O}\{i,j\}-\mathrm{orbit}} \; \sum_{D\in\mathcal{O}} \Big(\frac{1}{r_{ij}(D)} - \frac{1}{v_{ij}(D) r_{ij}(D)}\Big) \\
&= \sum_{0\le i<j\le 2} \; \sum_{\mathcal{O}\{i,j\}-\mathrm{orbit}} \#\mathcal{O}\Big(\frac{1}{r_{ij}(D\in\mathcal{O})} - \frac{1}{v_{ij}}(D\in\mathcal{O}) r_{ij}(D\in\mathcal{O})\Big) \\
&= \sum_{0\le i<j\le 2} \; \sum_{\mathcal{O}\{i,j\}-\mathrm{orbit}} 2\Big(1 - \frac{1}{v_{ij}(D\in\mathcal{O})}\Big) \\
&\quad (\#\mathcal{O} = 2 r_{ij}(D\in\mathcal{O}), \text{ because } \mathcal{D} \text{ is orientable})
\end{aligned}$$

holds. In particular, $\chi(\mathcal{D}) > 0$ in view of $v_{01}(D) = \infty$ for $D \in \mathcal{D}_\infty \neq \emptyset$.

So property (S3) holds if and only if the n non-trival branching numbers $v_1, \ldots, v_n$ belonging to $(\mathcal{D}; m)$ satisfy

$$\sum_{i=1}^{n} \Big(1 - \frac{1}{v_i}\Big) = 2$$

If we assume $v_1 = \infty$, then $\sum_{i=2}^{n}(1 - \frac{1}{v_i}) = 1$.

From $(1 - \frac{1}{v_i}) > \frac{1}{2}$ for $v_i > 2$ it follows that either $v_2 = \infty$ and $n = 2$ or $v_2 = v_3 = 2$ and $n = 3$. If $\mathcal{D}$ is non-orientable, then by considering the oriented covering $\mathcal{D}'$ we obtain the other cases. □

Lemma 7.3 *Let $(\mathcal{D} = \mathcal{D}_{tiles} \cup \mathcal{D}_{holes} \cup \mathcal{D}_\infty; m)$ be a Delaney-Dress symbol with properties (S1)—(S3). Then $\#\mathcal{D} \le 24$.*

Proof. First assume $\#\mathcal{D}_\infty \ge 2$. Let

$$\mathcal{D}_3 := \{D \in \mathcal{D} \mid m_{12}(D) = 3\} \quad \text{and} \quad \mathcal{D}_4 := \{D \in \mathcal{D} \mid m_{12}(D) \ge 4\}.$$

We have

$$\begin{aligned}
0 = \mathbf{K}_{(\mathcal{D},m)} &= \sum_{D\in\mathcal{D}} \Big(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} + \frac{1}{m_{02}(D)} - 1\Big) \\
&= \sum_{D\in\mathcal{D}} \Big(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} - \frac{1}{2}\Big)
\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{D\in\mathcal{D}}\left(\frac{1}{2}-\frac{1}{m_{12}(D)}\right)=&\sum_{D\in\mathcal{D}_{tiles}}\frac{1}{m_{01}(D)}+\sum_{D\in\mathcal{D}_{holes}\cup\mathcal{D}_\infty}\frac{1}{m_{01}(D)}\\ \le&2+\frac{1}{3}(\#(\mathcal{D}_{holes}\cup\mathcal{D}_\infty)-2)\\ =&2+\frac{1}{3}(\#((\mathcal{D}_{holes}\cup\mathcal{D}_\infty)\cap\mathcal{D}_3)\\ &+\#((\mathcal{D}_{holes}\cup\mathcal{D}_\infty)\cap\mathcal{D}_4)-2)\\ \le&2+\frac{1}{9}\#\mathcal{D}_3+\frac{1}{6}\#\mathcal{D}_4-\frac{2}{3}\end{aligned}$$

because $\#((\mathcal{D}_{holes}\cup\mathcal{D}_\infty)\cap\mathcal{D}_3)\le\frac{1}{3}\#\mathcal{D}_3$ and $\#((\mathcal{D}_{holes}\cup\mathcal{D}_\infty)\cap\mathcal{D}_4)\le\frac{1}{2}$ in view of $(\mathcal{D}_{holes}\cup\mathcal{D}_\infty)\sigma_2\subseteq\mathcal{D}_{tiles}$. So it follows from

$$\begin{aligned}\sum_{D\in\mathcal{D}}\left(\frac{1}{2}-\frac{1}{m_{12}(D)}\right)=&\frac{1}{2}\#\mathcal{D}-\sum_{D\in\mathcal{D}}\frac{1}{m_{12}(D)}\\ \ge&\frac{1}{2}\#\mathcal{D}-\frac{1}{3}\#\mathcal{D}_3-\frac{1}{4}\#\mathcal{D}_4\end{aligned}$$

that

$$\begin{aligned}\frac{4}{3}\ge&-\frac{1}{9}\#\mathcal{D}_3-\frac{1}{6}\#\mathcal{D}_4+\frac{1}{2}\#\mathcal{D}-\frac{1}{3}\#\mathcal{D}_3-\frac{1}{4}\#\mathcal{D}_4\\ =&\frac{1}{18}\#\mathcal{D}_3+\frac{1}{12}\#\mathcal{D}_4\ge\frac{1}{18}\#\mathcal{D}\end{aligned}$$

and thus, $24\ge\#\mathcal{D}$.

Now assume $\#\mathcal{D}_\infty=1$, then $\mathcal{D}$ is not orientable and the orientated covering $(\mathcal{D}'=\mathcal{D}'_{tiles}\cup\mathcal{D}'_{holes}\cup\mathcal{D}'_\infty;m')$ has properties (DS1)—(DS5) and also (S1)—(S3). If $D'\in\mathcal{D}_\infty$ then $D'\sigma_0=D'\sigma_1=D'$. So for $D=D'\sigma_2\in\mathcal{D}$ we have $D\sigma_0=D'\sigma_2\sigma_0=D'\sigma_0\sigma_2=D'\sigma_2=D$. Thus $\mathcal{D}_{tiles}$ is a $\{0,1\}$-chain and hence $\mathcal{D}'=\big\{(D,\epsilon)\mid D\in\mathcal{D},\ \epsilon\in\{+1,-1\}\big\}$ is a single $\{0,1\}$-orbit in $\mathcal{D}'$, i.e. $(\mathcal{D}';m')$ also has property (DS6). Thus we can apply the first part of this proof to $(\mathcal{D}';m')$ and get $24\ge\#\mathcal{D}'=2\#\mathcal{D}$. □

Using a computer program STRIPES we have produced a representative for each isomorphism type of Delaney-Dress symbol $(\mathcal{D};m)$ satisfying (S1)—(S3). Furthermore, for each such representative we have determined the frieze type of the stabilizer group Γ_H of an associated component $(\mathcal{T}_H,\Gamma_H)$. We thus obtain the following result

Theorem 7.4 *There are exactly 93 different equivariant equivalence classes of components* $(\mathcal{T}_H,\Gamma_H)$ *of stripey tilings* [see Figure 6 at the end of this paper].

In [12], Grünbaum and Shephard give a classification of the "periodic strip patterns" into 26 different "pattern types" SPS1—SPS26. Comparing definitions makes it obvious how to associate to each stripey tiling $(\mathcal{T},\Gamma)$ a "pattern type" (i.e. by "engulfing" each component of the stripey tiling in an appropriate "marked strip tile") in such a way that the following result holds

Theorem 7.5 *Let* $(\mathcal{T},\Gamma)$ *and* $(\mathcal{T}',\Gamma')$ *be two stripey tilings with components* $(\mathcal{T}_H,\Gamma_H)$ *and* $(\mathcal{T}'_{H'},\Gamma'_{H'})$*. Then* $(\mathcal{T},\Gamma)$ *and* $(\mathcal{T}',\Gamma')$ *are equivariantly equivalent if and only if*
(1) $(\mathcal{T}_H,\Gamma_H)$ *and* $(\mathcal{T}'_{H'},\Gamma'_{H'})$ *are equivariantly equivalent ,*
(2) $(\mathcal{T},\Gamma)$ *and* $(\mathcal{T}',\Gamma')$ *give rise to the same "pattern type" SPSi* $(i \in \{1,\dots,26\})$*.*

We can now obtain an enumeration of the stripey tilings. Let $(\mathcal{T}_H,\Gamma_H)$ be a representative of one of the 93 classes of components of stripey tilings. Consider Table 6.5.1 in [12]. If for "pattern type" SPSi ($i \in \{1,\dots,26\}$) it holds that the "induced tile group" in column (6) of the table and the frieze type of Γ_H coincide, then there exists a stripey tiling $(\mathcal{T},\Gamma)$ with component $(\mathcal{T}_H,\Gamma_H)$ that is associated with the "pattern type" SPSi. In this way, Table 6.5.1 and the above statements lead to the main result of this Section

Theorem 7.6 *There exist exactly 248 equivariant equivalence classes of stripey tilings* [see Table 2].

Table 2: For each of the seven possible frieze groups we list the number of types of stripey components, the names and number of appropriate patterns types and the resulting number of stripey tilings.

Consider a strip in the euclidean plane bounded by two infinite parallel lines. In [17] it was shown that exactly 24 equivariant types of isohedral tilings of such a strip exist. These correspond to the following 24 types of stripey components depicted in Figure 6: SC 3, SC 5, SC 10, SC 11, SC 13, SC 15, SC 21, SC 23, SC 36, SC 37, SC 38, SC 39, SC 42, SC 50, SC 52, SC 53, SC 57, SC 62, SC 63, SC 65, SC 82, SC 85, SC 92 and SC 93.

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Figure 5: The 54 types of components of non-discrete patchy partial tilings.

Figure 5 (continued)

Figure 5 (continued)

Figure 5 (continued)

Figure 6: The 93 types of components of stripey patchy tilings.

Figure 6 (continued)

Figure 6 (continued)

Figure 6 (continued)

Figure 6 (continued)

Figure 6 (continued)

Figure 6 (continued)