Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 34 (1993), No. 1, 57-62.

A Criterion for the Ideal of a Projectively Embedded Toric Surface to be Generated by Quadrics

Robert Jan Koelman *
Fakultät für Mathematik, Ruhr-Universität Bochum,
Universitätsstraße 150, W-4630 Bochum 1

Abstract. We show that the ideal of a projectively embedded toric surface is generated by quadrics if and only if the polygon which corresponds to the embedding has more than three lattice points on its boundary.

MSC 1991: Primary 14J25

1. Introduction

Let P be an integral convex polygon in $\mathbf{R}^2$. Let $\triangle_P$ be the corresponding fan (see [Da], 5.8), and let $X_P = T_{\text{emb}}(\triangle_P)$. The polygon P corresponds to a very ample line bundle $\mathcal{L}_P$. We can embed X_P with the complete linear system $|\mathcal{L}_P|$ in $\mathbf{P}^{n-1}$ where $n = h^0(X_P, \mathcal{L}_P)$.

In [Ko], it is shown that the homogeneous ideal $I(X_P) \subset \mathbf{C}[x_1, \ldots, x_n]$ is generated by polynomials of degrees 2 and 3.

In the case of curves it is well known (see [Ar], Petri's Theorem) that the ideal of the canonical image is generated by polynomials of degrees 2 and 3 and that one needs a polynomial of degree 3 if and only if the curve is either trigonal or a plane quintic.

We will prove a similar theorem in the case of toric surfaces. If we have more that three lattice points on the boundary of the polygon P, then the ideal $I(X_P)$ is generated by quadrics. If we only have the vertices of the polygon P as the lattice points on the boundary, then we will explicitly give a polynomial of degree 3 which we need as a generator.

2. The theorem and its proof

We use the notation as in the above introduction. By [Od] we know that to each lattice point $x \in P$ there corresponds a global section in $H^0(X_P, \mathcal{L}_P)$. By abuse of notation we will call this corresponding global section x too. Hence the set $\{x_1, \ldots, x_n\}$ is the collection of lattice points in P and also a basis for the vector space $H^0(X_P, \mathcal{L}_P)$. At any point it should be clear what is meant. As a guide, the operation on the lattice points is addition while we can multiply global sections.

0138-4821/93 \$ 2.50 — * Supp. by Heinrich-Hertz-Stiftung

The homogeneous part I_d of $I(X_P)$ is given by

$$0 \longrightarrow I_d \longrightarrow Sym^d(H^0(X_P, \mathcal{L}_P)) \longrightarrow H^0(X_P, \mathcal{L}_P^{\otimes d}) \longrightarrow 0, \tag{1}$$

which is exact for each d because X_P is a surface.

Definition 2.1 Let $f \in I(X_P)$ be a polynomial of the form $f = \mathbf{x}^{m_1} - \mathbf{x}^{m_2}$, where $m_i \in \mathbf{Z}^n_{\geq 0}$, $m_i = \{m_{i1}, \ldots, m_{in}\}$, $\mathbf{x}^{m_i} = x_1^{m_{i1}} \cdots x_n^{m_{in}}$, $i = 1, 2$. Let $\mathrm{supp}(f) = \{x_j \in P \,|\, m_{1j} > 0 \text{ or } m_{2j} > 0,\ j = 1, \ldots, n\}$ and let $P_f = \mathrm{Conv}(\mathrm{supp}(f))$.

Example 2.2 Let P be as in Figure 1.a and let $f = x_2x_7x_4 - x_2x_3x_8$ (for numbering see Figure 1.b). Then $\mathrm{supp}(f) = \{x_2, x_3, x_4, x_7, x_8\}$ and P_f is as in Figure 1.c.

Figure 1.a Figure 1.b Figure 1.c

Definition 2.3 Let P be a polygon, let $\delta(P)$ be the number of lattice points on the boundary of P. Then $\delta(P) = \#(\partial P \cap \mathbf{Z}^2)$.

Let $I_Q \subset I(X_P)$ be the ideal generated by the quadrics in $I(X_P)$. We have the following

Lemma 2.4 *Let P be an integral convex polygon and let $f \in I(X_P)$ such that $\delta(P_f) > 3$. Then there exists a polynomial $f' \in I(X_P)$ such that $f' \equiv f \bmod I_Q$ and one of the following is true*

1. $f' = 0$;
2. *The points in supp(f') are collinear;*
3. $\delta(P_{f'}) = 3$ *and* $\mathrm{Area}(P_{f'}) < \mathrm{Area}(P_f)$.

Proof. Without loss of generality we may assume that one vertex v of P is $(0, 0)$. Choose a point R on the edge of P such that the line segment from v to R is not an edge of P. This is possible since $\delta(P) > 3$. Let L be the line through R and v and let H^+ and H^- be the two halfspaces determined by L (hence L is contained in H^+ and in H^-). Using polynomials of degree 2 we can change f into g such that supp(g) is either contained in $H^+ \cap P_f$ or in $H^- \cap P_f$. To be more precise, if $f = \mathbf{x}^{m_1} - \mathbf{x}^{m_2}$ then supp(g) is contained in that halfspace which contains $Q = \sum_{i=1}^n m_{1i}x_i = \sum_{i=1}^n m_{2i}x_i$. The proof for this fact is completely analogous to the proof of [Ko], Lemma 2.9. Now it is trivial that Area(P_g) <Area(P_f). Continue this process with g and P_g until we transform g into the zeropolynomial or supp(g) are collinear points or P_g does not contain more than three points on the boundary. This process terminates because the area of P_g decreases. $\square$

Example 2.5 Let $P = P_f$ be as in Figure 2.a-c, where $f = x_1x_6x_{17} - x_8x_{11}x_{12}$. Then the steps of the proof of Lemma 2.4 can also be seen in Figure 2, where $f' = x_1x_{11}x_{16} - x_8x_{11}x_{12}$ and the final result $f'' = x_1x_{11}x_{16} - x_8x_{10}x_{13}$.

Figure 2.a P_f Figure 2.b $P_{f'}$ Figure 2.c $P_{f''}$

We wish to prove that if $\delta(P) > 3$ then any polynomial $f \in I(X_P)$ is contained in I_Q. Hence we want that $f \equiv 0 \bmod I_Q$. By relations of degree 2 we can transform any polynomial as in the above lemma. However we want to be able to transform any polynomial to the zeropolynomial.

In [Ko], it is proved that the ideal $I(X_P)$ is generated by equations of degrees 2 and 3. In order to prove that the ideal is generated by polynomials of degree 2 only, we have to prove that each polynomial of degree 3 is contained in I_Q. Therefore the following

Lemma 2.6 *Let P be an integral convex polygon. Let $f \in I(X_P)$ such that P_f is a triangle. Let $f = \mathbf{x}^{m_1} - \mathbf{x}^{m_2}$ such that $deg(f) = 3$. Let x_1, x_2, x_3 be the vertices of P_f. Then either $\mathbf{x}^{m_1} = x_1x_2x_3$ or $\mathbf{x}^{m_2} = x_1x_2x_3$.*

Proof. Suppose the lemma is not true. Then without loss of generality we may assume that $\mathbf{x}^{m_1} = x_1x_2x_4$ and $\mathbf{x}^{m_2} = x_3x_5x_6$ with x_4, x_5, x_6 (not necessarily distinct) elements of $\mathrm{supp}(f)$. Let $Q = x_1 + x_2 + x_4 = x_3 + x_5 + x_6$. Then we see that $Q \in x_1 + x_2 + P$ and $Q \in x_3 + 2P$. But these two polygons only intersect at the point $x_1 + x_2 + x_3$. $\square$

Lemma 2.7 *Let P be an integral convex polygon such that $\delta(P) \geq 4$. Let $f \in I(X_P)$ such that $\mathrm{supp}(f)$ is not collinear, $\delta(P_f) = 3$ and f irreducible. Then there exists a polynomial f' such that $f' \equiv f \bmod I_Q$ and such that $\mathrm{Area}(P'_f) < \mathrm{Area}(P_f)$.*

Proof. Without loss of generality we may assume that the vertices of P_f are $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (k,d)$ with $\gcd(k,d) = 1$, $\gcd(k-1,d) = 1$, $0 \leq k < d$. Let L_i be the lines determined by the edges of P_f. Because $\delta(P) > 3$ we have a vertex $v = (a,b) \in P, v \notin P_f$. Without any loss of generality, assume that v is contained in the area given by (see Figure 3). We obtain inequality (3) because we may assume, again without loss of generality, that if v is contained in the area between L_1 and L_3 that the lattice distance from v to L_3 is less than or equal to the lattice distance from v to L_1. For instance, we see that $b \leq -1$.

$$(x,y) \in \mathbf{R}^2 \begin{cases} y \leq 0 & (1) \\ dx-(k-1)y \leq d & (2) \\ dx-(k+1)y \geq 0 & (3) \end{cases}$$

Figure 3

Of course we have $P_1 = \mathrm{Conv}(x_1, x_2, v) \subset P$ and P_1 is given by (see Figure 4).

$$(x,y) \in \mathbf{R}^2 \begin{cases} y \leq 0 & (4) \\ bx-ay \leq 0 & (5) \\ bx-(a-1)y \geq b & (6) \end{cases}$$

Figure 4

Now we are going to prove that $(0,-1) \in P_1$ if $k < d-1$. For this we have to show that $(0,-1)$ satisfies the inequalities $(4), (5), (6)$.

Inequality (4) is satisfied trivially, since $-1 \leq 0$. From inequality (2) it follows that $a \leq 0$ if $b \leq -1$ hence inequality (5) is fulfilled. From (3) it follows that $a \geq \frac{k+1}{d}b$. But if $k < d-1$ then $\frac{k+1}{d} < 1$. Thus we see that $a > b$ from which follows that $a \geq b+1$. Hence (6) is also fulfilled.

Suppose that $k+1 = d$. As in the above, $(0,-1)$ satisfies inequalities (4) and (5). Suppose that (6) is not fulfilled. Then v is of the form (a,a) from which it follows that $\delta(\mathrm{Conv}(v, x_1, x_2, x_3)) = 3$ (see Figure 5.b). Therefore, P contains yet another vertex which makes it clear that the picture can be chosen in such a way that $(0,-1) \in P$.

Figure 5.a

Figure 5.b

Since f is assumed irreducible we may assume (by Lemma 2.6) that $f = x_1x_2x_3 - x_4x_5x_6$, where x_4, x_5, x_6 are (not necessarily distinct) elements of $\mathrm{supp}(f)$ and each of them distinct from $x_i, i = 1, 2, 3$! Because $v = (0, -1)$ and $w = (1, 1) \in P$ (see Figure 5.a) we have the relation $vw - x_1x_2 \in I_Q$. Let $f' = vwx_2 - x_3x_4x_5$, then $P_{f'} \subset \mathrm{Conv}(\{\mathrm{Conv}(v, x_1, x_2, x_3) \cap \mathbf{Z}^2)\} \backslash \{x_1, x_2\})$. Now we see that $Area(P_{f'}) \leq Area(P_f) + \frac{1}{2} - \frac{1}{2} - \frac{1}{2}$. □

With the help of the above lemmas we are able to transform a polynomial in $I(X_P)$ as in the following

Lemma 2.8 *Let P be an integral convex polygon such that $\delta(P) \geq 4$. Let $f \in I(X_P)$ such that $deg(f) = 3$. Then there exists a polynomial $f' \in I(X_P)$ such that $f' \equiv f \bmod I_Q$ and either $f' = 0$ or $\mathrm{supp}(f')$ is collinear.*

Proof. Apply to f the following algorithm:

(1) Apply Lemma 2.4 to find an equivalent polynomial f' such that $\delta(P'_f) = 3$ or $f' = 0$ or $\mathrm{supp}(f')$ is collinear. Put $f := f'$.

(2) If f is reducible then f is of the form $f = x_i g$ where g is quadratic, hence let $f' = 0$. If f is irreducible and $f \neq 0$ and $\mathrm{supp}(f)$ is not collinear then apply Lemma 2.7 to find an equivalent polynomial f' such that $\mathrm{Area}(P_{f'}) < \mathrm{Area}(P_f)$. Let $f := f'$.

(3) If $f \neq 0$ and $\mathrm{supp}(f)$ is not collinear do the following: If $\delta(P_f) > 3$ then go to (1) else go to (2).

The algorithm stops because in each step the area of P_f goes down. Thus for the resulting f we either have $f = 0$ or $\mathrm{supp}(f)$ is collinear. □

To complete the process we need the following lemma. The lemma is elementary and was proved by several people (i.e. [Ba], Theorem 1, $n = 1$). The following lemma, however, shows an easy algorithmic proof.

Lemma 2.9 *Let P be a polytope of dimension 1. Embed X_P with $\mathcal{L}_P$ into $\mathbf{P}^{s-1}$. Then the ideal $I(X_P)$ is generated by quadrics.*

Proof. We know $X_P \cong \mathbf{P}^1$ and $\mathcal{L}_P \cong \mathcal{O}(s-1)$. Let $f = \mathbf{x}^{m_1} - \mathbf{x}^{m_2} \in I(X_P)$. Using quadratic relations we can rewrite f into $f' = x_1^{a_1} x_s^{a_2} x_\alpha - x_1^{b_1} x_s^{b_2} x_\beta$ with $a_1, a_2, b_1, b_2, \alpha, \beta \in \mathbf{Z}_{\geq 0}$, $1 < \alpha, \beta < s$ and $a_1 + a_2 = b_1 + b_2$. Let $x_1 = 0$ and $x_s = s - 1$ (so now we consider x_i to be an element of the lattice $\mathbf{Z}$, see Figure 6).

Figure 6

Hence $a_2(s-1) + \alpha - 1 = b_2(s-1) + \beta - 1$. It follows that $a_2 = b_2$, $\alpha = \beta$ and hence $f' = 0$. □

We are now able to state the following

Theorem 2.10 *Let P be a integral convex polygon such that $\delta(P) \geq 4$. Then the ideal $I(X_P)$ is generated by quadrics.*

Proof. Let $f \in I(X_P)$. By Lemma 2.8, transform f with into f' with $f' \equiv f \bmod I_Q$ such that either $f' = 0$ or $\mathrm{supp}(f)$ is collinear. If $f' = 0$ then $f \in I_Q$. If f' is not zero then we know, by Lemma 2.9, that f' is in the ideal generated by the quadratic relations with support on the line given by $\mathrm{supp}(f)$. Hence, certainly $f \in I_Q$. □

We wish to prove that the ideal $I(X_P)$ is not generated by quadrics only, if $\delta(P) = 3$. Therefore we show the following

Lemma 2.11 *Let P be an integral convex polygon such that $\delta(P) = 3$. Then there exists a polynomial $f \in I(X_P)$ such that $f \notin I_Q$.*

Proof. Without loss of generality we may assume that the vertices of P are $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (k,d)$ with $\gcd(k,d) = 1$, $\gcd(k-1,d) = 1$ and $0 \leq k < d$. If we can show that there exists a polynomial of the form $f = x_1x_2x_3 - \mathbf{x}^m$ where $\mathbf{x}^m$ is not divisible by x_1, x_2 or x_3 then we are done since f can never be in the ideal generated by quadrics because there are no points on the edges different from the vertices. Let $Q = x_1+x_2+x_3 = (k+1,d)$ and let $x_4 = (1,1)$. Clearly, $x_4 \in P$. But we see that $(k, d-1) \in 2P$ hence there exists x_5, x_6, not necessarily distinct, such that $x_5 + x_6 = (k, d-1)$. But x_5 and x_6 are both distinct from x_1, x_2 and x_3 because $(k, d-1)$, $(k-1, d-1)$ and $(0,-1)$ are not points in P. For this we must have that $k > 1$, which is the case since $\gcd(k-1,d)$=1. □

We combine the preceding lemma and theorem in the following following

Corollary 2.12 *Let P be an integral convex polygon. Then $I(X_P)$ is generated by quadrics if and only if $\delta(P) \geq 4$.*

Proof. Reformulation of Theorem 2.10 and Lemma 2.11. □

References

[Ar] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of algebraic curves, Volume 1* (Springer-Verlag, New York etc., 1985).

[Ba] V. Batyrev, *Letter to B. Sturmfels*, 1992.

[Da] V.I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys 33:2 (1978), 97 - 154; Uspekhi Mat. Nauk 33:2 (1978), 85 - 134. F. Knudsen, D. Mumford and B. Saint-Donat,

[Ko] R.J. Koelman, *Generators of the ideal of a projectively embedded toric surface*, to appear in the Tohoku Math. Journal.

[Od] T. Oda, *Convex Bodies and Algebraic Geometry* (Springer-Verlag, Berlin etc., 1988).