Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 34 (1993), No. 1, 1-8.

On the General Brunn-Minkowski Theorem

Rolf Schneider
Mathematisches Institut, Albert-Ludwigs-Universität
Albertstr. 23 b, 7800 Freiburg i. Br.

Abstract. Some stability results are obtained in connection with the general Brunn-Minkowski theorem for mixed volumes of convex bodies.

1. Introduction and statement of results

For convex bodies K, L (non-empty, compact, convex subsets) in n-dimensional Euclidean space $\mathbf{E}^n$, the Minkowski linear combination is defined by

$$K_\lambda := (1-\lambda)K + \lambda L := \{(1-\lambda)x + \lambda y : x \in K,\ y \in L\}$$

for $0 \le \lambda \le 1$. The classical Brunn-Minkowski theorem states that the nth root of the volume of K_λ is a concave function of λ. In particular,

$$V_n(K_\lambda)^{1/n} \ge (1-\lambda)V_n(K)^{1/n} + \lambda V_n(L)^{1/n} \tag{1}$$

for $0 \le \lambda \le 1$, where V_n denotes the volume in $\mathbf{E}^n$. If we assume that K and L are n-dimensional, then equality in (1), for some $\lambda \in (0,1)$, implies that K and L are homothetic. Improvements of this assertion in the form of stability results have been obtained by Diskant [3] (see also [6], Theorem 2.3.1) and Groemer [9]. For a precise explanation of what is meant here by 'stability', and for a survey over stability results for geometric inequalities, we refer to Groemer [11].

The general Brunn-Minkowski theorem is an extension of (1) to mixed volumes. (We assume here that the reader is familiar with the theory of mixed volumes; see, e.g., Leichtweiß [14] or Schneider [17].) Let a number $m \in \{1, \dots, n-1\}$ and convex bodies $K, L, K_{m+1}, \dots, K_n$ in $\mathbf{R}^n$ be given. The general Brunn-Minkowski theorem states that the function

$$\lambda \mapsto V(\underbrace{K_\lambda, \dots, K_\lambda}_{m}, K_{m+1}, \dots, K_n)^{1/m},$$

where V denotes the mixed volume, is concave for $0 \le \lambda \le 1$ (see Theorem 6.4.3 in [17], where references are given in the corresponding notes). Denoting the $(n-m)$-tuple $(K_{m+1}, \dots, K_n)$ by $\mathcal{C}$, we can write the general Brunn-Minkowski inequality in the form

$$V(\underbrace{K_\lambda, \dots, K_\lambda}_{m}, \mathcal{C})^{1/m} \ge (1-\lambda)V(\underbrace{K, \dots, K}_{m}, \mathcal{C})^{1/m} + \lambda V(\underbrace{L, \dots, L}_{m}, \mathcal{C})^{1/m} \tag{2}$$

0138-4821/93 \$ 2.50

for $0 \le \lambda \le 1$. Equality in (2) holds trivially if $\lambda = 0$ or 1, if K and L lie in parallel $(m-1)$-dimensional affine subspaces, or if K and L are homothetic. It is known that there are also non-trivial cases of equality, depending on the nature of $\mathcal{C}$, but the exact conditions for equality are unknown. However, if K and L are of dimension at least m and if the bodies $K_{m+1}, \dots, K_n$ are smooth (that is, have unique supporting hyperplanes at all boundary points), then equality in (2) for some $\lambda \in (0,1)$ implies that K and L are homothetic (Theorem 6.6.9 in [17]). In particular, this is true if $K_{m+1} = \dots = K_m = B$, where B denotes the unit ball of $\mathbf{R}^n$. Note that

$$V(\underbrace{K_\lambda, \dots, K_\lambda}_{m}, B, \dots, B) = W_{n-m}(K_\lambda)$$

is the $(n-m)$th quermassintegral of K_λ; in particular, nW_1 is the surface area.

It is the purpose of the present note to strengthen the mentioned uniqueness assertions by proving corresponding stability results. A brief hint to such stability results was given in [17], Section 6.4, note 3. While [17] was in the process of publication, two papers of Diskant [6] and Groemer [10], treating stability for particular cases of the general Brunn-Minkowski theorem, came to our knowledge. Since our method, combined with some more recent results, yields stronger and more general stability estimates, it seems of some interest to present the complete argument.

For the formulation of our results, some notation is required. By $\mathcal{K}^n$ we denote the set of convex bodies in $\mathbf{E}^n$ and by $\mathcal{K}(r,R)$ the set of convex bodies in $\mathcal{K}^n$ with inradius at least r and circumradius at most R; here $r, R > 0$ are given numbers. On $\mathcal{K}^n$ we introduce a metric $\delta_{\alpha,\beta}$ by

$$\delta_{\alpha,\beta}(K,L)^2 := \frac{\alpha}{n} \int\limits_{S^{n-1}} (h_K - h_L)^2 \, d\omega + \frac{\beta}{n} \int\limits_{S^{n-1}} |\nabla_{\mathrm{S}}(h_K - h_L)|^2 \, d\omega$$

for $K, L \in \mathcal{K}^n$. Here S^{n-1} denotes the unit sphere of $\mathbf{E}^n$, ω is spherical Lebesgue measure on S^{n-1}, h_K denotes the support function of K, and ∇_{S} is the gradient on S^{n-1} (the integral of $|\nabla_{\mathrm{S}}(h_K - h_L)|^2$ exists without additional assumptions on K or L, see Heil [13], Section 4). Finally, α, $\beta \ge 0$ are two parameters that have to satisfy the inequalities

$$\alpha + 2n\beta \le \frac{n+1}{n-1}, \quad \beta \le \frac{1}{n-1}, \quad \alpha + \beta > 0$$

(in order that (9) below be valid). The stability estimates below will be stated in terms of the metric $\delta_{\alpha,\beta}$. By specialization and application of additional inequalities, this yields estimates in terms of other metrics. The special choice $\alpha = (n+1)/(n-1)$ and $\beta = 0$ gives

$$\delta_{\alpha,\beta}(K,L)^2 = \frac{n+1}{n(n-1)} \, \delta_2(K,L)^2,$$

where δ_2 is the L_2-metric on $\mathcal{K}^n$. If δ denotes the Hausdorff metric on $\mathcal{K}^n$, then

$$\delta_2(K,L)^2 \ge C_n \mathrm{diam}\,(K \cup L)^{1-n} \delta(K,L)^{n+1} \tag{3}$$

and

$$\delta_2(K, B_K)^2 \geq C_n' b(K)^{(1-n)/2} \delta(K, B_K)^{(n+3)/2}. \tag{4}$$

Here diam denotes the diameter, $b(K)$ is the mean width of K, and B_K is the Steiner ball of K, that is, the ball that has the same Steiner point and mean width as K. The constants C_n, C_n' depend only on n and are explicitly known (see [17], Lemmas 6.6.4 and 6.6.5). Inequalities (3) and (4) are due, respectively, to Vitale [18] and to Groemer & Schneider [12]. Fuglede [7] has a result (Lemma 1.4) that can be used in the special case $\alpha = 0$. The choice

$$\alpha = \beta = \frac{n+1}{(n-1)(2n+1)}$$

gives

$$\delta_{\alpha,\beta}(K, L)^2 = \frac{n+1}{n(n-1)(2n+1)} \, \delta_W(K, L)^2$$

where the 'Sobolev distance' δ_W is defined by

$$\delta_W(K, L)^2 := \int_{S^{n-1}} (h_K - h_L)^2 \, d\omega + \int_{S^{n-1}} |\nabla_S (h_K - h_L)|^2 \, d\omega.$$

For $n = 2$ and $K, L \in \mathcal{K}^2$, Arnold & Wellerding [2] have proved that

$$\delta_W(K, L)^2 \geq (\pi \coth \pi)^{-1} \delta(K, L)^2$$

and that the constant $(\pi \coth \pi)^{-1} \approx 0.3171$ is best possible.

As we want to estimate distances not between individual convex bodies but between homothety classes, we introduce a suitable normalization. For a convex body $K \in \mathcal{K}^n$ of positive dimension we define

$$\bar{K} := \frac{1}{b(K)}(K - s(K)),$$

where $b(K)$ is the mean width and $s(K)$ is the Steiner point of K.

We assume in the following that a number $0 \leq \epsilon_0 < 1$, positive numbers r, R and an integer $m \in \{2, \ldots, n-1\}$ are given. The constants $c_1, \ldots, c_{11}$ appearing below depend on n, ϵ_0, r, R. The existence of these constants will in each case be easy to see, due to the fact that the convex bodies occurring are either elements of $\mathcal{K}(r, R)$ or balls and that the mixed volume is monotone in each argument. We don't care keeping track of the size of these constants, because it is the order of the estimates that is essential, and as long as this is not optimal, the multiplicative constants are of little interest.

The common assumption for the following stability estimates is that the general Brunn-Minkowski inequality (2) holds approximately with equality, more precisely, that

$$\begin{aligned}&V(K_\lambda, \ldots, K_\lambda, K_{m+1}, \ldots, K_n)^{1/m} \leq \\ &(1-\lambda)V(K, \ldots, K, K_{m+1}, \ldots, K_n)^{1/m} + \lambda V(L, \ldots, L, K_{m+1}, \ldots, K_n)^{1/m} + \epsilon\end{aligned} \tag{5}$$

for some $\epsilon \in [0, \epsilon_0]$ and some $\lambda \in (0,1)$.

Theorem 1. *If $K, L \in \mathcal{K}(r,R)$, $K_\lambda = (1-\lambda)K + \lambda L$, $K_{m+1} = \ldots = K_n = B$ and if* (5) *holds, then*

$$\delta_{\alpha,\beta}(\bar{K}, \bar{L})^2 \leq c_1 \left(\frac{\epsilon}{\lambda^m (1-\lambda)^m} \right)^{(1/2)^{m-2}}. \tag{6}$$

If we choose, e.g., $\lambda = 1/2$ (or if we assume that (5) holds for all $\lambda \in [0,1]$), then (6) together with (3) yields an estimate of the form

$$\delta(\bar{K}, \bar{L}) \leq c_2 \epsilon^q \qquad \text{with} \qquad q = \frac{1}{(n+1)2^{m-2}}. \tag{7}$$

Such an estimate is said to be *of order q*. The only general result of this type known before is an estimate of Diskant ([6], Theorem 3.3.1), which is of order

$$q = \frac{(m-1)!m!}{4(m-1)^{n+1-m}(n!)^2}$$

and thus considerably weaker if n is large compared to m. Diskant's estimate is only proved under the additional assumptions that $n \geq 5$, $m \geq 4$ and that K and L have orthogonal projections on some hyperplane that are Hausdorff distance at most ϵ apart. For the special case $m = 2$, an estimate of type (6) was proved by Groemer [10]. For $m = n-1$, an estimate of type (7) that is of order $1/4n^3$, but obtained under additional assumptions only, is due to Diskant [5] (see also [6], Theorem 3.5.1).

If the body L, too, is a ball, then (6) can be improved considerably.

Theorem 2. *If $K \in \mathcal{K}(r,R)$, $K_\lambda = (1-\lambda)K + \lambda B$, $K_{m+1} = \ldots = K_n = B$ and if* (5) *holds, then*

$$\delta_{\alpha,\beta}(\bar{K}, \bar{B})^2 \leq c_3 \frac{\epsilon}{\lambda^m (1-\lambda)^m}. \tag{8}$$

Together with (4), inequality (8) yields an estimate

$$\delta(K, B_K) \leq c_4 \left(\frac{\epsilon}{\lambda^m (1-\lambda)^m} \right)^{2/(n+3)}.$$

A weaker estimate, namely of order $1/(n!2^{n-2})$, was obtained by Diskant [4] (see also [6], Theorem 3.2.2).

So far we have assumed that the bodies $K_{m+1}, \ldots, K_n$ are balls. Without any assumptions on $K_{m+1}, \ldots, K_n$, equality in (2) for some $\lambda \in (0,1)$ does not imply that K and L are homothetic; a fortiori, no stability result can hold without strong restrictions on $K_{m+1}, \ldots, K_n$. A suitable restriction that does lead to stability is η-smoothness. The convex body K is said to be η-smooth, for $\eta > 0$, if the ball ηB is a summand of K, equivalently if ηB can roll freely in K (see, e.g., [17], Theorems 3.2.2 and 3.2.9).

Theorem 3. *If $K, L, K_{m+1}, \dots, K_n \in \mathcal{K}(r, R)$, K_i is η_i-smooth for some $\eta_i > 0$ ($i = m+1, \dots, n$), and if (5) holds, then*

$$\delta_{\alpha,\beta}(\bar{K}, \bar{L})^2 \leq \frac{c_5}{\eta_{m+1} \cdots \eta_n} \left(\frac{\epsilon}{\lambda^m (1-\lambda)^m} \right)^{(1/2)^{m-2}}.$$

2. Proofs

For given convex bodies $K, L \in \mathcal{K}^n$ (of positive dimension, without loss of generality), we use the abbreviation

$$V_{ij} := V(\underbrace{K, \dots, K}_{i}, \underbrace{L, \dots, L}_{j}, \underbrace{B, \dots, B}_{n-i-j}).$$

We need some results from Schneider [15]. From Lemma 1 of that paper it follows as there on p. 55, but avoiding the special choice $\beta = 0$, that

$$\frac{V_{20}}{b(K)^2} - \frac{2V_{11}}{b(K)b(L)} + \frac{V_{02}}{b(L)^2} + \delta_{\alpha,\beta}(\bar{K}, \bar{L})^2 \leq 0. \tag{9}$$

To put this in a more convenient form, one may either use the identity $2(b - \sqrt{ac}) + (a - 2b + c) = (\sqrt{a} - \sqrt{c})^2$ to obtain

$$V_{11} - \sqrt{V_{20} V_{02}} \geq \frac{1}{2} b(K) b(L) \delta_{\alpha,\beta}(\bar{K}, \bar{L})^2, \tag{10}$$

or the identity $b^2 - ac + c(a - 2b + c) = (b - c)^2$ to obtain

$$V_{11}^2 - V_{20} V_{02} \geq V_{02} b(K)^2 \delta_{\alpha,\beta}(\bar{K}, \bar{L})^2. \tag{11}$$

For $\beta = 0$ this inequality was independently obtained by Goodey & Groemer [8], in essentially the same way. Of course, the bodies K and L may be interchanged here, so that

$$V_{11}^2 - V_{20} V_{02} \geq \max\{V_{02} b(K)^2, V_{20} b(L)^2\} \delta_{\alpha,\beta}(\bar{K}, \bar{L})^2.$$

Now let $K, L, K_{m+1}, \dots, K_n \in \mathcal{K}(r, R)$ be given, set $\mathcal{C} := (K_{m+1}, \dots, K_n)$ and define

$$V_{(i)} := V(\underbrace{K, \dots, K}_{m-i}, \underbrace{L, \dots, L}_{i}, \mathcal{C})$$

for $i = 0, \dots, m$. If (5) holds, then the inequality $x^m - y^m \leq m x^{m-1}(x - y)$ for $x \geq y \geq 0$ yields

$$\begin{aligned} F_m(K, L, \mathcal{C}, \lambda) :=& V(K_\lambda, \dots, K_\lambda, \mathcal{C}) \\ & - \left[(1-\lambda) V(K, \dots, K, \mathcal{C})^{1/m} + \lambda V(L, \dots, L, \mathcal{C})^{1/m} \right]^m \\ \leq& m V(K_\lambda, \dots, K_\lambda, \mathcal{C})^{(m-1)/m} \epsilon \leq c_6 \epsilon. \end{aligned} \tag{12}$$

Expanding the mixed volume $V(K_\lambda,\ldots,K_\lambda,\mathcal{C})$ and using the binomial theorem we obtain

$$F_m(K,L,\mathcal{C},\lambda)=\sum_{i=0}^{m}\binom{m}{i}(1-\lambda)^{m-i}\lambda^i\left[V_{(i)}-V_{(0)}^{(m-i)/m}V_{(m)}^{i/m}\right].$$

It is a consequence of the Aleksandrov-Fenchel inequality (in particular, inequality (6.4.5) in [17]) that each summand on the right-hand side is non-negative, hence

$$F_m(K,L,\mathcal{C},\lambda)\geq\binom{m}{i}(1-\lambda)^{m-i}\lambda^i\left[V_{(i)}-V_{(0)}^{(m-i)/m}V_{(m)}^{i/m}\right] \tag{13}$$

for $i=1,\ldots,m-1$.

In the special case $m=2$ we even have an equality,

$$F_2(K,L,\mathcal{C},\lambda)=2(1-\lambda)\lambda\left[V_{(1)}-V_{(0)}^{1/2}V_{(2)}^{1/2}\right].$$

If we further specialize this to $\mathcal{C}=(B,\ldots,B)$, then the right-hand side is equal to

$$2(1-\lambda)\lambda\left[V_{11}-V_{20}^{1/2}V_{02}^{1/2}\right],$$

hence (10) yields

$$F_2(K,L,\mathcal{C},\lambda)\geq(1-\lambda)\lambda b(K)b(L)\delta_{\alpha,\beta}(\bar{K},\bar{L})^2.$$

For $\delta_{\alpha,\beta}=\sqrt{(n+1)/n(n-1)}\,\delta_2$, this is inequality (10) of Groemer [10].

Returning to the general case, we deduce from (12) and (13) that

$$V_{(i)}\leq V_{(0)}^{(m-i)/m}V_{(m)}^{i/m}+\frac{c_6\,\epsilon}{\binom{m}{i}(1-\lambda)^{m-i}\lambda^i}.$$

Multiplying the two inequalities that result for $i=1$ and $i=m-1$, we deduce

$$V_{(1)}V_{(m-1)}\leq V_{(0)}V_{(m)}+c_7\frac{\epsilon}{\lambda^m(1-\lambda)^m}. \tag{14}$$

As shown in Schneider [15], pp. 56 - 57, this implies

$$V_{11}^2-V_{20}V_{02}\leq c_8\left(\frac{\epsilon}{\lambda^m(1-\lambda)^m}\right)^{(1/2)^{m-2}}. \tag{15}$$

Together with (11), this proves (6) and thus Theorem 1.

It was also shown in Schneider [15] (see inequality (4.2) for $j=p=m-1$) that (14) implies

$$V_{(m-1)}^2-V_{(m-2)}V_{(m)}\leq c_9\frac{\epsilon}{\lambda^m(1-\lambda)^m}.$$

If now not only $K_m, \dots, K_n$ but also L is equal to the unit ball B, then the left-hand side is equal to $V_{11}^2 - V_{20}V_{02}$. This together with (11) proves Theorem 2.

For the proof of Theorem 3 we define

$$\begin{aligned}&\Delta(K, L, C_1, \dots, C_{n-2})\\ &:= V(K, L, C_1, \dots, C_{n-2})^2 - V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2})\end{aligned}$$

for $K, L, C_1, \dots, C_{n-2} \in \mathcal{K}^n$. Under the assumptions of Theorem 3, one can show that

$$\Delta(K, L, \underbrace{B, \dots, B}_{m-2}, K_{m+1}, \dots, K_n) \leq c_{10} \left(\frac{\epsilon}{\lambda^m (1-\lambda)^m} \right)^{(1/2)^{m-2}}. \tag{16}$$

This inequality is obtained precisely in the way that (15) was obtained in the proof of Theorem 1. For the passage from (14) to (15) we referred to Schneider [15], where the special assumption $K_{m+1} = \dots = K_n = B$ was made. However, one checks easily that this assumption is not used in the proof, hence (16) holds. If now $\eta_i B$ is a summand of K_i for $i = m+1, \dots, n$, then it follows from Theorem 3 of Arnold [1][1] that

$$\Delta(K, L, \underbrace{B, \dots, B}_{m-2}, K_{m+1}, \dots, K_n) \geq c_{11} \left(\prod_{i=m+1}^{n} \eta_i \right) \Delta(K, L, B, \dots, B).$$

A weaker inequality of this type was proved by Schneider [16], Theorem 1. Since

$$\Delta(K, L, B, \dots, B) = V_{11}^2 - V_{20}V_{02},$$

another application of (11) completes the proof of Theorem 3.

References

[1] R. Arnold, On the Aleksandrov-Fenchel inequality and the stability of the sphere. Preprint.

[2] R. Arnold and A. Wellerding, On the Sobolev distance of convex bodies. *Aequationes Math.* **44** (1992), 72 - 83.

[3] V. I. Diskant, Stability of the solution of the Minkowski equation. (Russian) *Sibirskii Mat. Zh.* **14** (1973), 669 - 673. English translation: *Siberian Math. J.* **14** (1973), 466 - 469.

[4] V. I. Diskant, The stability of the solutions of the generalized Minkowski equations for a ball. (Russian) *Ukrain. Geom. Sb.* **18** (1975), 53 - 59.

[5] V. I. Diskant, Stability of the solution of the Minkowski equation for the surface area of convex bodies. (Russian) *Ukrain. Geom. Sb.* **25** (1982), 43 - 51.

[1] The author thanks Randolf Arnold for sending him a preprint of his paper [1].

[6] V. I. Diskant, Refinements of an isoperimetric inequality and stability theorems in the theory of convex bodies. (Russian) *Trudy Inst. Mat. (Novosibirsk)* **14** (1989), Sovrem. Probl. Geom. Analiz., 98 - 132.

[7] B. Fuglede, Stability in the isoperimetric problem for convex or nearly spherical domains in R^n. *Trans. Amer. Math. Soc.* **314** (1989), 619 - 638.

[8] P. R. Goodey and H. Groemer, Stability results for first order projection bodies. *Proc. Amer. Math. Soc.* **109** (1990), 1103- 1114.

[9] H. Groemer, On the Brunn-Minkowski theorem. *Geom. Dedicata* **27** (1988), 357 - 371.

[10] H. Groemer, On the stability of a Brunn-Minkowski type inequality. Preprint.

[11] H. Groemer, Stability of geometric inequalities. In: *Handbook of Convex Geometry* (eds. P. M. Gruber and J. M. Wills), North-Holland, Amsterdam 1993 (to appear).

[12] H. Groemer and R. Schneider, Stability estimates for some geometric inequalities. *Bull. London Math. Soc.* **23** (1991), 67 - 74.

[13] E. Heil, Extensions of an inequality of Bonnesen to D-dimensional space and curvature conditions for convex bodies. *Aequationes Math.* **34** (1987), 35 - 60.

[14] K. Leichtweiß, *Konvexe Mengen.* Springer, Berlin etc. 1980.

[15] R. Schneider, Stability in the Aleksandrov-Fenchel-Jessen theorem. *Mathematika* **36** (1989), 50 - 59.

[16] R. Schneider, A stability estimate for the Aleksandrov-Fenchel inequality, with an application to mean curvature. *Manuscripta Math.* **69** (1990), 291 - 300.

[17] R. Schneider, *Convex bodies: the Brunn-Minkowski theory.* (Encyclopedia of Math. and its Appl., vol. 44) Cambridge University Press 1993 (to appear).

[18] R. A. Vitale, L_p metrics for compact, convex sets. *J. Approx. Th.* **45** (1985), 280 - 287.