Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 34 (1993), No. 1, 63-75.

Cube Tilings as Contributions of Algebra to Geometry

To the memory of Professor Ott-Heinrich Keller

Sándor Szabó
Department of Mathematics, University of Bahrain
P. O. Box 32038, State of Bahrain

1. Introduction

Talking about space filling cube systems a family of cubes abutting face to face comes to mind first. If the cubes are unit cubes, then in a suitable coordinate system the cube centers have integer coordinates. In two dimension a family of squares that tiles the plane is the analogous structure. The most common square tiling which probably occurs our mind is a checker board like arrangement of squares. The above systems consist of parallel cubes of equal side length. Further these systems are extremely regular. We might wonder: As the other extreme how irregular how chaotic such a cube system can be? Roughly speaking this is the central problem of this paper. What good may come out of wasting time on this not particularly exciting looking question? Following an argument of H. Minkowski we will describe how solving a system of linear inequalities in integers is related to high dimensional space filling lattice like cube systems.

In 1900 Minkowski formulated a challenging conjecture. An n-dimensional lattice like cube tiling cannot be arbitrarily irregular, it must contain at least two cubes having an $(n-1)$-dimensional face in common. We call two such cubes a *twin*.

Consider a family of cubes with the following properties:

(i) the cubes are translates of each other,
(ii) the translation vectors form a lattice,
(iii) the interiors of the cubes are disjoint,
(iv) each point of the space which is not on the boundary of any cube is contained by exactly one cube.

Minkowski's conjecture now reads that properties (i)-(iv) force the existence of a twin.

Two further conjectures motivated by Minkowski's problem will be considered, Keller's and Furtwängler's conjecture. The occurrence of a twin in a cube tiling can be verified in dimensions 1,2, and 3 even if we drop the condition of latticity using a collection of model squares and cubes.

In 1930 O.-H. Keller sharpened Minkowski's conjecture dropping the condition of latticity. In 1936 Ph. Furtwängler suggested to replace condition (iv) by

0138-4821/93 \$ 2.50

(v) each point not lying on the boundary of any cubes is contained by exactly k cubes. In other words Furtwängler conjectured that in a multiple lattice like cube system there is at least one twin.

An interesting algebraic tool due to G. Hajós made possible to settle the above problems. Advertising this application of the algebra in geometry, which is analogous to the algebraic solution of ruler and compass constructions, is also a purpose of this paper.

2. Minkowski's conjecture

We start with a nice geometric version of the pigeon hole principle known as Minkowski's convex body theorem. The ingredients of the 2-dimensional case of this theorem are a lattice whose basic parallelogram is of area 1 and a centrally symmetric closed convex disk of area 4 centered at a lattice point. The conclusion is that there is at least one additional lattice point in the disk.

A closed square of side length 2 centered at a lattice point satisfies the conditions of the theorem and so the square cannot contain only one lattice point. Reinterpreting this fact in arithmetical terms we get an interesting result on nontrivial solution of a system of linear inequalities.

Suppose that the lattice is spanned by the vectors

$$l_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad l_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}.$$

As the area of the parallelogram spanned by these vectors is 1, $\det(l_1, l_2) = \pm 1$. A typical lattice point is of form $x_1 l_1 + x_2 l_2$, where x_1, x_2 are integers. So if $\det(l_1, l_2) = \pm 1$, then there are integers x_1, x_2 such that $(x_1, x_2) \neq (0, 0)$ and

$$\begin{aligned} |a_{11}x_1 + a_{12}x_2| &\leq 1 \\ |a_{21}x_1 + a_{22}x_2| &\leq 1. \end{aligned}$$

In certain cases the existing nontrivial lattice point is on the boundary of the square. Then the $\leq$ signs cannot be replaced by $<$ signs. Call these types of lattices *critical.* Turn to the question: How can the critical lattices be characterized? Place a unit square parallel to the coordinate axes to each lattice point centered at the lattice point. The resulting square system contains two cubes with common interiors if and only if the interior of the large square contains a nontrivial lattice point.

Therefore a lattice is critical if and only if the corresponding square system is a tiling.

If Minkowski's conjecture holds then an n-dimensional lattice like cube tiling has a hierarchical structure. If the tiling contain a twin, then by the latticity it contains an infinite column. In addition the whole tiling is a union of infinite columns. Intersecting the tiling by an $(n-1)$-dimensional plane perpendicular to the columns we get a smaller dimensional lattice like cube tiling and we can repeat the procedure. Turning back to the critical lattices we may say that a lattice is critical if and only if it has a "triangular shape" system of basis vectors. This is the arithmetical background of Minkowski's conjecture on lattice like cube tilings.

Figure 1

3. Algebraic reformulation

To handle high dimensional cube systems we badly need a good book-keeping tool. Consider a lattice like tiling of unit squares. Suppose that there are positive integers q_1, q_2 such that each square is a union of $q_1 q_2$ rectangles of side length $1/q_1$ and $1/q_2$. In addition the rectangles are members of a standard rectangle tiling. Let $l_1' = (1/q_1)e_1$, $l_2' = (1/q_2)e_2$ be basis vectors of the lattice L' of the rectangle tiling, where e_1, e_2 are the coordinate unit vectors. Let L be the lattice of the squares.

Figure 2

Each square and each rectangle can be described by its left lower corner. If each

$l' \in L'$ is uniquely expressible in the form

$$l' = l + (i_1/q_1)e_1 + (i_2/q_2)e_2, \qquad 0 \le i_1 \le q_1 - 1, \qquad 0 \le i_2 \le q_2 - 1,$$

if and only if the square system is a tiling. In other words the square system is a tiling if and only if the sum $L' = L + S_1 + S_2$ is direct, where

$$S_1 = \{0, (1/q_1)e_1, \ldots, ((q_1 - 1)/q_1)e_1\},$$

$$S_2 = \{0, (1/q_2)e_2, \ldots, ((q_2 - 1)/q_2)e_2\}.$$

Here L' and L are abelian groups with respect to the vector addition. Direct sum of subsets is a bit uncommon construction contrary to the direct sum of subgroups.

Consider the factor group $G = L'/L$ (written multiplicatively with identity element e) and let a_1 and a_2 be the cosets containing $(1/q_1)e_1$ and $(1/q_2)e_2$ respectively. Applying the natural homomorphism $f : L' \to G$ to the direct sum $L' = L + S_1 + S_2$ we have the direct product $G = A_1 A_2$, where

$$A_1 = \{e, a_1, a_1^2, \ldots, a_1^{q_1 - 1}\}, \qquad A_2 = \{e, a_2, a_2^2, \ldots, a_2^{q_2 - 1}\}.$$

Given G, A_1, and A_2 reversing the previous argument we can construct a lattice like square system. This system is a tiling if and only if the product is $A_1 A_2$ is direct and gives G. If the square system contains a twin then e_1 or e_2 belongs to L, that is, $a_1^{q_1}$ or $a_2^{q_2}$ is equal to e.

Call the set $A_i = \{e, a_i, a_i^2, \ldots, a_i^{q_i - 1}\}$ a *cyclic subset*. The algebraic reformulation of Minkowski's conjecture reads as follows:
If the finite abelian group G is a direct product of cyclic subsets $A_1, \ldots, A_n$, then $a_i^{q_i} = e$ for some i, $1 \le i \le n$.

We have seen the equivalence only for rational lattices. Hajós [5] showed that in fact it is enough to consider only rational lattices.

4. Furtwängler's conjecture

The algebraic reformulation gives an algebraic equivalent for Furtwängler's conjecture as well.

If each element g of a finite abelian group G is expressible in precisely k distinct ways as a product of elements coming from the cyclic subsets $A_1, \ldots, A_n$ respectively, then $a_i^{q_i} = e$ for some i, $1 \le i \le n$.

Here elements of G occur with multiplicity so introducing the group ring $Z(G)$ looks justified. As $Z(G)$ consists of all formal linear combinations of elements of G with integer coefficients, the subset A of G corresponds to the element

$$\overline{A} = \sum_{a \in A} a$$

of $Z(G)$. The equation

$$k\overline{G} = \overline{A}_1 \cdots \overline{A}_n \tag{1}$$

in $Z(G)$ concisely expresses the fact that each g in G can be expressed in k distinct ways as products of elements of $A_1, \ldots, A_n$. The extra addition operation also looks useful when working in $Z(G)$.

To illustrate this we verify Furtwängler's conjecture for $n = 3$. Start with the equation

$$k\overline{G} = \overline{A}_1\overline{A}_2\overline{A}_3 \tag{2}$$

and multiply both sides by $(e - a_1)(e - a_2)(e - a_3)$. From the resulting

$$0 = (e - a_1^{q_1})(e - a_2^{q_2})(e - a_3^{q_3})$$

equation after introducing the notations $x_1 = a_1^{q_1}$, $x_2 = a_2^{q_2}$, $x_3 = a_3^{q_3}$ and multiplying out we have

$$x_1 + x_2 + x_3 + x_1x_2x_3 = e + x_1x_2 + x_1x_3 + x_2x_3.$$

The identity element e on the right side must be equal to an element on the left side. If this is one of x_1, x_2, x_3, then we are done. Thus we assume that $e = x_1x_2x_3$ and so

$$x_1 + x_2 + x_3 = x_1x_2 + x_1x_3 + x_2x_3.$$

Here x_1 on the left side must be equal to an element on the right side. If this is x_1x_2 or x_1x_3, then we are done. So we assume that $x_1 = x_2x_3$. From $e = x_1x_2x_3$ and $x_1 = x_2x_3$ we have $x_1^2 = e$ and similarly $x_2^2 = e$ and $x_3^2 = e$. To get

$$0 = \overline{A}_1(e - a_2^{q_2})(e - a_3^{q_3}) = \overline{A}_1(e - x_2)(e - x_3)$$

multiply (2) by $(e - a_2)(e - a_3)$. After reordering we obtain $\overline{A}_1(x_2 + x_3) = \overline{A}_1(e + x_2x_3)$. As $x_2x_3 = x_1$, the generatum of a_1, $\langle a_1\rangle$ contains each term occurring on the right side. So it contains each term on the left side. In particular $x_2, x_3 \in \langle a_1\rangle$.

The equation $x^2 = e$ has one or two solutions in a cyclic group depending on the order of the group is odd or even. Apply this observation in the case $x_2^2 = e$ and $x_3^2 = e$. If $x_2 = e$ or $x_3 = e$, then we are done. Therefore we may assume that both x_2 and x_3 are equal to the second order element of $\langle a_1\rangle$. Then $x_1 = x_2x_3 = e$ is the desired result. This completes the proof.

L. Rédei recognized that characters of G are also effective tools to study equation (1). If $A \in Z(G)$

$$A = \sum_{g\in G} \alpha(g)g,$$

and χ is a character of G, then let

$$\chi(A) = \sum_{g\in G} \alpha(g)\chi(g).$$

Let χ_i be the i-th character and g_j the j-th element of G. By the standard orthogonality relations of the characters the columns of the matrix $\chi_i(g_j)$ are linearly independent.

From this we have that if $A, B \in Z(G)$ and $\chi(A) = \chi(B)$ for each character χ of G, then the corresponding coefficients $\alpha(g)$ and $\beta(g)$ in A and B are equal, that is, $A = B$.

Equation (1) holds if and only if for each character χ of G

$$\chi(k\overline{G}) = \chi(\overline{A}_1 \cdots \overline{A}_n) = \chi(\overline{A}_1) \cdots \chi(\overline{A}_n).$$

For the principal character χ of G we have $k|G| = |A_1| \cdots |A_n| = q_1 \cdots q_n$. For other characters we have $0 = \chi(\overline{A}_1) \cdots \chi(\overline{A}_n)$. The set of characters χ of G for which $\chi(\overline{A}) = 0$ we call the *annihilator* of the set A and we denote it by $\mathrm{Ann}(A)$. Thus (1) holds with some integer k if and only if each nonprincipal character is in $\mathrm{Ann}(A_1) \cup \cdots \cup \mathrm{Ann}(A_n)$ and $q_1 \cdots q_n$ is a multiple of $|G|$.

The annihilator of a cyclic subset A_i can be described easily. Since

$$\chi(A_i) = \chi(e) + \chi(a_i) + \cdots + \chi(a_i^{q_i - 1}),$$

$\chi(A_i) = q_i$ if $\chi(a_i) = 1$ and

$$\chi(A_i) = \frac{1 - \chi(a_i^{q_i})}{1 - \chi(a_i)}$$

if $\chi(a_i) \neq 1$. Thus $\mathrm{Ann}(A_i)$ consists of all characters χ of G for which $\chi(a_i) \neq 1$ and $\chi(a_i^{q_i}) = 1$.

We have the following character test for Furtwängler's conjecture:
If $a_1, \ldots, a_n$ are elements of a finite abelian group G and $q_1, \ldots, q_n$ are positive integers such that
(i) $q_1 \cdots q_n$ is a multiple of $|G|$,
(ii) for each nonprincipal character χ of G there is an i, $1 \leq i \leq n$ with $\chi(a_i) \neq 1$ and $\chi(a_i^{q_i}) = 1$,
then $a_i^{q_i} = e$ for some i, $1 \leq i \leq n$.

This test helps to find counter examples for Furtwängler's conjecture. Let G be a finite abelian group of invariants $(2, 2, 3)$ with basis elements x, y, z, where $|x| = |y| = 2$ and $|z| = 3$. Choose the a_i elements and q_i numbers in the next way.

i	1	2	3	4	5
a_i	xy	yz	xz	yz	xyz
q_i	2	2	3	3	3

Here $q_1 \cdots q_5 = 9|G|$ and so (i) is satisfied.

To verify (ii) choose a nonprincipal character χ of G. First compute $a_i^{q_i}$ and $\chi(a_i^{q_i})$.

i	1	2	3	4	5
$\chi(a_i^{q_i})$	$\chi(z^2)$	$\chi(z^2)$	$\chi(x)$	$\chi(y)$	$\chi(xy)$

If $\chi(z) = 1$, then $\chi(x) \neq 1$ or $\chi(y) \neq 1$ and so $\chi(a_i) \neq 1$ and $\chi(a_i^{q_i}) = 1$ for $i = 1$ or $i = 2$.

If $\chi(z) \neq 1$, then one of $\chi(x), \chi(y), \chi(xy)$ is 1 and so $\chi(a_i) \neq 1$ and $\chi(a_i^{q_i}) = 1$ for $i = 3$, $i = 4$, $i = 5$. Thus (ii) is satisfied. However $a_i^{q_i} \neq e$ for each i, $1 \leq i \leq 5$.

The first counterexample for Furtwängler's conjecture is from 1938 and due to G. Hajós. Three years later in 1941 he proved Minkowski's conjecture. The outline of his proof is the subject of the next section.

In 1979 R. M. Robinson characterized all k, n values for which Furtwängler's conjecture fails. For these values $n \geq 4$.

- If $n = 4$, then k is a multiple of a square of an odd prime.
- If $n = 5$, then $k = 3$ or $k \geq 5$.
- If $n \geq 6$, then $k \geq 2$.

In 1990 S. Szabó showed that the algebraic form of Furtwängler's conjecture does not hold for finite abelian groups except for cyclic groups and groups of invariants $(p^\alpha, p, \ldots, p)$, where p is a prime and α is a positive integer.

5. Proof of Minkowski's conjecture

From the equation

$$\overline{G} = \overline{A}_1 \cdots \overline{A}_n, \tag{3}$$

where $A_1, \ldots, A_n$ are cyclic subsets of the finite abelian group G we would like to deduce that $a_i^{q_i} = e$ for some i, $1 \leq i \leq n$. The $n = 1$ case is trivial so we proceed by induction on n. Let $n \geq 2$. We may assume that $a_i^{q_i} \neq e$, for each i, $1 \leq i \leq n$ since otherwise we are done. Multiplying (3) by $(e - a_n)$ we get

$$0 = \overline{A}_1 \cdots \overline{A}_{n-1}(e - a_n^{q_n}) = \overline{A}_1 \cdots \overline{A}_{n-1}(e - x_n).$$

This equation may hold after canceling some factors. After canceling and reindexing the factors if this is necessary we get

$$0 = \overline{A}_1 \cdots \overline{A}_t(e - x_n), \tag{4}$$

where none of the factors can be canceled without violating the equation. As $e - x_n \neq 0$, $t \geq 1$ and $e - x_n$ cannot be canceled since $0 = \overline{A}_1 \cdots \overline{A}_t$ contradicts to (3).

Let $H = A_1 \cdots A_t$. If by chance H were a subgroup of G, then we had an equation $\overline{H} = \overline{A}_1 \cdots \overline{A}_t$, where $t \leq n - 1$ and so by the inductive assumption we were done. In any case $\langle A_1, \ldots, A_t \rangle = \langle a_1, \ldots, a_t \rangle$ is a subgroup K of G. We would like to show that $H = K$. We will show that $|H|$ divides $|K|$ and $|K|$ has at most as many prime factors as does $|H|$. As $H \subset K$ it will be enough.

Let $B = A_{t+1} \cdots A_n$. The product HB is direct and gives G. Restricting the factorization $G = HB$ to K we have $K = G \cap K = H(B \cap K)$. Since the product $H(B \cap K)$ is direct, we have that $|H|$ divides $|K|$.

In order to show that $|K|$ has at most as many prime factors as $|H|$ Hajós studied vanishing products of form

$$0 = B\overline{A}_1 \cdots \overline{A}_u(e - x_1) \cdots (e - x_v), \tag{5}$$

where B is an element of $Z(G)$, $A_1,\dots,A_u$ are cyclic subsets and $x_1,\dots,x_v$ are elements of G. Further the product is minimal in the sense that none of the factors $\overline{A}_1,\dots,\overline{A}_u, e-x_1,\dots,e-x_v$ can be canceled without destroying the equation.

Assign a positive integer m_i to each $\overline{A}_i$. Namely let m_i be the number of the not necessarily distinct prime factors of q_i. We denote the set of elements of G whose coefficients in B are not zero, that is, the support of B by B'. Finally let $r(B',a_1,\dots,a_u,x_1,\dots,x_v)$ be the number of the not necessarily distinct prime factors of the order of the subgroup $\langle B',a_1,\dots,a_u,x_1,\dots,x_v\rangle$.

Hajós proved that from (5) it follows

$$r(B',a_1,\dots,a_u,x_1,\dots,x_v)-r(B')<m_1+\cdots+m_u+v.$$

In fact this is the most essential part of the whole proof. This result can be applied to (4) with $B=e$ and provides $r(a_1,\dots,a_t,x_n)<m_1+\cdots+m_t+1$, that is, $r(a_1,\dots,a_t)\le m_1+\cdots+m_t$. Hence $|K|$ has not more prime factors than does $|H|$. This completes the proof.

Since Hajós proved Minkowski's conjecture a theory of factoring finite abelian groups by subsets has developed. Surveying this field could be the subject of another paper. However we cannot resist to mention a nice result of L. Rédei.

Note that A_i coincides with $\langle a_i\rangle$ if and only if $a_i^{q_i}=e$. Thus by Hajós' theorem from factorization $G=A_1\cdots A_n$ it follows that one of the factors is a subgroup of G. Secondly note that a nonsubgroup cyclic subset of composite order is a direct product of two smaller nonsubgroup cyclic subsets. (This decomposition is not uniquely determined.) Therefore Hajós' theorem is equivalent to its special case when the factors are of prime cardinality. This form of the theorem was generalized by L. Rédei in 1965 removing the condition of cyclicity of the factors.

If a finite abelian group is factored into subsets of prime cardinality each of which contains the identity element, then at least one of the factors is a subgroup of the group.

6. Keller's conjecture

Turn to systems of n-dimensional parallel unit cubes that tile n-space. By Keller's conjecture such a system cannot be arbitrarily chaotic it must contain at least one twin. This conjecture also can be formulated in an algebraic way. We describe the details in connection with 2-dimensional tilings. Consider a family of parallel unit squares. We code each square by the coordinates of its left lower corner. Taking the integer parts of these coordinates we assign a member of the standard square tiling to each square. A family of squares tiles the plane if and only if

(i) the interiors are disjoint,

(ii) each element of the standard square tiling is assigned to exactly one square of the family.

Take the four squares of the standard square tiling whose left lower corners are $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

Call these four squares *selectors*. The selectors can be extended to a standard square tiling by translating the union of the four squares. More generally any four unit squares

Figure 3

with disjoint interiors can be extended to a periodic tiling of the plane if the selectors are assigned to them. In particular from a square tiling the selectors select four squares that can be extended to a periodical tiling. The resulting tiling will contain twins only if the original did. Hence we may restrict our attention to periodical tilings and in a periodical tiling it is enough to deal with the four squares produced by the selectors.

Suppose that there are positive integers q_1, q_2 such that each square of the tiling is a union of $q_1 q_2$ rectangles of side lengths $1/q_1$, $1/q_2$ that are from a standard tiling of rectangles. The selected four squares can also be described in terms of finite abelian groups.

Let G be an abelian group with basis elements a_1, a_2 of orders $2q_1, 2q_2$ respectively. A factorization of form $G = HA_1A_2$, where $|H| = 4$ and

$$A_1 = \{e, a_1, a_1^2, \dots, a_1^{q_1-1}\}, \qquad A_2 = \{e, a_2, a_2^2, \dots, a_1^{q_2-1}\}$$

corresponds to a periodic square tiling and conversely. The tiling contains a twin if and only if $a_1^{q_1}$ or $a_2^{q_2}$ is of form $h'h^{-1}$, where $h', h \in H$.

To complete the reformulation we show that each square tiling can be rationalized. Each square has joining squares that block the square against a translation parallel with the first coordinate axis. This property generates an equivalence relation among the squares. Equivalence classes slide freely on each other parallel to the first coordinate axis. In addition the differences between the first coordinates of the squares within an equivalence class are integers and so rational numbers. We can repeat the argument in connection with the second coordinates as well.

The algebraic form of Keller's conjecture is the following:
Let G be an abelian group with basis elements $a_1, \dots, a_n$ of orders $2q_1, \dots, 2q_n$ respectively. If $G = HA_1 \cdots A_n$ is a factorization, where $|H| = 2^n$ and $A_i = \{e, a_i, a_i^2, \dots, a_i^{q_i-1}\}$, then $HH^{-1} \cap \{a_1^{q_1}, \dots, a_n^{q_n}\} \neq \emptyset$.

We should remark that a factorization $G = HA_1 \cdots A_n$ of any finite abelian group G corresponds to a periodical tiling of n-dimensional cubes. In general the periods form a less regular lattice and the number of the cubes translated periodically is not necessarily 2^n. For instance if H is a subgroup of G, then the corresponding tiling is lattice like.

In 1940 O. Perron verified Keller's conjecture for $n \leq 6$. Although he did not use the algebraic reformulation, the next algebraic argument for $n = 3$ well illustrates the flavor of his proof.

From the factorization
$$G = HA_1A_2A_3 \tag{6}$$
we would like to deduce that $a_i^{q_i} = h'h^{-1}$, where $h', h \in H$. We assume that for $n = 2$ this has already been established. Intersecting $G = (HA_2)A_1A_3$ by $K = \langle a_1, a_3 \rangle$ we have the factorization $K = G \cap K = (HA_2 \cap K)A_1A_3$. As the $n = 2$ case is settled, we may assume that $h_1a_1^{q_1} = h_2a_2^{\alpha_2}$ since this is only a matter of indexing the factors. Here $0 \leq |\alpha_2| \leq q_2 - 1$ and $h_1, h_2 \in H$. As $a_1^{q_1} = a_1^{-q_1}$, we may assume that $0 \leq \alpha_2 \leq q_2 - 1$.

If $\alpha_2 = 0$, then we are done. So assume that $\alpha_2 \neq 0$. Multiplying by $a_2^{q_2-\alpha_2}$ we have $h_1a_1^{q_1}a_2^{q_2-\alpha_2} = h_2a_2^{q_2}$. As (6) is a factorization of G, $h_2a_2^{q_2}$ is uniquely expressible in the form
$$h_2a_2^{q_2} = h_3a_1^{\beta_1}a_2^{\beta_2}a_3^{\beta_3}, \qquad h_3 \in H,$$
$$0 \leq \beta_1 \leq q_1 - 1, \quad 0 \leq \beta_2 \leq q_2 - 1, \quad 0 \leq \beta_3 \leq q_3 - 1.$$
If $\beta_2 \neq 0$, then $h_2a_2^{q_2-\beta_2} = h_3a_1^{\beta_1}a_3^{\beta_3}$ is a contradiction. Thus $\beta_2 = 0$. If $\beta_1 \neq 0$, then $h_1a_1^{q_1-\beta_1}a_2^{q_2-\alpha_2} = h_3a_3^{\beta_3}$ is a contradiction. Hence $\beta_1 = 0$.

If $\beta_3 = 0$, then we are done. So assume that $\beta_3 \neq 0$. Multiplying by $a_3^{q_3-\beta_3}$ we have
$$h_1a_1^{q_1}a_2^{q_2-\alpha_2}a_3^{q_3-\beta_3} = h_2a_2^{q_2}a_3^{q_3-\beta_3} = h_3a_3^{q_3}.$$
Since (6) is a factorization $h_3a_3^{q_3}$ is uniquely expressible in the form
$$h_3a_3^{q_3} = h_4a_1^{\gamma_1}a_2^{\gamma_2}a_3^{\gamma_3}, \quad h_4 \in H,$$
$$0 \leq \gamma_1 \leq q_1 - 1, \quad 0 \leq \gamma_2 \leq q_2 - 1, \quad 0 \leq \gamma_3 \leq q_3 - 1.$$
As before we get $\gamma_3 = 0$, $\gamma_2 = 0$, $\gamma_1 = 0$. Therefore $h_3a_3^{q_3} = h_4$. This completes the proof.

The algebraic form of Keller's conjecture was verified for certain classes of finite abelian groups. For cyclic groups of order p^α by K. Seitz. For cyclic groups of order $p^\alpha q^\beta$ by A. D. Sands. For groups of invariants (p^α, p^β), $(p^\alpha, p, \ldots, p)$, $(p^\alpha, q, \ldots, q)$ by K. Corrádi and S. Szabó. For groups of type $(p^\alpha, p^\beta, p, \ldots, p)$ and $(p^\alpha, p^\beta, q, \ldots, q)$ by D. Moews. Here p and q are distinct primes and in the last case $p \geq 5$; furthermore α and β are positive integers.

In 1986 S. Szabó pointed out that to decide Keller's conjecture we may restrict our attention to abelian groups of invariants $(4, \ldots, 4)$. Geometrically speaking we may assume that the translation vectors between cubes are not only rational but the denominators of their coordinates are 1 or 2. The rationalization procedure that provides the integers $q_1, \ldots, q_n$ can be carried out such that the resulting $q_1, \ldots, q_n$ are powers of 2. In the corresponding factorization $G = HA_1 \cdots A_n$ the cyclic subsets $A_1, \ldots, A_n$ are direct products of cyclic subsets of cardinality 2. Thus we have a factorization $G = H\{e, a_1\} \cdots \{e, a_m\}$, where $m \geq n$. An m-dimensional cube tiling corresponds to this factorization. The selector cubes select 2^m cubes of this system that can be extended periodically to tile the whole m-space. The resulting tiling is described by the factorization $G' = H'\{e, a'_1\} \cdots \{e, a'_m\}$, where $|H'| = 2^m$ and the elements $a'_1, \ldots, a'_m$ of order 4 are basis elements of G'. Thus we have the following equivalent of Keller's conjecture.

Let G be the direct product of the cyclic groups of order 4 generated by $a_1, \ldots, a_n$ respectively. If $G = H\{e, a_1\} \cdots \{e, a_n\}$ is a factorization, then $HH^{-1} \cap \{a_1^2, \ldots, a_n^2\} \neq \emptyset$.

This factorization suggests a combinatorial approach. Let $B = \{e, a_1\} \cdots \{e, a_n\}$. Define a graph Γ_n on the elements of G. Let the elements g' and g of G be adjacent if a 2 occurs among the exponents in the basis representation of $g'g^{-1}$. If one exponent in this basis representation is 2 and the others are 0, then call the edge a twin edge. The exponents in the basis representation of elements of BB^{-1} can only be $0, 1, 3$. Call a subgraph a clique if it does not contain not adjacent vertices. Let H be the set of vertices of a clique of size 2^n in Γ_n. Then $HH^{-1} \cap BB^{-1} = \{e\}$ which means that the product HB is direct. On the other hand the cardinalities force that HB gives G. Therefore cliques of size 2^n in Γ_n correspond to cube tilings. Twin edges correspond to twin cubes.

Together with Γ_n it will be convenient to use Γ_n^* which is Γ_n without the twin edges. Keller's conjecture states that each clique of size 2^n contains at least one twin edge or equivalently if Γ_n^* contains a clique of size 2^n, then there is a counter example for the conjecture.

By inspection K. Corrádi and S. Szabó found, which has already been established by O. Perron, that each clique of size 2^n in Γ_n contains at least one twin edge for $n \leq 6$. A "typical" 2^n-clique in Γ_n is abundant in twin edges. This inspection brought the following extreme cliques to light.

$n=2$	$n=3$	$n=4$	$n=5$	
00	000	0000	00000	02322
12	102	0012	00012	21011
	210	0120	11132	21113
20	021	1320	00332	20303
32	222	1032	03132	21323
		2130	01020	23001
	230	0201	02100	23103
	023	2111	02112	23022
	302	0222	00213	22211
		2300	10211	23231
		2312	13020	23223
		2232	00230	
			02220	30211
		3032	21130	32303
		3320	21331	33020
		2113	02301	31132
		0203	12303	

Each element below the gap has an element above the gap such that the two nodes are connected with a twin edge and these are all the twin edges in the clique. Hence these cliques also defines "large" cliques in Γ_n^*. The next table summarizes the data.

n	2	3	4	5	6
number of twins	2	3	4	4	4
clique size in Γ_n^*	2	5	12	28	60
n	4	8	16	32	64

In 1992 J. C. Lagarias and P. W. Shor refuted Keller's conjecture by a clever construction. Mark the three zeros by a prime in the last three rows in the given 2^3-clique in Γ_3 and denote the resulting list by A. The 2^5-clique in Γ_5 listed above has 12 vectors having 0 in the first column, another 12 vectors having 2 in the first column, and 4 vectors having value 1 in the first column. Deleting the first column the resulting sets of vectors are denoted by S_0, $S_{0'}$, S_2 respectively. Adding 1 modulo 4 to the first column in S_0 and S_2 we get S_1 and S_3. The Γ_5^* clique property guarantees that

(i) $S_0, S_{0'}, S_1, S_2, S_3$ are disjoint cliques in Γ_4^*,

(ii) $S_0 \cup S_2$, $S_{0'} \cup S_2$, $S_1 \cup S_3$ are cliques in Γ_4.

Now construct a set of vectors B in the next way. If (a, b, c) is an element of A, then let $(u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3)$ be an element of B, where $(u_1, u_2, u_3) \in S_a$, $(v_1, v_2, v_3) \in S_b$, $(w_1, w_2, w_3) \in S_c$. The claim is that B is a 2^{12}-clique in Γ_{12}^* and so Keller's conjecture fails in dimension 12. They also constructed a counter example in 10-space and consequently there are counter examples for each dimension n, $n \geq 10$. The $7 \leq n \leq 9$ cases are undecided.

References

[1] K. Corrádi and S. Szabó, Some special cases of Keller's conjecture, *Journal of Pure and Appl. Alg.* **49** (1987), 247–252.

[2] K. Corrádi and S. Szabó, Keller's conjecture for certain p-groups, *Journal of Alg.* **119** (1988), 213–217.

[3] K. Corrádi and S. Szabó, A combinatorial approach for Keller's conjecture, *Periodica Math. Hungar.* **21** (1990), 95–100.

[4] Ph. Furtwängler, Über Gitter konstanter Dichte, *Monatsch. Math. Phys.* **43** (1936), 281–288.

[5] G. Hajós, Covering high dimensional spaces by cube lattices, *Mat. Fiz. Lapok* **45** (1938), 171–190. (in Hungarian)

[6] G. Hajós, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, *Math. Zeitschr.* **47** (1941), 427–467.

[7] O. H. Keller, Über die lückenlose Erfüllung des Raumes mit Würfeln, *J. Reine Angew. Math.* **163** (1930), 231–248.

[8] J. C. Lagarias and P. W. Shor, Keller's cube-tiling conjecture is false in high dimension, *Bulletin Amer. Math. Soc.* **27** (1992), 279–283.

[9] J. Lawrence, Tiling R^d by translates of the orthants, *Proc. of the 2nd Univ. Oklahoma Conf. Convexity and Related Combinatorial Geom.* 1982.

[10] H. Minkowski, *Geometrie der Zahlen*, Leipzig, Teubner 1896.

[11] D. Moews, Kellers's conjecture for certain finite groups, *Journal of Alg.* **142** (1991), 435–440.

[12] O. Perron, Über lückenlose Ausfüllung des n-dimensionalen Raumes durch kongruente Würfel, I, II, *Math. Zeitschr.* **46** (1940), 1–26; **46** (1940), 161–180.

[13] L. Rédei, Die neue Theorie der endlichen Abelschen Gruppen und Verallgemeinerung des Hauptsatzes von Hajós, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 329–373.

[14] R. M. Robinson, Multiple tilings of n-dimensional space by unit cubes, *Math. Zeitschr.* **166** (1979), 225–274.

[15] A. D. Sands, On Keller's conjecture for certain cyclic groups, *Proc. Edinburgh Math. Soc.* **22** (1979), 17–21.

[16] K. Seitz, Investigations in the Hajós-Rédei theory of finite abelian groups, *K. Marx Univ. Economics, Dept. Math.* Budapest, 1975.

[17] S. K. Stein, Algebraic tiling, *Amer. Math. Monthly* **81** (1974), 445–462.

[18] S. Szabó, A reduction of Keller's conjecture, *Periodica Math. Hungar.* **17** (1986), 265–277.

[19] S. Szabó, Multiple factorizations by cyclic subsets, *Journal of Alg.* **134** (1990), 28–35.