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Transnormal Graph Deformations

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In [5] Stewart Robertson proved a theorem which implies that every transnormal plane curve can be deformed into a circle through transnormal curves. At about the same time M.C. Irwin showed that under a certain assumption transnormal embeddings of S^1 into R^n are transnormally isotopic to spherical ones. The assumption that Irwin needed was soon removed by Bernd Wegner in [6]. This kind of results motivated the present work.

The curves we are dealing with are graphs of smooth maps defined either on R or S^1. We shall show that if the graph of such a map is transnormal then it can be deformed through transnormal graphs to the graph of a constant map. The proofs for R and S^1 are similar but for the sake of the exposition we shall treat the two cases separately.

0. Introduction

Let M be an m-dimensional, smooth ($=C^\infty$) manifold and $f: M \longrightarrow R^n$ a smooth embedding such that $f(M)$ is closed. We denote by $N(x)$ the normal affine $(n-m)$-plane to f at x. The embedding f is *transnormal* if whenever $f(y) \in N(x)$ we have $N(x) = N(y)$.

The cardinality of $f^{-1}(N(x))$ does not depend on x and it is called the *order of transnormality*.

Transnormality is an invariant property under affine isometries.

1. The real line

Let $f: R \longrightarrow R^n$ be a smooth map. We consider the map $F: R \longrightarrow R^{n+1}$ given by $F(x) = (x, f(x))$. Obviously $F(R)$ is the graph of f. We shall say that f has *transnormal graph* if F is a transnormal embedding. Transnormal graphs of this type were studied in [1]. We recall that F is transnormal if and only if every normal hyperplane to F meets $F(R)$ in just one point.

Proposition 1.1 *Let $f: R \longrightarrow R^n$ be a smooth map with transnormal graph. Then, for $0 \leq t \leq 1$, $f_t: R \longrightarrow R^n$ given by $f_t(x) = t \cdot f(x)$ also has transnormal graph.*

Proof. We assume that there exists $t \in R$, $0 < t < 1$, such that the corresponding graph is not transnormal. Then there are $y_0, y_1 \in R$, $y_0 \neq y_1$, such that the normal hyperplane to

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F_t at y_0 meets F_t at $(y_1, tf(y_1))$. Without loss of generality we may assume that $f_i'(y_0) \geq 0$, $i = 1, \ldots, n$, where the f_i's are the components of f. We consider $y_1 < y_0$ the arguments being similar if $y_1 > y_0$.

If $(y_1, tf(y_1))$ belongs to the normal hyperplane to F_t at y_0 we have

$$y_1 - y_0 + t^2 \sum_{i=1}^{n} (f_i(y_1) - f_i(y_0)) f_i'(y_0) = 0$$

and also

$$y_1 - y_0 + \sum_{i=1}^{n} (f_i(y_1) - f_i(y_0)) f_i'(y_0) > 0.$$

This means that $(y_1, f(y_1))$ belongs to the open half-space given by the inequality

$$x_1 - y_0 + \sum_{i=1}^{n} (x_{i+1} - f_i(y_0)) f_i'(y_0) > 0.$$

Observe now that $f_i'(y_0) > 0$ for at least one $i = 1, \ldots, n$. Therefore we can find z_1, $z_1 < y_0$, obeying the following condition: For those j's such that $f_j'(y_0) > 0$ we have $f_j(z_1) < f_j(y_0)$. It is obvious that

$$z_1 - y_0 + \sum_{i=1}^{n} (f_i(z_1) - f_i(y_0)) f_i'(y_0) < 0.$$

We see that the points $(y_1, f(y_1))$, $(z_1, f(z_1))$ belong to different half-spaces determined by the normal hyperplane to F at y_0. Therefore between y_1 and z_1 the graph of f must intersect that normal hyperplane and F is not transnormal. $\square$

2. The circle S^1

Let $f: S^1 \longrightarrow R^n$ be a smooth map and as before consider the embedding $F: S^1 \longrightarrow R^{n+2}$ given by $F(x) = (x, f(x))$. The results we are going to mention next can be found in [2].

Assume that f has transnormal graph, that is, F is transnormal. Then the order of transnormality is 2 and the two points where the normal hyperplane to F at x meets $F(R)$ are $(x, f(x))$ and $(-x, f(-x))$. It follows that $f(x) - f(-x)$ belongs to the orthogonal complement of the vector subspace $(f_*)_x(T_x S^1)$. Also, for every $x \in S^1$, $(f_*)_x = -((f \circ A)_*)_x$, where $A: S^1 \longrightarrow S^1$ is the antipodal map and $(f_*)_x$ is the linear map induced by f in the tangent space of S^1 at x.

Proposition 2.1 *Let $f: S^1 \longrightarrow R^n$ be a smooth map with transnormal graph. Then, for $0 \leq t \leq 1$, $f_t: S^1 \longrightarrow R^n$ given by $f_t(x) = tf(x)$ also has transnormal graph.*

Proof. Since $f(x) - f(-x)$ belongs to the orthogonal complement of $(f_*)_x(T_x S^1)$ it follows easily that, for $0 \leq t \leq 1$, the normal hyperplane to F_t at x passes through $(-x, tf(-x))$.

Moreover the normal hyperplanes to F_t at x and $-x$ coincide. In fact $((F_t)_*)_x(T_xS^1) = ((F_t)_*)_{-x}(T_{-x}S^1)$ since from $(f_*)_x = -((f \circ A)_*)_x$ we can establish that $((tf)_*)_x = -((tf \circ A)_*)_x$.

From now on the pattern of the proof is as for Proposition 1.1. Let us assume that there is $0 < t < 1$ such that the corresponding graph is not transnormal. Then there are $y_0, y_1 \in S^1$, $y_0 \neq \pm y_1$, such that the normal hyperplane to F_t at y_0 contains $(y_1, tf(y_1))$. We consider the stereographic projection from $-y_0$ and denote by $\psi\colon R \longrightarrow S^1 \setminus \{-y_0\}$ its inverse. Without loss of generality we may assume that $(f_i \circ \psi)'(0) \geq 0$, $i = 1, \ldots, n$, where the f_i's are the components of f and $\psi(0) = y_0$.

Consider the open half-line determined by 0 which contains s_1, with $\psi(s_1) = y_1$. In that half-line $(\psi(s) - \psi(0) \mid \psi'(0))$ never vanishes, with $(\cdot \mid \cdot)$ standing for the standard inner product in R^2. It is positive if $s_1 > 0$ and negative if $s_1 < 0$. We shall assume, for instance, that $s_1 > 0$. Then

$$(\psi(s_1) - \psi(0) \mid \psi'(0)) + \sum_{i=1}^{n} (f_i(\psi(s_1)) - f_i(y_0))(f_i \circ \psi)'(0) < 0.$$

As in Proposition 1.1. it is also possible to obtain $t_1 > 0$ such that

$$(\psi(t_1) - \psi(0) \mid \psi'(0)) + \sum_{i=1}^{n} (f_i(\psi(t_1)) - f_i(y_0))(f_i \circ \psi)'(0) > 0.$$

That the graph of f is not transnormal follows immediately. $\square$

We would like to point out that Proposition 2.1 is trivial in the case $n = 1$. In fact if $f\colon S^1 \longrightarrow R$ has transnormal graph then f is constant (see [2]).

3. Remark on self-parallelism

The notion of self-parallelism is a generalization of the notion of transnormality, see [8] for example. Given a smooth map $f\colon M \longrightarrow R^n$ $(M = R, S^1)$, f has *self-parallel graph* if there is a non-trivial (that is, different from the identity) diffeomorphism δ such that, for every $x \in M$, $N(x) = N(\delta(x))$, with $N(y)$ standing for the normal hyperplane to F at y. Here, as before, F is given by $F(x) = (x, f(x))$. While there are no self-parallel graphs if $M = R$ (see [8]) self-parallel graphs for $M = S^1$ have in fact been characterized in [3]. A smooth map $f\colon S^1 \longrightarrow R^n$ has self-parallel graph if and only if it is spherical and preserves antipodal points. It happens that δ is unique and equal to the antipodal map A.

A statement analogous to Proposition 2.1 holds with transnormal replaced by self-parallel. The proof is just the first paragraph of the proof for that proposition. In fact if the graph is self-parallel we still have $f(x) - f(-x)$ in the orthogonal complement of $(f_*)_x(T_xS^1)$ and $(f_*)_x = -((f \circ A)_*)_x$. Alternatively one could observe that f_t is spherical and preserves antipodal points.

Thanks are due to the referee for drawing our attention to [7] where the present isotopy has been described for a spherical and antipodal point preserving transnormal curve on the 2-sphere in suitable spherical polar coordinates.

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