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Representation of the Hirzebruch-Kleinschmidt Varieties by Quadrics

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Abstract. Let $X_n(a_1, \ldots, a_k)$ be an n-dimensional smooth projective toric variety which Kleinschmidt [5] introduced as a higher-dimensional generalization of Hirzebruch surfaces [3]. We show that for $r := n - k + \sum_{i=1}^{k} \binom{a_i+1+n-k}{n-k}$ there is a projective embedding $\phi : X_n(a_1, \ldots, a_k) \hookrightarrow \mathbf{P}^r$ such that the ideal of $\phi(X_n(a_1, \ldots, a_k))$ is generated by quadratic binomials.

1. Introduction and result

If $P \subset \mathbf{R}^n$ is a lattice polytope (vertices in $\mathbf{Z}^n$) one associates with P an algebraic variety as follows. Let $\{m_0, \ldots, m_r\} = P \cap \mathbf{Z}^n$ be the set of all lattice points of P (in some order). We define parameter functions

$$x_i := z^{m_i} := z_1^{m_{i1}} \ldots z_n^{m_{in}}, i = 0, \ldots, r \tag{1}$$

where $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ ($\mathbf{C}$ can be replaced by a more general field). Let $[x_0 : x_1 : \cdots : x_r]$ be homogeneous coordinates of the complex projective r-space $\mathbf{P}^r$. Any affine relation

$$\begin{aligned} \alpha_{i_0} m_{i_0} + \cdots + \alpha_{i_s} m_{i_s} &= \alpha_{i_{s+1}} m_{i_{s+1}} + \ldots + \alpha_{i_r} m_{i_r} \\ \alpha_{i_0} + \cdots + \alpha_{i_s} &= \alpha_{i_{s+1}} + \ldots + \alpha_{i_r} \end{aligned} \tag{2}$$

where $\alpha_{i_j} \geq 0$ $(j = 0, \ldots, r)$ are integers induces a homogeneous monomial equation

$$x_{i_0}^{\alpha_{i_0}} \cdots x_{i_s}^{\alpha_{i_s}} = x_{i_{s+1}}^{\alpha_{i_{s+1}}} \cdots x_{i_r}^{\alpha_{i_r}}. \tag{3}$$

The set of common solutions of all equations (3) is an algebraic variety X_P. The binomials obtained from (3) by subtracting one side from the other generate the ideal of X_P. The number of equations (3) can be assumed to be finite. The algebraic torus $T = \mathbf{C}^{*n}$ is naturally embedded into X_P as a dense subset, and it acts on X_P by diagonal matrices

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($z_i \longrightarrow c_i z_i, c_i \neq 0, i = 1, \ldots, n$). Each k-dimensional face of P defines a (complex) k-dimensional subvariety of X_P with a k-dimensional torus acting on it. If a toric varity X is defined in more general terms (compare [2], [4], or [7]) then X is compact (complete) and projective if and only if there exists a polytope P and an equivariant algebraic isomorphism $\phi : X \longrightarrow X_P$. The lattice polytope P hereby is only determined up to strict combinatorial isomorphism.

We contribute to the problem of finding a generating system for the equations (3) with monomials whose degree is as low as possible. R.J Koelman [6] has shown that for any 2-dimensional variety X_P the equations (3) can all be deduced from a finite set of quadratic monomial equations

$$x_{i_1} x_{i_2} = x_{i_3} x_{i_4} \tag{4}$$

provided P possesses at least 4 lattice points on its boundary. Batyrev [1] has proved that for arbitrary n it is possible to choose P such that the degree of the equations (3) can be restricted to be $\leq n+1$.

In the present paper we extend Koelman's result to a class of higher-dimensional, smooth projective toric varieties. The smoothness of X_P is characterized by the following property of P (see, for example, [7], p. 15):

(5) *P is simple, that is, from each vertex v_0 of P there emanate precisely n edges of P, and for the adjacent lattice points $v_1, \ldots, v_n$ of v_0 on such edges we have*

$$\det(v_1 - v_0, \ldots, v_n - v_0) = \pm 1.$$

Figure 1 Figure 2 Figure 3

We restrict the Picard number of X_P to 2, or, equivalently, P to have $n+2$ facets. As Kleinschmidt [5] has shown such varieties can be described explicitely. The smallest polytopes which occur for a representation X_P can be characterized as follows (which is readily seen from Kleinschmidt's analysis):

(6) *There exist an integer k, $0 < k < n$, and integers $a_i \geq 0$, $i = 1, \ldots, k$, such that, up to a unimodular transformation, P is obtained from the simplices $\Delta :=$*

$\mathrm{conv}\{0, e_1, \ldots, e_k\}$, $\Delta' := \mathrm{conv}\{0, e_{k+1}, \ldots, e_n\}$ *($e_1, \ldots, e_n$ the canonical basis of* $\mathbf{R}^n$*) as*

$$P = \mathrm{conv}[\Delta' \cup (e_1 + (a_1+1)\Delta') \cup \cdots \cup (e_k + (a_k+1)\Delta')].$$

For $n = 2$ we have $\Delta = [0, e_1]$ (line segment), $\Delta' = [0, e_2]$ (Figure 1). For $n = 3$, we have either $k = 1$ or $k = 2$ and obtain polytopes as are illustrated in Figure 2 or Figure 3, respectively.

Each lattice point lies in one of the simplices $\Delta', e_1 + (a_1+1)\Delta', \ldots, e_k + (a_k+1)\Delta'$. So the total number of lattice points of P is

$$r + 1 = 1 + n - k + \binom{a_1 + 1 + n - k}{n-k} + \cdots + \binom{a_k + 1 + n - k}{n-k}. \tag{7}$$

We shall prove:

Theorem. *For each Hirzebruch-Kleinschmidt variety $X_n(a_1, \ldots, a_k)$ and the embedding $\phi : X_n(a_1, \ldots, a_k) \hookrightarrow \mathbf{P}^r$ according to (1), (3) and (6) the ideal of $\phi(X_n(a_1, \ldots, a_k))$ is generated by quadratic binomials*

$$x_{i_1}x_{i_2} - x_{i_3}x_{i_4}, \quad 0 \le i_j \le r, j = 1,2,3,4, \tag{8}$$

and hence $\phi(X_n(a_1, \ldots, a_k))$ is an intersection of quadrics.

Examples. For $X_2(2)$ (Figure 1) we obtain $r = 5$ and the equations $x_4^2 = x_3x_5$, $x_3^2 = x_2x_4$, $x_2x_5 = x_3x_4$, $x_1x_3 = x_0x_2$, $x_1x_4 = x_0x_3$, $x_1x_5 = x_0x_4$. In case $X_3(2)$ (Figure 2) we find $r = 12$ and 35 equations (8). For $X_3(1,2)$ (Figure 3) we obtain $r = 8$ and 15 equations.

Remark. It is an open question if our theorem can be extended to all smooth projective toric varieties. Koelman's result for $n = 2$ takes care of all smooth cases. For $n \ge 3$ it is not sufficient to have enough lattice prints on the boundary of P, as is true for $n = 2$. This can be seen in the following example: Let $P := \mathrm{conv}\{m_0 := 0, m_1 := e_1, m_2 := e_2, m_i := e_1 + e_2 + (i-2)e_3, i = 3, \ldots, k; k \ge 4\} \subset \mathbf{R}^3$. Then $m_0 + 2m_3 = m_1 + m_2 + m_4$ implies a cubic equation which cannot be reduced to quadratic ones.

2. Proof of the theorem by a "game"

The set of equations (8) is finite since there are only finitely many lattice points in P. We consider $P \cap \mathbf{Z}^n$ as a game board. The proof of our theorem can in the following way be reduced to a game on $P \cap \mathbf{Z}^n$, as follows.

In order to build up a monomial equation (3) by equations deduced from (8) we apply two operations:

(a) Multiply each side of a monomial equation by the same variable x_i.

(b) Given two monomial equations set the product of the left sides equal to the product of the right sides.

The establishing of (3) is achieved heuristically by applying the following operations to the given affine relation (2):

(a$'$) Delete an m_i if it occurs on both sides of the equation.

(b$'$) If $m_i + m_p = m_j + m_q$ for $m_i, m_q, m_j, m_p \in P \cap \mathbf{Z}^n$ and if $m_i + m_p$ occurs on one side of (2), replace it by $m_j + m_q$.

If 0 can be established on both sides, the inverse operations lead to (2) and hence (3).

(a$'$), (b$'$) can now be turned into game rules: If m_i occurs with coefficient $\alpha_i > 0$ on the left side of (2) we place α_i "red" figures on m_i. If m_j occurs on the right side of (2) with coefficient $\alpha_j > 0$, we place α_j "green" figures on m_j. By assumption there are, at the beginning of the game, on each m_i either only red figures or only green figures or no figures at all. (a$'$), (b$'$) are equivalent to the following rules:

(a$''$) If a red figure moves to a point where there is a green one it "catches" the green figure and leaves with it the game board. The same applies, conversely, to any green figure.

(b$''$) A red figure may move from m_i to m_j if another red figure moves from m_p to m_q where $m_i + m_p = m_j + m_q$ (or $m_j - m_i = m_p - m_q$), provided $m_i, m_j, m_p, m_q \in P \cap \mathbf{Z}^n$ (compare Figure 4). The same rule applies to green figures.

Figure 4

We assert that the game can always be played until the board is empty. (The player who catches the most figures wins the game; one moves alternately unless a player catches a figure in which case he may continue. In the following proof, however, each player keeps moving as long as he can).

In fact: Let $\alpha := \alpha_{i_0} + \cdots + \alpha_{i_s} = \alpha_{i_{s+1}} + \cdots + \alpha_{i_r}$. Then $\alpha^{-1}(\alpha_{i_0} m_{i_1} + \cdots + \alpha_{i_s} m_{i_s})$ can be considered to be the centroid of the red figures, $\alpha^{-1}(\alpha_{i_{s+1}} + \cdots + \alpha_{i_r} m_{i_r})$ that of the green figures. Clearly:

(9) *The centroid of the red figures equals that of the green figures; equality of the centroids remains unchanged if an operation (a$''$) or (b$''$) is applied.*

We set $T_0 := \Delta'$, $T_1 := e_1 + (a_1 + 1)\Delta', \ldots, T_k := e_k + (a_k + 1)\Delta'$. Any set $T_0 \cup \cdots \cup T_{i-1} \cup T_{i+1} \cup \cdots \cup T_k$ is contained in a facet of $P, i = 0, \ldots, k$. Therefore: If there is a red figure in T_i, there is also a green one in T_i, since, otherwise, the centroids were different. The same is true for a green figure in T_i.

Let u be a red figure and v a green figure in T_i.

Let $H_i^1, \dots, H_i^{n-k+1}$ be the hyperplanes relative to affT_i which are parallel to the facets $F_i^1, \dots, F_i^{n-k+1}$ of T_i, respectively, and which pass through u. If v lies in one of these hyperplanes, we can, up to renumeration, assume v to lie in $H_i^1, \dots, H_i^h$, but not in $H_i^{h+1}, \dots, H_i^{n-k+1}$. One of the hyperplanes $H_i^{h+1}, \dots, H_i^{n-k+1}$, H_i^{h+1} say, separates v from the facet F_i^{h+1} or contains F_i^{k+1} (compare Figure 5 for $n = 4, k = 2, h = 1, i = 2$).

We denote the regular $(n-k-h)$-simplex $H_i^1 \cap \dots \cap H_i^h \cap T_i$ by $T_i^{(h)}$, and set $T_i^{(0)} = T_i$ in case v is not contained in any of the $H_i^j, j = 1, \dots, n-k-h$.

Figure 5

Now two possibilities occur:

(A) There is a red figure $w \neq u$ in one of the $T_j, j = 0, \dots, h$ which does not lie in the facet of T_j parallel to F_i.

(B) All red figures $\neq u$ lie in the facets F_j parallel to F_i.

In case (A) we apply (b'') to $u, u', w, w+(u-u')$ instead of m_p, m_q, m_i, m_j, respectively, where u' is chosen to have a shorter distance from v than u has, however, such that $\tilde{H}_i^{h+1} := H_i^{h+1} + u' - u$ still separates v from F_j^{h+1}. If v lies on $\tilde{H}_i^{h+1}$ we set $T_i^{(h+1)} := T_i^{(h)} \cap \tilde{H}_i^{h+1}$. If not we repeat the same procedure until we either arrive at a $T_i^{(h+1)}$ or (B) becomes true.

In the latter case we interchange the roles of u and v and proceed analogously. Going on with red and green figures we either arrive at $T_i^{(n-k)}$ so that a red and a green figure are on the same place, or (B) becomes valid. In the former case, we apply (a$''$) and take both figures from the board.

So suppose (B) is true for red figures and, analogously, for green figures, where the green figures $\neq v$ lie in facets $\tilde{F}_j \neq F_j$ of $T_j, j = 0, \dots, k$. We set $P_u := \operatorname{conv}(\{u\} \cup F_0 \cup F_1 \cup \dots \cup F_k)$ and $P_v := \operatorname{conv}(\{v\} \cup \tilde{F}_0 \cup \tilde{F}_1 \cup \dots \cup \tilde{F}_k)$. Clearly, P_u and P_v do not have a relative interior point in common. Therefore, the centroid of the red figures and the centroid of the green figures do not coincide, a contradiction.

Hence we can take off two figures and continue the game until the game board is empty.

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