Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 34 (1993), No. 2, 163-171.

Perturbations and Approximations of Support Functions of Convex Bodies

H. Groemer*
Department of Mathematics, The University of Arizona, Tucson, Arizona 85721, USA

Abstract. Let h be the support function of a d-dimensional convex body and Φ a real valued function on the unit sphere. The primary objective of this note is to find conditions on Φ that imply that $h + \Phi$ is again a support function. It is shown that for a large class of convex bodies such a condition can be formulated as an inequality involving Φ and its first and second order partial derivatives. Furthermore, some applications regarding the approximation of support functions by sums of spherical harmonics are discussed.

1. Introduction

Let $\mathbf{E}^d$ denote the euclidean d-dimensional space and let K be a convex body in $\mathbf{E}^d$, i. e. a compact convex subset of $\mathbf{E}^d$ whose interior is not empty. It is always assumed that $d \geq 2$. The *support function* of K can be defined by

$$h(u) = \sup\{\langle x, u\rangle : \ x \in K\},$$

where $\langle \cdot\,, \cdot\rangle$ denotes the euclidean inner product. Although this relation defines $h(u)$ for all $u \in \mathbf{E}^d$, in this note the support function will always be viewed as a function on the unit sphere S^{d-1} in $\mathbf{E}^d$ (centered at the origin o of $\mathbf{E}^d$). Hence, geometrically interpreted, $h(u)$ is the appropriately signed distance between o and the support plane of K associated with the direction u. In view of the dominant role that support functions play in the theory of convex sets the following question is of interest:

If a support function h is given, which conditions guarantee that a function Φ on S^{d-1} has the property that the perturbed function $h + \Phi$ is again a support function (of some convex body in $\mathbf{E}^d$)?

Simple examples show that an inequality of the form $|\Phi| < \epsilon$ cannot be sufficient; but it will be shown that for a large class of convex bodies a relatively simple condition on Φ and its first and second derivatives will be sufficient. This result and its consequences

* Supported by National Science Foundation Research Grant DMS 8922399

are of particular value in connection with applications of spherical harmonics in convexity. For example, in the work of Meissner [8] and Schneider [10] on rotors in convex polytopes, and of Goodey and Groemer [4] and Groemer [6] on certain geometric inequalities there appear hypothetical convex bodies whose support functions are linear combinations of finitely many spherical harmonics of prescribed order, and the question arises whether such bodies do actually exist. Our main theorem or, more directly, Corollary 1 provide an immediate answer to such questions.

A related problem that will be considered concerns the question whether the terms H_i of a sequence of functions on S^{d-1} are, for sufficiently large values of i, support functions if the sequence converges to a support function. This problem is again of particular interest in connection with applications of spherical harmonics in convexity.

Although results that partially overlap with those that are proved here can be found imbedded in the literature on geometric applications of spherical harmonics (see in particular Schneider [9], [11], [12], [13], Shephard [15], and the survey [5]) our unified presentation, elementary methods, and explicitly stated weak assumptions should be of interest. The proof of the main theorem is obtained by the reduction of the general situation to the case $d = 2$ by restricting the given support function to suitable great circles on S^{d-1}. Apparently the only other method that has been employed for the treatment of such problems (for $d > 2$) involves an argument concerning the eigenvalues of a certain matrix associated with the given support function (cf. Schneider [11] and [13]). However methods of this kind appear not to be suitable for proving sufficient conditions of such an explicit nature as those given here.

2. Results

To facilitate the formulation of our results we first introduce some definitions and notations.

If F is a (real valued) function on S^{d-1} it can always be extended to $\mathbf{E}^d \setminus \{o\}$ by setting

$$\bar{F}(x) = F(x/\|x\|),$$

where $\|\cdot\|$ denotes the euclidean norm. If $u \in S^{d-1}$, $x = (x_1, \cdots, x_d)$ and $i, j \in \{1, \cdots, d\}$ we set

$$D_i F(u) = \left(\frac{\partial \bar{F}(x)}{\partial x_i}\right)_{x=u}$$

and

$$D_{ij} F(u) = \left(\frac{\partial^2 \bar{F}(x)}{\partial x_i \partial x_j}\right)_{x=u},$$

provided that these derivatives exist. If they exist, or if they exist and are continuous, F will be said to be *twice differentiable* or *twice continuously differentiable*, respectively. Analogously one defines the property that F is n *times differentiable*.

A convex body K in $\mathbf{E}^d$ is defined to be ϵ-*smooth* if for every boundary point p of K there exists a ball B of radius ϵ such that $p \in B \subset K$. It can be shown that K is ϵ-smooth if its support function is twice continuously differentiable and the principal radii of curvature of the boundary of K exist and are at least ϵ. (For the case $d = 3$ see Blaschke [1] §24 and for the general case Koutroufiotis [7] or Firey [3].)

The following theorem contains the principal result of this note.

Theorem. *Let an $\epsilon > 0$ be given and let K be an ϵ-smooth convex body whose support function h is twice differentiable. If Φ is a function on S^{d-1} that is also twice differentiable and such that for all $u \in S^{d-1}$ and all $i, j \in \{1, \cdots, d\}$*

$$2d^3|D_{ij}\Phi(u)| + \sqrt{2d^3}|D_i\Phi(u)| + |\Phi(u)| < \epsilon\,, \tag{1}$$

then $h + \Phi$ is again the support function of a convex body in $\mathbf{E}^d$.

Before we present the proof of this theorem we formulate several consequences as corollaries.

If one takes as K the ball of radius ρ centered at o, then it is obviously ρ-smooth and one obtains immediately the following corollary that answers questions of the type mentioned in the introduction.

Corollary 1. *If Φ is a twice continuously differentiable function on S^{d-1} and ρ is a constant such that for all $u \in S^{d-1}$ and all $i, j \in \{1, \cdots, d\}$*

$$2d^3|D_{ij}\Phi(u)| + \sqrt{2d^3}|D_i\Phi(u)| + |\Phi(u)| < \rho\,,$$

then $\rho + \Phi$ is the support function of a convex body in $\mathbf{E}^d$. In particular, if $Q_{k_1}, \cdots Q_{k_n}$ are spherical harmonics (with Q_i having order i) there exists a constant c_0 such that for all $c \geq c_0$

$$c + Q_{k_1} + \cdots + Q_{k_n}$$

is the support function of a convex body in $\mathbf{E}^d$.

For $d = 2$ a result of this type has been proved by Shephard [15] and the general case has been considered by Schneider [11]. Clearly, the above theorem shows not only the existence of c_0 but it also enables one to determine a suitable value for c_0 .

We now formulate a corollary which shows that under appropriate assumptions the terms of a sequence of functions that converge to a support function will eventually consist of support functions.

Corollary 2. *For some $\epsilon > 0$ let K be an ϵ-smooth convex body in $\mathbf{E}^d$ with support function h and let H_n be a sequence of twice differentiable functions on S^{d-1} with the property that H_n converges pointwise to h and that for all fixed $i, j \in \{1, \cdots, d\}$ the sequences D_iH_n converge pointwise and $D_{ij}H_n$ converge uniformly. Then, there exists an n_0 such that for all $n > n_0$ the functions H_n are support functions of convex bodies.*

To verify this corollary one only needs to consider the corresponding sequence $\Phi_n = H_n - h$. From well-known convergence theorems for infinite series it follows that under the assumptions of the corollary the sequences Φ_n , $D_i\Phi_n$, and $D_{ij}\Phi_n$ converge uniformly to 0. Hence, $2d^3|D_{ij}\Phi_n| + \sqrt{2d^3}|D_i\Phi_n| + |\Phi_n|$ converges (for given i , j) uniformly to 0, and the above theorem shows that there is an n_0 such that for $n > n_0$ the functions $H_n = h + \Phi_n$ are the support functions of convex bodies.

The next corollary is obtained by combining our main theorem with known facts regarding the expansion

$$h \sim \sum_{k=0}^{\infty} Q_k$$

of a support function h in terms of d-dimensional spherical harmonics Q_k (where Q_k is supposed to have order k). Seeley [14] has shown that for all $u \in S^{d-1}$ and $i, j \in \{1, \cdots d\}$

$$|Q_k(u)| \leq c_1 k^{\frac{d-2}{2}} \|Q_k\|_2 \,, \tag{2}$$

$$|D_i Q_k(u)| \leq c_2 k^{\frac{d}{2}} \|Q_k\|_2 \,, \tag{3}$$

and

$$|D_{ij} Q_k(u)| \leq c_3 k^{\frac{d+2}{2}} \|Q_k\|_2 \,, \tag{4}$$

where c_1 , c_2 , c_3 depend only on d, and the norm $\|F\|_2$ of a function F on S^{d-1} is defined by

$$\|F\|_2 = \left(\int_{S^{d-1}} F(u)^2 d\sigma(u) \right)^{1/2} ,$$

with $d\sigma(u)$ indicating integration with respect to Lebesgue measure on S^{d-1}. It is also known that if Δ^m denotes the Laplace-Beltrami operator, m times repeated, and if h is $2m$ times continuously differentiable, then

$$\|\Delta^m h\|_2^2 = \sum_{k=0}^{\infty} k^{2m}(k+d-2)^{2m} \|Q_k\|_2^2$$

and therefore, if $k > 0$,

$$\|Q_k\|_2 \leq k^{-2m} \|\Delta^m h\|_2 \,.$$

From this inequality in conjunction with (2), (3), and (4) it follows that for $k > 0$

$$|Q_k(u)| \leq c_1 k^{\frac{d-2}{2} - 2m} \|\Delta^m h\|_2 \,,$$

$$|D_i Q_k(u)| \leq c_2 k^{\frac{d}{2} - 2m} \|\Delta^m h\|_2 \,,$$

and

$$|D_{ij} Q_k(u)| \leq c_3 k^{\frac{d+2}{2} - 2m} \|\Delta^m h\|_2 \,.$$

Thus, the series

$$\sum_{k=0}^{\infty} Q_k(u), \quad \sum_{k=0}^{\infty} D_i Q_k(u), \quad \sum_{k=0}^{\infty} D_{ij} Q_k(u)$$

converge uniformly if $\frac{d+2}{2} - 2m < -1$, i. e., if $m > \frac{d+4}{4}$, and the first one converges to h since it converges to h in mean. Hence, under the condition $m > \frac{d+4}{4}$ the assumptions of Corollary 2 are satisfied for the partial sums of these series and one obtains the following result.

Corollary 3. *For some $\epsilon > 0$ let K be an ϵ-smooth convex body in $\mathbf{E}^d$ whose support function h is $2m$ times continuously differentiable, where $m > \frac{d+4}{4}$. If*

$$h \sim \sum_{k=0}^{\infty} Q_k$$

is the expansion of h in terms of spherical harmonics Q_k (with Q_k having order k), then there is an n_0 such that for any $n > n_0$ the partial sum

$$Q_0 + Q_1 + \cdots + Q_n$$

is the support function of a convex body in $\mathbf{E}^d$.

From this corollary one can deduce a useful fact regarding the relationship of the space of all convex bodies in $\mathbf{E}^d$ (equipped with the Hausdorff metric) and the subspace of those convex bodies whose support functions are finite sums of spherical harmonics. Let δ denote the Hausdorff metric. If a convex body K in $\mathbf{E}^d$ and an $\epsilon > 0$ are given it is well-known that there exists a convex body, say K', whose support function is infinitely often differentiable and such that

$$\delta(K, K') < \frac{\epsilon}{3}.$$

Moreover, if K_o is the parallel body of K' at distance $\epsilon/3$, then its support function is again infinitely often differentiable, it is ϵ-smooth, and

$$\delta(K', K_o) \leq \frac{\epsilon}{3}.$$

Finally, Corollary 3 shows that there is a convex body $\tilde{K}$ whose support function is a finite sum of spherical harmonics and such that

$$\delta(K_o, \tilde{K}) < \frac{\epsilon}{3}.$$

Combining the these three inequalities one obtains the following result.

Corollary 4. *The space of convex bodies in $\mathbf{E}^d$ whose support functions are finite sums of spherical harmonics is dense (with respect to the Hausdorff metric) in the space of all convex bodies in $\mathbf{E}^d$.*

3. Proof of the Theorem

The proof is accomplished in three steps. First the case $d = 2$ is considered, then an auxiliary statement regarding a characterization of support functions in terms of their restrictions to great circles on S^{d-1} is proved, and finally these two results are combined to establish the desired theorem.

1. If h is a twice differentiable function on $(-\infty, \infty)$ of period 2π, then h is the support function of a convex body in $\mathbf{E}^2$ if for all $\omega \in [0, 2\pi]$

$$h''(\omega) + h(\omega) > 0\,. \tag{5}$$

This fact is actually known (see for example Shephard [15]) but for the sake of completeness we give here a simple proof.

Consider the closed curve C whose position vector in a cartesian coordinate system of $\mathbf{E}^2$ is given by

$$p(\omega) = h'(\omega)u'(\omega) + h(\omega)u(\omega) \quad (0 \le \omega \le 2\pi)\,, \tag{6}$$

where

$$u(\omega) = (\cos\omega, \sin\omega)\,.$$

(6) implies obviously that

$$p'(\omega) = (h''(\omega) + h(\omega))\, u'(\omega)\,, \tag{7}$$

and this, together with (5), shows that for any $q \in S^1$ the equality $\langle p'(\omega), q\rangle = 0$ holds if and only if $\langle u'(\omega), q\rangle = 0$. Thus, for any $q \in S^1$ the curve C has exactly two tangent lines orthogonal to q and this implies obviously that C is convex. The support function of this convex curve is, as one finds from (6),

$$\langle p(\omega), u(\omega)\rangle = h(\omega)\,.$$

Hence h is the support function of a convex body in $\mathbf{E}^2$, namely the body whose boundary is C.

For later use we also note that the radius of curvature of C, say $R(\omega)$, at the point $p(\omega)$ is given by

$$R(\omega) = h''(\omega) + h(\omega)\,. \tag{8}$$

This can be obtained from (5), (7) and the fact that (if s denotes the arclength of C) $R(\omega) = |ds/d\omega| = \langle p'(\omega), p'(\omega)\rangle^{1/2}$.

2. Now the following statement will be proved: A function h on S^{d-1} is the support function of some convex body in $\mathbf{E}^d$ if for every one-dimensional concentric subsphere S of S^{d-1} the function h_S, i. e. the restriction of h to S, is the support function of a convex body in $\mathbf{E}^2$.

To show this consider the function $\hat{h}(x) = \|x\| h(x/\|x\|)$ $(h(o) = 0)$. It is well-known (cf. Bonnesen-Fenchel [2]) that h is the support function of a convex body if and only if $\hat{h}$

is a convex function on $\mathbf{E}^d$. Thus, one only has to show that if $\hat{h}$ is not convex there is a two-dimensional plane, say E, through o such that the restriction of $\hat{h}$ to E is not convex. Assuming that $\hat{h}$ is not convex one can find two points $p, q \in \mathbf{E}^d$ and an $\alpha \in (0,1)$ such that

$$\hat{h}\big((1-\alpha)p + \alpha q\big) > (1-\alpha)\hat{h}(p) + \alpha\hat{h}(q)\,. \tag{9}$$

As E we now choose the two-dimensional plane spanned by o, p and q, and let $\hat{h}_E$ denote the restriction of $\hat{h}$ to E. Clearly, (9) implies that

$$\hat{h}_E\big((1+\alpha)p + \alpha q\big) > (1-\alpha)\hat{h}_E(p) + \alpha\hat{h}_E(q)$$

and this is already the desired conclusion.

3. Let $e_1, \cdots, e_d$ be an orthonormal basis of $\mathbf{E}^d$. Furthermore, let H be a two-dimensional plane in $\mathbf{E}^d$ that contains o, and assume that v_o, w_o are two given mutually orthogonal unit vectors in H. Obviously there are real numbers s_i, t_i such that

$$v_o = s_1 e_1 + \cdots + s_d e_d\,, \quad w_o = t_1 e_1 + \cdots + t_d e_d$$

and

$$s_1^2 + \cdots + s_d^2 = 1\,, \quad t_1^2 + \cdots + t_d^2 = 1\,. \tag{10}$$

We set $S^{d-1} \cap H = S$ and note that if $u \in S$ and if ω is the angle between u and v_o, then

$$u = v_o \cos\omega + w_o \sin\omega = \sum_{k=1}^{d} (s_k \cos\omega + t_k \sin\omega) e_k\,.$$

Hence, if Φ_S denotes the restriction of Φ to S, then $\Phi_S(u)$ can be viewed as a function of ω and we find that

$$\begin{aligned}\frac{d\Phi_S}{d\omega} &= \frac{d}{d\omega}\Phi(s_1\cos\omega + t_1\sin\omega, \cdots, s_d\cos\omega + t_d\sin\omega)\\ &= \frac{d}{d\omega}\bar{\Phi}(s_1\cos\omega + t_1\sin\omega, \cdots, s_d\cos\omega + t_d\sin\omega)\\ &= \sum_{i=1}^{d}\big(D_i\Phi(u)\big)(-s_i\sin\omega + t_i\cos\omega)\end{aligned}$$

and similarly

$$\begin{aligned}\frac{d^2\Phi_S}{d\omega^2} = \sum_{i=1}^{d}\Bigg[&\Big(\sum_{j=1}^{d}\big(D_{ij}\Phi(u)\big)(-s_j\sin\omega + t_j\cos\omega)\Big)(-s_i\cos\omega + t_i\sin\omega)\\ &- \big(D_i\Phi(u)\big)(s_i\cos\omega + t_i\sin\omega)\Bigg]\,.\end{aligned}$$

Using the inequality $|s\sin\omega + t\cos\omega| \le \sqrt{s^2+t^2}$ one finds that

$$\left|\frac{d^2\Phi_S}{d\omega^2}\right| + |\Phi_S| \le \sum_{i=1}^{d}\left[\sum_{j=1}^{d} |D_{ij}\Phi(u)|\sqrt{s_j^2+t_j^2}\right]\sqrt{s_i^2+t_i^2} + \sum_{i=1}^{d}\left(|D_i\Phi(u)|\sqrt{s_i^2+t_i^2}\right) + |\Phi(u)|\,.$$

Since (10) in conjunction with Hölder's inequality shows that $\sum_{i=1}^{d}\sqrt{s_i^2+t_i^2} \le \sqrt{2d}$ it follows that

$$\begin{aligned}\left|\frac{d^2\Phi_S}{d\omega^2}\right| + |\Phi_S| &\le 2d\sum_{i=1}^{d}\sum_{j=1}^{d}|D_{ij}\Phi(u)| + \sqrt{2d}\sum_{i=1}^{d}|D_i\Phi(u)| + |\Phi(u)| \\ &\le \frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left(2d^3\,|D_{ij}\Phi(u)| + \sqrt{2d^3}\,|D_i\Phi(u)| + |\Phi(u)|\right).\end{aligned}$$

This inequality and (1) imply that

$$\left|\frac{d^2\Phi_S}{d\omega^2}\right| + |\Phi_S| < \epsilon\,.$$

Consequently, writing again h_S for the restriction of h to S we find

$$(h_S+\Phi_S)'' + (h_S+\Phi_S) \ge h_S'' + h_S - (|\Phi_S''| + |\Phi_S|) > h_S'' + h_S - \epsilon\,, \tag{11}$$

where the derivatives are taken with respect to ω. We now note that h_S is the support function of the orthogonal projection of K onto H and that this projection is evidently again ϵ-smooth. In consideration of (8) this shows that $h''+h \ge \epsilon$ and it follows from (11) that

$$(h_S+\Phi_S)'' + (h_S+\Phi_S) > 0\,.$$

As proved in paragraphs 1 and 2 this implies that $h+\Phi$ is the support function of a convex body in $\mathbf{E}^d$.

If $d=2$ paragraph 1 of this proof shows that the condition (1) can be replaced by the simpler inequality

$$|\Phi''| + |\Phi| < \epsilon\,,$$

where Φ is viewed as a function of ω, i. e., of the angle whose corresponding point on S^1 is $(\cos\omega, \sin\omega)$.

References

[1] Blaschke, W.: *Kreis und Kugel* (Veit & Co., 1916; second Ed.: Walter de Gruyter & Co., 1956).

[2] Bonnesen, T. and Fenchel W.: *Theorie der konvexen Körper.* Ergebn. d. Math. Bd. 3 (Springer Verl., 1934, engl. transl.: Theory of Convex Bodies, BCS Assoc., 1987).

[3] Firey, W. J.: *Inner contact measures.* Mathematika 26, 106-112 (1979).

[4] Goodey, P. R. and Groemer, H.: *Stability results for first order projection bodies.* Proc. Am. Math. Soc. 109, 1103-1114 (1990).

[5] Groemer, H.: *Fourier series and spherical harmonics in convexity.* Handbook of Convex Geometry, Section 4.8 (North Holland, 1993).

[6] Groemer, H.: *On circumscribed cylinders of convex sets.* Geom. Dedicata 46, 331-338 (1993).

[7] Koutroufiotis, D.: *On Blaschke's rolling theorem.* Arch. Math. 23, 655-660 (1972).

[8] Meissner, E.: *Über die durch reguläre Polyeder nicht stützbaren Körper.* Vierteljahresschr. d. naturforschenden Gesellsch. Zürich 63, 544-551 (1918).

[9] Schneider, R.: *Zu einem Problem von Shephard über die Projektionen konvexer Körper.* Math. Z. 101, 71-82 (1967).

[10] Schneider, R.: *Gleitkörper in konvexen Polytopen.* J. reine angew. Math. 248, 193-220 (1971).

[11] Schneider, R.: *On Steiner points of convex bodies.* Israel J. Math. 9, 241-249 (1971).

[12] Schneider, R.: *Smooth approximations of convex bodies.* Rend. Circ. Mat. Palermo II, 33, 436-440 (1984).

[13] Schneider, R.: *Convex Bodies: the Brunn-Minkowski Theory* (Cambridge University Press, Cambridge 1992).

[14] Seeley, R. T.: *Spherical harmonics.* Am. Math. Monthly 73, 115-121 (1966).

[15] Shephard, G. C.: *A uniqueness theorem for the Steiner point of a convex region.* J. London Math. Soc. 43, 439-444 (1968).