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# Specht Modules for Weyl Groups

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## 1. Introduction

Over fields of characteristic zero, there are well known construction of the irreducible representations and of irreducible modules, called Specht modules for the symmetric groups $S_n$ which are based on elegant combinatorial concepts connected with Young tableaux etc. (see, e.g. [10]). James [8] extended these ideas to construct irreducible representations and modules over arbitrary field. Al-Aamily, Morris and Peel [1] showed how this construction could be extended to deal with the Weyl groups of type $B_n$. In [11] the second author described a possible extension of James' work for Weyl groups in general, where Young tableaux are interpreted in terms of root systems. We now modify these results and give an alternative generalisation of James' work which is an improvement and extension of the original approach suggested by Morris.

## 2. Some General Results On Weyl Groups

In this section we establish the notation and state some results on Weyl groups which are required later . Standard references for this material are N. Bourbaki [3], R. W. Carter [4] and J. E. Humphreys [7] .

Let $\Phi$ be a root system in an $l$-dimensional real space $V$ and $\pi$ be a simple system in $\Phi$ with corresponding positive system $\Phi^+$ and negative system $\Phi^-$ . Let

$$\mathcal{W} = \mathcal{W}(\Phi) =< \tau_\alpha \mid \alpha \in \Phi >$$

be the *Weyl group* of $\Phi$, where $\tau_\alpha$ is the reflection corresponding to $\alpha$. Let $l(w)$ denote the *length* of $w$ and the *sign* of $w$, $s(w)$, is defined by $s(w) = (-1)^{l(w)}$, $w \in \mathcal{W}$.

We note the following facts which are required later.

**2.1** There are $\mid \mathcal{W} \mid$ simple systems (positive systems) in $\Phi$ given by $w\pi$ $(w\Phi^+)$, $w \in \mathcal{W}$. The group $\mathcal{W}$ acts transitively on the set of simple systems.

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**2.2** Let $\Gamma$ be the *Dynkin diagram* (or *Coxeter graph*) of $\Phi$. A Weyl group is *irreducible* if its Dynkin diagram is connected. Irreducible Weyl groups have been classified and correspond to root systems of type $A_l (l \geq 1)$, $B_l (l \geq 2)$, $C_l (l \geq 3)$, $D_l (l \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. For example $\mathcal{W}(A_l) \cong S_{l+1}$, the symmetric group on the set $\{1, 2, \ldots, l+1\}$ . As our aim in this paper is to generalise ideas from the symmetric groups, the root system and Dynkin diagram are given in this case. The Dynkin diagram is

and if $\{\epsilon_1, \ \epsilon_2, \ldots, \ \epsilon_{l+1}\}$ is the standard basis for $R^{l+1}$ , then

$$\begin{aligned}
\pi &= \{ \ \alpha_1 = \epsilon_1 - \epsilon_2, \ \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \ \alpha_l = \epsilon_l - \epsilon_{l+1} \} \\
\Phi &= \{ \ \epsilon_i - \epsilon_j \ \mid \ 1 \leq i, \ j \leq l+1 \} \\
\Phi^+ &= \{ \ \epsilon_i - \epsilon_j \ \mid \ 1 \leq i < j \leq l+1 \}
\end{aligned}$$

**2.3** A *subsystem* $\Psi$ of $\Phi$ is a subset of $\Phi$ which is itself a root system in the space which it spans. A subsystem $\Psi$ is said to be *additively closed* if $\alpha, \beta \in \Psi$, $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Psi$. From now on we assume that $\Psi$ is additively closed subsystem of $\Phi$. A *Weyl subgroup* $\mathcal{W}(\Psi)$ of $\mathcal{W}$ corresponding to a subsystem $\Psi$ is the subgroup of $\mathcal{W}$ generated by the $\tau_\alpha, \alpha \in \Psi$.

**2.4** The graphs which are Dynkin diagrams of subsystems of $\Phi$ may be obtained up to conjugacy by a standard algorithm due independently to E. B. Dynkin, A. Borel and J. de Siebenthal (see e.g. [4]).

**2.5** If $w \in \mathcal{W}$ and $U$ is the subspace of $V$ composed of all vectors fixed by $w$, then $w$ is a product of reflections corresponding to roots in the orthogonal complement $U^\perp$ of $U$ [4].

**2.6** The simple system $J$ of $\Psi$ can always be chosen such that $J \subset \Phi^+$ [12].

**2.7** The set $D_\Psi = \{w \in \mathcal{W} \mid w(j) \in \Phi^+ \text{ for all } j \in J\}$ is a *distinguished set of coset representatives* of $\mathcal{W}(\Psi)$ in $\mathcal{W}$ , that is, each element $w \in \mathcal{W}$ has unique expression of the form $d_\Psi w_\Psi$, where $d_\Psi \in D_\Psi$ and $w_\Psi \in \mathcal{W}(\Psi)$ and furthermore $d_\Psi$ is the element of minimal length in the coset $d_\Psi \mathcal{W}(\Psi)$ [12].

**2.8** We now extend the above to cover some further reflection subgroups of Weyl groups. This work is due to Steinberg and we follow the description given by Carter [5] where the results highlighted below are proved.

Let $\rho$ be a non-trivial symmetry of the Dynkin diagram of $\Phi$. Then there is a unique isometry $\tau$ of $V$ such that $\tau(r)$ is a positive multiple of $\rho(r) = \bar{r}$ for all simple roots $r \in \pi$.

The isometry $\tau$ satisfies the conditions :

$$\tau(r) = \bar{r} \quad \text{if all the roots of } \Phi \text{ have same length}$$

$$\tau(r) = \begin{cases} \frac{1}{\sqrt{2}}\bar{r} & \text{if } r \text{ is short} \\ \sqrt{2}\bar{r} & \text{if } r \text{ is long} \end{cases} \qquad \text{for } \Phi = B_2 \text{ or } F_4$$

$$\tau(r) = \begin{cases} \frac{1}{\sqrt{3}}\bar{r} & \text{if } r \text{ is short} \\ \sqrt{3}\bar{r} & \text{if } r \text{ is long} \end{cases} \qquad \text{for} \Phi = G_2.$$

Clearly the order of $\tau$ as an isometry of $V$ is equal to the order of $\rho$ as a permutation of $\pi$. Now let

$$V^1 = \{v \in V \mid \tau(v) = v\}.$$

For each $v \in V$ let $v^1$ denote the projection of $v$ onto the subspace $V^1$ . Then $v^1$ is the average of the vectors in the orbit of $v$ under $\tau$.

Since $\tau(r)$ is a positive multiple of $\rho(r)$ for all $r \in \pi$ we have

$$\tau w_r \tau^{-1} = w_{\rho(r)}, \quad \text{where } r \in \pi.$$

Let

$$\mathcal{W}^1 = \{w \in \mathcal{W} \mid \tau w \tau^{-1} = w\},$$

then $\mathcal{W}^1$ operates faithfully on $V^1$. Let $J$ be an orbit of $\pi$ under $\rho$. Let $\mathcal{W}(J)$ be the subgroup of $\mathcal{W}$ generated by the elements $\tau_r$ for $r \in J$. Let $w_0^J$ be the element of $\mathcal{W}(J)$ which transforms every positive root in $\Phi_J$ into a negative root. Then $w_0^J \in \mathcal{W}^1$ and $\mathcal{W}^1$ is generated by the elements $w_0^J$ for the different $\rho$-orbits of $\pi$. Since $w_0^J$ coincides with $w_{r^1}$ on $V^1$ for each root $r \in J$ the elements $w_0^J$ are reflections when restricted to $V^1$. Then the reflections $w_{r^1}$ of $V^1$, for all $r \in \pi$, generate the group $\mathcal{W}^1$ of isometries of $V^1$. Let $\Phi^1 = \{r^1 \in V^1 \mid r \in \Phi\}$ and $\pi^1 = \{r^1 \in V^1 \mid r \in \pi\}$. The sets $w\ (\Phi_J^+)$ form a partition of $\Phi$ as $w$ runs through the elements of $\mathcal{W}^1$ and $J$ runs through the $\rho$-orbits of $\pi$. The roots $r$ and $s$ are in the same set if and only if $r^1$ is a positive multiple of $s^1$.

The sets $\Phi^1$ and $\pi^1$ almost act as a root system and simple system for $\mathcal{W}^1$ acting on $V^1$, since

(i) $\Phi^1$ spans $V^1$,

(ii) Every element of $\Phi^1$ is a linear combination of elements of $\pi^1$ with coefficients all non-negative or all non-positive,

(iii) A basis of $V^1$ may be obtained by picking one element of $\pi^1$ out of each set of positive multiples,

(iv) If $r^1 \in \Phi^1$ then there is an element of $\mathcal{W}^1$ which coincides with $w_{r^1}$ on $V^1$,

(v) If $r^1, s^1 \in \Phi^1$ then $w_{r^1}(s^1) \in \Phi^1$.

The groups $\mathcal{W}^1$ will be referred to as Steinberg subgroups from now on.

## 3. Specht Modules for Weyl Groups

Let $\Phi$ be a root system with simple system $\pi$ and Dynkin diagram $\Gamma$ and let $\Psi$ be a subsystem of $\Phi$ with simple system $J \subset \Phi^+$ and Dynkin diagram $\Delta$. If $\Psi = \bigcup_{i=1}^{r} \Psi_i$, where $\Psi_i$ are the indecomposable components of $\Psi$ , then let $J_i$ be a simple system in $\Psi_i$ $(i = 1, 2, \ldots, r)$ and $J = \bigcup_{i=1}^{r} J_i$. Let $\Psi^{\perp}$ be the largest subsystem in $\Phi$ orthogonal to $\Psi$ and let $J^{\perp} \subset \Phi^{+}$ the simple system of $\Psi^{\perp}$.

Let $\Psi'$ be a subsystem of $\Phi$ which is contained in $\Phi \setminus \Psi$, with simple system $J' \subset \Phi^{+}$ and Dynkin diagram $\Delta'$. If $\Psi' = \bigcup_{i=1}^{s} \Psi'_i$, where $\Psi'_i$ are the indecomposable components of $\Psi'$ then let $J'_i$ be a simple system in $\Psi'_i$ $(i = 1, 2, \ldots, s)$ and $J = \bigcup_{i=1}^{s} J'_i$. Let $\Psi'^{\perp}$ be the largest subsystem in $\Phi$ orthogonal to $\Psi'$ and let $J'^{\perp} \subset \Phi^{+}$ the simple system of $\Psi'^{\perp}$.

Let $\bar{J}$ stand for the ordered set $\{J_1, J_2, \ldots, J_r; J'_1, J'_2, \ldots, J'_s\}$, where in addition the elements in each $J_i$ and $J'_i$ are ordered. Let

$$\mathcal{T}_{J,J'} = \{w\bar{J} \mid w \in \mathcal{W}\}$$

Now , we consider under what conditions the elements in the set $\mathcal{T}_{J,J'}$ are distinct. We obtain the following lemma.

**Lemma 3.1** $\mid \mathcal{T}_{J,J'} \mid = \mid \mathcal{W} \mid$ *if and only if* $\mathcal{W}(J^{\perp}) \cap \mathcal{W}(J'^{\perp}) = < e >$.

*Proof.* Suppose that $w_1\bar{J} = w_2\bar{J}$, where $w_1, w_2 \in \mathcal{W}$. Then $w_1\bar{J} = w_2\bar{J}$, $w_1J' = w_2J'$ and since the elements of each component of $J$ and $J'$ are also ordered it follows that $w_2^{-1}w_1(\alpha) = \alpha$ for all $\alpha \in J$ and $w_2^{-1}w_1(\alpha) = \alpha$ for all $\alpha \in J'$, that is, by (2.5) $w_2^{-1}w_1 \in \mathcal{W}(J^{\perp}) \cap \mathcal{W}(J'^{\perp})$. Thus we have that the elements of $\mathcal{T}_{J,J'}$ are distinct if and only if $\mathcal{W}(J^{\perp}) \cap \mathcal{W}(J'^{\perp}) = < e >$. □

Now we can give our principal definition.

**Definition 3.2** Let $\Psi$ and $\Psi'$ be subsystems of $\Phi$ with simple systems $J$ and $J'$ respectively such that $\Psi' \subseteq \Phi \setminus \Psi$ and $J \subset \Phi^{+}$, $J' \subset \Phi^{+}$. The pair $\{J, J'\}$ is called a *useful system* in $\Phi$ if $\mathcal{W}(J) \cap \mathcal{W}(J') = < e >$ and $\mathcal{W}(J^{\perp}) \cap \mathcal{W}(J'^{\perp}) = < e >$.

**Remark 1** If $\{J, J'\}$ is a useful system in $\Phi$ , then $\{wJ, wJ'\}$ is also a useful system in $\Phi$, for $w \in \mathcal{W}$. Thus , from now on $\mathcal{T}_{J,J'}$ will be denoted by $\mathcal{T}_{\Delta}$.

**Remark 2** If $\{J, J'\}$ is a useful system in $\Phi$ then $\Psi \cap \Psi' = \emptyset$ and $\Psi^{\perp} \cap \Psi'^{\perp} = \emptyset$. However the converse is not true in general.

**Definition 3.3** Let $\{J, J'\}$ be a useful system in $\Phi$ . Then the elements of $\mathcal{T}_{\Delta}$ are called $\Delta$-*tableaux*, the $J$ and $J'$ are called the *rows* and the *columns* of $\{J, J'\}$ respectively.

We see that this is a natural extension of the concept of a Young tableaux in the representation theory of symmetric groups. In the case $A_l$, the subsystems are given up to conjugacy by root systems of type $A_{l_1} + \cdots + A_{l_s}$, where the $l_i \geq 0$ and $\sum(l_i+1) = l+1$. Thus the corresponding Weyl subgroup is $\mathcal{W}(A_{l_1}) \times \cdots \times \mathcal{W}(A_{l_s})$ which is isomorphic to the Young subgroup $S_{l_1+1} \times \cdots \times S_{l_s+1}$ of $S_{l+1}$ and there is a Weyl subgroup corresponding to each partition of $l+1$. If $\Psi = A_{l_1} + \cdots + A_{l_s}$ is a subsystem of $A_l$, then $\lambda = (l_1+1, \ldots, l_s+1)$ is a partition of $l+1$. If we put $k_i = l_1 + \cdots + l_i + i$ $(i = 1, 2, \ldots, s)$ and $k_0 = 0$ then $\{\epsilon_{k_{i-1}+1} - \epsilon_{k_{i-1}+2}, \ldots, \epsilon_{k_i - 1} - \epsilon_{k_i}\}$ is a simple system for $A_{l_i}$. This may be represented by the $\lambda$-tableau

$$\begin{array}{llllllllll}
1 & 2 & 3 & . & . & . & . & . & . & k_1 \\
k_1+1 & k_1+2 & k_1+3 & . & . & . & . & . & k_2 & \\
k_2+1 & k_2+2 & k_2+3 & . & . & . & . & k_3 & & \\
. & . & . & . & . & . & & & & \\
k_{s-1}+1 & k_{s-1}+2 & k_{s-1}+3 & . & . & l+1 & & & &
\end{array}$$

the other n! $\lambda - tableau$ being obtained by allowing the elements of $S_n$ to act on this $\lambda - tableau$.

The orthogonal subsystem $\Psi^{\perp}$ is the root system determined by the elements in rows of length one in the $\lambda - tableau$. The ordering of the rows is important, for example the $3^2 - tableaux$ $\substack{123\\456}$ and $\substack{456\\123}$ are regarded as distinct tableaux.

**Definition 3.4** Two $\Delta$-tableaux $\bar{J}$ and $\bar{K}$ are *row-equivalent*, written $\bar{J} \sim \bar{K}$, if there exists $w \in \mathcal{W}(J)$ such that $\bar{K} = w\bar{J}$. The equivalence class which contains the $\Delta$-tableau $\bar{J}$ is $\{\bar{J}\}$ and is called a $\Delta$-*tabloid.*

Let $\tau_\Delta$ be the set of all $\Delta$-tabloids. It is clear that the number of distinct elements in $\tau_\Delta$ is $[\mathcal{W} : \mathcal{W}(J)]$ and by (2.7) we have

$$\tau_\Delta = \{\{d\bar{J}\} \mid d \in D_\Psi\}.$$

We note that if $\bar{J} = \{J; J'\}$ then $dJ \subset \Phi^+$ but $dJ'$ need not be a subset of $\Phi^+$.

We now give an example to illustrate the construction of a $\Delta$-tabloid. In this example and later examples we use the following notation. If $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a simple system in $\Phi$ and $\alpha \in \Phi$, then $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, where $a_i \in \mathbf{Z}$. From now on $\alpha$ is denoted by $a_1a_2\cdots a_n$ and $\tau_{\alpha_1}, \tau_{\alpha_2}, \ldots, \tau_{\alpha_n}$ are denoted by $\tau_1, \tau_2, \ldots, \tau_n$ respectively.

**Example 3.5** Let $\Phi = \mathbf{A_3}$ with simple system $\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3 - \epsilon_4\}$. Let $\Psi = \mathbf{2A_1}$ be the subsystem of $\Phi$ with simple system $J = \{100, 001\}$. Then the Dynkin diagram for $\Psi$ is

Let $\Psi' = \mathbf{2A_1}$ be the subsystem of $\Phi$ which is contained in $\Phi \setminus \Psi$, with simple system $J' = \{110, 011\}$. Since $\mathcal{W}(J) \cap \mathcal{W}(J') =< e >$ and $\mathcal{W}(J^{\perp}) \cap \mathcal{W}(J'^{\perp}) =< e >$, then $\{J, J'\}$

is a useful system in $\Phi$ . Then $\tau_\Delta$ contains the $\Delta-$ *tabloids* (we also give these in the traditional notation as in [9]);

The group $\mathcal{W}$ acts on $\tau_\Delta$ according to

$$\sigma\{\overline{wJ}\} = \{\overline{\sigma wJ}\} \quad \text{for all } \sigma \in \mathcal{W}.$$

This action is well defined, for if $\{\overline{w_1J}\} = \{\overline{w_2J}\}$, then there exists $\rho \in \mathcal{W}(w_1J)$ such that $\overline{\rho w_1J} = \overline{w_2J}$. Hence since $\sigma\rho\sigma^{-1} \in \mathcal{W}(\sigma w_1J)$ and $\overline{\sigma w_2J} = \overline{\sigma\rho w_1J} = (\sigma\rho\sigma^{-1})(\overline{\sigma w_1J})$, we have $\{\overline{\sigma w_1J}\} = \{\overline{\sigma w_2J}\}$.

Now if K is arbitrary field , let $M^\Delta$ be the K-space whose basis elements are the $\Delta$-tabloids. Extend the action of $\mathcal{W}$ on $\tau_\Delta$ linearly on $M^\Delta$, then $M^\Delta$ becomes a $K\mathcal{W}$-module. Then we have the following lemma.

**Lemma 3.6** *The $K\mathcal{W}$-module $M^\Delta$ is a cyclic $K\mathcal{W}$-module generated by any one tabloid and $dim_K M^\Delta = [\mathcal{W} : \mathcal{W}(J)]$.*

Now we proceed to consider the possibility of constructing a $K\mathcal{W}$-module which corresponds to the Specht module in the case of symmetric groups. In order to do this we need to define a $\Delta$-polytabloid.

**Definition 3.7** Let $\{J, J'\}$ be a useful system in $\Phi$. Let

$$\kappa_{J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma \quad \text{and} \quad e_{J,J'} = \kappa_{J'}\{\bar{J}\}$$

where $s$ is the sign function defined in Section 2 . Then $e_{J,J'}$ is called the generalized $\Delta$-*polytabloid* associated with $J$.

If w $\in \mathcal{W}(\Phi)$ , then

$$\begin{aligned} w\kappa_{J'} &= \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)w\sigma \\ &= \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)(w\sigma w^{-1})w \\ &= \{\sum_{\sigma \in \mathcal{W}(wJ')} s(\sigma)\sigma\}w. \end{aligned}$$

Hence, for all $w \in \mathcal{W}(\Phi)$, we have

$$w\, e_{J,J'} = \kappa_{wJ'}\{\overline{wJ}\} = e_{wJ,wJ'}. \tag{3.1}$$

Let $S^{J,J'}$ be the subspace of $M^{\Delta}$ generated by $e_{wJ,wJ'}$ where $w \in \mathcal{W}$. Then by (3.1) $S^{J,J'}$ is a $K\mathcal{W}$-submodule of $M^{\Delta}$, which is called a *generalized Specht module*. Then we have the following theorem .

**Theorem 3.8** *The $K\mathcal{W}$-module $S^{J,J'}$ is a cyclic submodule generated by any $\Delta$-polytabloid.*

The following proposition notes some isomorphisms between Specht modules.

**Proposition 3.9** *Let $\{J, J'\}$ be a useful system in $\Phi$. Then we have the following isomorphisms:*

*(i) If $w \in \mathcal{W}$, then $S^{J,J'} \cong S^{wJ,wJ'}$.*

*(ii) If $w \in \mathcal{W}(J)$, then $S^{J,J'} \cong S^{J,wJ'}$.*

*(iii) If $w \in \mathcal{W}(J')$, then $S^{J,J'} \cong S^{wJ,J'}$.*

*Proof.* (i) If $w \in \mathcal{W}$, then define $T_w : S^{J,J'} \to S^{wJ,wJ'}$ by

$$T_w(e_{J,J'}) = w(e_{J,J'}).$$

Then by (3.1) we have $w(e_{J,J'}) = e_{wJ,wJ'} \in S^{wJ,wJ'}$. Clearly $T_w$ is an isomorphism. The statements (ii), (iii) follow easily from (i). □

Proposition 3.9 says that a generalized Specht module is dependent only on the Dynkin diagram $\Delta$ and $\Delta'$ of $J$ and $J'$ respectively, thus, from now on it will be denoted by $S^{\Delta,\Delta'}$.

A Specht module is spanned by the $e_{wJ,wJ'}$ for all $w \in \mathcal{W}$ ; the next lemma shows that we need only consider a certain subset of $\mathcal{W}$ .

**Lemma 3.10** *Let $\{J, J'\}$ be a useful system in $\Phi$ . Then $S^{\Delta,\Delta'}$ is generated by $e_{dJ,dJ'}$ , where $d \in D_{\Psi'}$.*

*Proof.* If $w \in \mathcal{W}$, then by (2.7), $w = d\rho$, where $d \in D_{\Psi'}$ and $\rho \in \mathcal{W}(J')$ and since

$$\rho e_{J,J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\rho\sigma\{\bar{J}\} = s(\rho)e_{J,J'},$$

we have

$$e_{wJ,wJ'} = we_{J,J'} = d\rho e_{J,J'} = s(\rho)de_{J,J'} = s(\rho)e_{dJ,dJ'}.$$ □

**Lemma 3.11** *Let $\{J, J'\}$ be a useful system in $\Phi$ and let $d \in D_{\Psi}$. If $\{\overline{dJ}\}$ appears in $e_{J,J'}$ then it appears only once.*

*Proof.* If $\sigma, \sigma^{'} \in \mathcal{W}(J^{'})$ and suppose that $\sigma = dw, \sigma^{'} = dw^{'}$ where $w, w^{'} \in \mathcal{W}(J)$. Then $d = \sigma w^{-1} = \sigma^{'} w^{'^{-1}}$ and $\sigma^{'^{-1}} \sigma = w^{'^{-1}} w \in \mathcal{W}(J) \cap \mathcal{W}(J^{'}) =< e >$. Hence we have $w = w^{'}$ and $\sigma = \sigma^{'}$. Then $\{\overline{dJ}\}$ appears in $e_{J,J'}$ only once. □

**Corollary 3.14** *If $\{J, J^{'}\}$ be a useful system in $\Phi$, then $e_{J,J'} \neq 0$.*

*Proof.* By Lemma 3.11 if $\{\overline{\sigma J}\}$ appears in $e_{J,J'}$ then all the $\{\overline{\sigma J}\}$ are different, where $\sigma \in \mathcal{W}(J^{'})$. But $\{\{\overline{\sigma J}\} \mid \sigma \in \mathcal{W}(J^{'})\}$ is a linearly independent subset of $\{\{\overline{dJ}\} \mid d \in D_{\Psi}\}$. If $e_{J,J'} = 0$ then $s(\sigma) = (-1)^{l(\sigma)} = 0$ for all $\sigma \in \mathcal{W}(J^{'})$. This is a contradiction and so $e_{J,J'} \neq 0$. □

**Remark 3** In [11] the second author has defined a partial dual of a subsystem $\Psi$ simply as a subsystem $\Psi^{'}$ of $\Phi$ which is contained in $\Phi \setminus \Psi$. However the following lemma shows that the extra condition $\mathcal{W}(J) \cap \mathcal{W}(J^{'}) =< e >$ in our definition of a useful system is also necessary. Unfortunately this condition which is a group theoretical one is not easily checked and it would be useful if it could be replaced by a criteria in terms of the root system only.

**Lemma 3.13** *If there exists $w \in \mathcal{W}(J) \cap \mathcal{W}(J^{'})$ such that $w$ has order 2, and $s(w) = -1$ then $e_{J,J'} = 0$.*

*Proof.* If $w \in \mathcal{W}(J) \cap \mathcal{W}(J^{'})$ and $w$ has order 2, then

$$(e - w)\{\bar{J}\} = \{\bar{J}\} - \{\bar{J}\} = 0$$

and also $\{e, w\}$ is a subgroup of $\mathcal{W}(J^{'})$. Thus we can select signed coset representatives $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_s$ for $\{e, w\}$ in $\mathcal{W}(J^{'})$ such that

$$e_{J,J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma\{\bar{J}\} = (\sum_{i=1}^{s} \sigma_i)(e - w)\{\bar{J}\} = 0.$$ □

**Example 3.14** Let $\Phi = \mathbf{B_3}$ and $\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3\}$. Let $\Psi = \mathbf{3A_1}$ be the subsystem of $\Phi$ with simple system $J = \{\alpha_1 = \epsilon_1 - \epsilon_2, \tilde{\alpha} = \epsilon_1 + \epsilon_2, \alpha_3 = \epsilon_3\}$ and let $\Psi^{'} = \mathbf{3A_1}$ be the subsystem of $\Phi$ with $J^{'} = \{\alpha_2 = \epsilon_2 - \epsilon_3, \alpha_1 + \alpha_2 + \alpha_3 = \epsilon_1, \alpha_2 + 2\,\alpha_3 = \epsilon_2 + \epsilon_3\}$. Then $\Psi \cap \Psi^{'} = \emptyset$. But

$\mathcal{W}(J)$ is given by the elements

$\{e, \tau_1, \tau_3, \tau_1\tau_3, \tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2, \tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1, \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2, \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1\}$,

$\mathcal{W}(J^{'})$ is given by the elements

$\{e, \tau_2, \tau_1\tau_2\tau_3\tau_2\tau_1, \tau_3\tau_2\tau_3, \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1, \tau_3\tau_2\tau_3\tau_2, \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1, \tau_1\tau_2\tau_3\tau_1\tau_2\tau_1\}$.

It follows that $w = \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1 \in \mathcal{W}(J) \cap \mathcal{W}(J^{'})$ and $e_{J,J'} = 0$. □

**Lemma 3.15** *Let $\{J, J_1'\}$ and $\{J, J_2'\}$ be useful systems in $\Phi$. If $\Psi_1' \subseteq \Psi_2'$, then $S^{J,J_2'}$ is a $K\mathcal{W}$-submodule of $S^{J,J_1'}$, where $J_1'$ and $J_2'$ are simple systems for $\Psi_1'$ and $\Psi_2'$ respectively.*

*Proof.* Since $\Psi_1' \subseteq \Psi_2'$, $\mathcal{W}(J_1')$ is a subgroup of $\mathcal{W}(J_2')$ and we can select coset representatives $a_1, a_2, \ldots, a_n$ for $\mathcal{W}(J_1')$ in $\mathcal{W}(J_2')$ such that $\mathcal{W}(J_2') = \bigcup_{i=1}^{n} a_i \mathcal{W}(J_1')$. Then we have

$$\begin{aligned} e_{J,J_2'} &= \sum_{\sigma \in \mathcal{W}(J_2')} s(\sigma)\sigma\{\bar{J}\} \\ &= \sum_{i=1}^{n} a_i s(a_i) \sum_{\sigma \in \mathcal{W}(J_1')} s(\sigma)\sigma\{\bar{J}\} \\ &= (\sum_{i=1}^{n} a_i s(a_i)) e_{J,J_1'}. \end{aligned}$$

□

Now we consider under what conditions $S^{\Delta,\Delta'}$ is irreducible.

**Lemma 3.16** *Let $\{J, J'\}$ be a useful system in $\Phi$ and let $d \in D_\Psi$. Then the following conditions are equivalent:*
*(i) $\{\overline{dJ}\}$ appears with non-zero coefficient in $e_{J,J'}$.*
*(ii) There exists $\sigma \in \mathcal{W}(J')$ such that $\sigma\{\bar{J}\} = \{\overline{dJ}\}$.*
*(iii) There exists $\rho \in \mathcal{W}(J)$ and $\sigma \in \mathcal{W}(J')$ such that $d = \sigma\rho$.*

*Proof.* The equivalence of (i) and (ii) follows directly from the formula

$$e_{J,J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma\{\bar{J}\}.$$

(ii) $\Rightarrow$ (iii): Suppose that there exists $\sigma \in \mathcal{W}(J')$ such that $\sigma\{\bar{J}\} = \{\overline{dJ}\}$. Then we have $\sigma^{-1}d\{\bar{J}\} = \{\bar{J}\}$. By the definition of equivalence there exists $\rho \in \mathcal{W}(J)$ such that $\sigma^{-1}d\bar{J} = \rho\bar{J}$. Then $\rho^{-1}\sigma^{-1}d \in \mathcal{W}(J^\perp) \cap \mathcal{W}(J'^\perp)$. Since $\{J, J'\}$ is a useful system in $\Phi$ then $d = \sigma\rho$, where $\sigma \in \mathcal{W}(J')$ and $\rho \in \mathcal{W}(J)$.
(iii) $\Rightarrow$ (ii): Let $d = \sigma\rho$, where $\sigma \in \mathcal{W}(J')$ and $\rho \in \mathcal{W}(J)$. Since $\rho \in \mathcal{W}(J)$, $\rho\bar{J} = \bar{J}$ then $\{\overline{dJ}\} = \{\overline{\sigma\rho J}\} = \{\overline{\sigma J}\}$. □

**Lemma 3.17** *Let $\{J, J'\}$ be a useful system in $\Phi$ and let $d \in D_\Psi$. If $\{\overline{dJ}\}$ appears in $e_{J,J'}$ then $d\Psi \cap \Psi' = \emptyset$.*

*Proof.* Let $\alpha \in d\Psi$. If $\{\overline{dJ}\}$ appears in $e_{J,J'}$ then by Lemma 3.16 $d = \sigma\rho$, where $\sigma \in \mathcal{W}(J')$ and $\rho \in \mathcal{W}(J)$. Then $\alpha \in \sigma\rho\Psi$. Since $\rho \in \mathcal{W}(J)$, then $\alpha \in \sigma\Psi$ and $\sigma^{-1}\alpha \in \Psi$. But $\Psi \cap \Psi' = \emptyset$, then $\sigma^{-1}\alpha \notin \Psi'$. Since $\sigma \in \mathcal{W}(J')$, $\sigma\Psi' = \Psi'$ then $\alpha \notin \Psi'$. □

**Lemma 3.18** *Let $\{J, J'\}$ be a useful system in $\Phi$ and let $d \in D_\Psi$. Let $d\Psi \cap \Psi' \neq \emptyset$. Then $\kappa_{J'}\{\overline{dJ}\} = 0$.*

*Proof.* If $d\Psi \cap \Psi' \neq \emptyset$, then let $\alpha \in d\Psi \cap \Psi'$ and so $\tau_\alpha \in \mathcal{W}(dJ) \cap \mathcal{W}(J')$. Thus

$$(e - \tau_\alpha)\{\overline{dJ}\} = \{\overline{dJ}\} - \{\overline{dJ}\} = 0.$$

Since $\{e, \tau_\alpha\}$ is a subgroup of $\mathcal{W}(J')$ then we can select signed coset representatives $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_s$ for $\{e, \tau_\alpha\}$ in $\mathcal{W}(J')$ such that

$$\kappa_{J'}\{\overline{dJ}\} = (\sum_{i=1}^{s} \sigma_i)(e - \tau_\alpha)\{\overline{dJ}\} = 0. \qquad \square$$

The converse of Lemma 3.17 is not true in general, which leads to the following definition.

**Definition 3.19** A useful system $\{J, J'\}$ in $\Phi$ is called a *good system* if $d\Psi \cap \Psi' = \emptyset$ for $d \in D_\Psi$ then $\{\overline{dJ}\}$ appears with non-zero coefficient in $e_{J,J'}$.

**Lemma 3.20** *Let $\{J, J'\}$ be a good system in $\Phi$ and let $d \in D_\Psi$.*
*(i) If $\{\overline{dJ}\}$ does not appear in $e_{J,J'}$, then $\kappa_{J'}\{\overline{dJ}\} = 0$.*
*(ii) If $\{\overline{dJ}\}$ appears in $e_{J,J'}$, then there exists $\sigma \in \mathcal{W}(J')$ such that*

$$\kappa_{J'}\{\overline{dJ}\} = s(\sigma)e_{J,J'}.$$

*Proof.* (i) If $\{\overline{dJ}\}$ does not appear in $e_{J,J'}$, then by definition of a good system we have $d\Psi \cap \Psi' \neq \emptyset$ and hence by Lemma 3.17 we have $\kappa_{J'}\{\overline{dJ}\} = 0$.
(ii) Since $\{\overline{dJ}\}$ appears in $e_{J,J'}$ it follows by Lemma 3.16 that there exists $\sigma \in \mathcal{W}(J')$ such that $\sigma\{\bar{J}\} = \{\overline{dJ}\}$. Then we have

$$\kappa_{J'}\{\overline{dJ}\} = (\sum_{\rho \in \mathcal{W}(J')} s(\rho)\rho)\{\overline{\sigma J}\} = s(\sigma)e_{J,J'}. \qquad \square$$

**Corollary 3.21** *Let $\{J, J'\}$ be a good system in $\Phi$. If $m \in M^\Delta$ then $\kappa_{J'}m$ is a multiple of $e_{J,J'}$.*

*Proof.* If $m \in M^\Delta$ then we have

$$m = \sum_{d \in D_\Psi} \alpha_d\{\overline{dJ}\}, \quad \text{where } \alpha_d \in K,$$

$$\kappa_{J'}m = \sum_{d \in D_\Psi} \alpha_d \kappa_{J'}\{\overline{dJ}\}.$$

But by Lemma 3.20 $\kappa_{J'} m = \lambda e_{J,J'}$, where $\lambda \in K$. $\square$

We now define a bilinear form $< \, , \, >$ on $M^{\Delta}$ by setting

$$< \{\bar{J}_1\}, \{\bar{J}_2\} >= \begin{cases} 1 & \text{if } \{\bar{J}_1\} = \{\bar{J}_2\} \\ 0 & \text{otherwise.} \end{cases}$$

This is a symmetric, non-singular, $\mathcal{W}$-invariant bilinear form on $M^{\Delta}$.

Now we can prove James' submodule theorem in this general setting.

**Theorem 3.22** *Let $\{J, J'\}$ be a good system in $\Phi$. Let $U$ be submodule of $M^{\Delta}$. Then either $S^{\Delta,\Delta'} \subseteq U$ or $U \subseteq S^{\Delta,\Delta'\perp}$, where $S^{\Delta,\Delta'\perp}$ is complement of $S^{\Delta,\Delta'}$ in $M^{\Delta}$.*

*Proof.* If $u \in U$ then

$$\begin{aligned} < u, e_{J,J'} > &=< u, \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma\{\bar{J}\} > \\ &= \sum_{\sigma \in \mathcal{W}(J')} < s(\sigma)\sigma^{-1}u, \{\bar{J}\} > \\ &=< \kappa_{J'} u, \{\bar{J}\} > . \end{aligned}$$

But by Corollary 3.21 $\kappa_{J'} u = \lambda e_{J,J'}$, for some $\lambda \in K$. If $\lambda \neq 0$ for some $u \in U$, then $e_{J,J'} \in U$, that is, $S^{\Delta,\Delta'} \subseteq U$. However, if $\lambda = 0$ for all $u \in U$, then $< u \, , \, e_{J,J'} > = 0$, that is, $U \subseteq S^{\Delta,\Delta'\perp}$. $\square$

We can now prove our principal result.

**Theorem 3.23** *Let $\{J, J'\}$ be a good system in $\Phi$ . The $K\mathcal{W}$-module*

$$D^{\Delta,\Delta'} = S^{\Delta,\Delta'} / S^{\Delta,\Delta'} \cap S^{\Delta,\Delta'\perp}$$

*is zero or irreducible.*

*Proof.* If U is a submodule of $S^{\Delta,\Delta'}$ then U is a submodule of $M^{\Delta}$ and by Theorem 3.22 either $S^{\Delta,\Delta'} \subseteq U$ in which case $U = S^{\Delta,\Delta'}$ or $U \subseteq S^{\Delta,\Delta'\perp}$ and $U \subseteq S^{\Delta,\Delta'} \cap S^{\Delta,\Delta'\perp}$, which completes the proof. $\square$

In the case of K = $\mathbf{Q}$ or any field of characteristic zero $< \, , \, >$ is an inner product and $D^{\Delta,\Delta'} = S^{\Delta,\Delta'}$. Thus if for a subsystem $\Psi$ of $\Phi$ a good system $\{J, J'\}$ can be found, then we have a construction for irreducible $K\mathcal{W}$-modules. Hence it is essential to show for each subsystem that a good system exists which satisfies Definition 3.19.

In the following example, we show how a good system may be constructed in most cases for the Weyl group of type $\mathbf{D_4}$. In a future publication, we shall present an algorithm for constructing a good system for certain subsystems; indeed this algorithm will give a good system with additional properties which will lead to the construction of a $K$-basis for our Specht modules $S^{\Delta,\Delta'}$, which correspond to the basis consisting of standard tableaux in the case of symmetric groups.

**Example 3.24** Let $\Phi = \mathbf{D_4}$ with simple system

$$\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \epsilon_3 + \epsilon_4\}.$$

Let $e$, $\tau_2$, $(\tau_1\tau_2)$, $(\tau_1\tau_3\tau_2)$, $(\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)$, $(\tau_1\tau_4\tau_2)$, $(\tau_1\tau_2\tau_1\tau_4\tau_2\tau_1)$, $(\tau_3\tau_4\tau_2)$, $(\tau_2\tau_3\tau_2\tau_4\tau_2\tau_3)$, $(\tau_1\tau_3\tau_4\tau_2)$, $(\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4)$, $(\tau_2\tau_1\tau_3\tau_4\tau_2)$, $(\tau_2\tau_1\tau_4\tau_2\tau_1\tau_3)$ be representatives of conjugate classes $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$, $C_7$, $C_8$, $C_9$, $C_{10}$, $C_{11}$, $C_{12}$, $C_{13}$ respectively of $\mathcal{W}(\mathbf{D_4})$. Then we get the character table of $\mathcal{W}(\mathbf{D_4})$ as shown in Table 1.

Table 1

The non-conjugate subsystems of $\mathbf{D_4}$ are:

(1) $\Psi_1 = \mathbf{A_3}$ with simple system $J_1 = \{1000, 0100, 0010\}$

(2) $\Psi_2 = \mathbf{A_3'}$ with simple system $J_2 = \{1000, 0100, 0001\}$

(3) $\Psi_3 = \mathbf{A_3''}$ with simple system $J_3 = \{0100, 0010, 0001\}$

(4) $\Psi_4 = \mathbf{4A_1}$ with simple system $J_4 = \{1000, 1211, 0010, 0001\}$

(5) $\Psi_5 = \mathbf{3A_1}$ with simple system $J_5 = \{1000, 0010, 0001\}$

(6) $\Psi_6 = \mathbf{A_2}$ with simple system $J_6 = \{1000, 0100\}$

(7) $\Psi_7 = \mathbf{2A_1}$ with simple system $J_7 = \{1000, 0010\}$

(8) $\Psi_8 = \mathbf{2A_1'}$ with simple system $J_8 = \{1000, 0001\}$

(9) $\Psi_9 = \mathbf{2A_1''}$ with simple system $J_9 = \{0010, 0001\}$

(10) $\Psi_{10} = \mathbf{A_1}$ with simple system $J_{10} = \{1000\}$

(11) $\Psi_{11} = \mathbf{D_4}$ with simple system $J_{11} = \{1000, 0100, 0010, 0001\}$

(12) $\Psi_{12} = \emptyset$ with simple system $J_{12} = \emptyset$.

Let $\Psi_1 = \mathbf{A_3}$ be the subsystem of $\mathbf{D_4}$ with $J_1 = \{1000, 0100, 0010\}$. Let $\Psi_1^{'} = \mathbf{2A_1}$ be the subsystem of $\Phi$ which is contained in $\Phi \setminus \Psi$, with simple system $J_1^{'} = \{1101, 0111\}$. Since $\mathcal{W}(J_1) \cap \mathcal{W}(J_1^{'}) =< e >$ and $\mathcal{W}(J_1^{\perp}) \cap \mathcal{W}(J_1^{'\perp}) =< e >$, then $\{J_1, J_1^{'}\}$ is a useful system in $\Phi$. Then $\tau_{\Delta_1}$ contains the $\Delta_1$-*tabloids*:

$$\begin{aligned}
\{\bar{J}_1\} &= \{1000, 0100, 0010; 1101, 0111\}\\
\{\tau_4\bar{J}_1\} &= \{1000, 0101, 0010; 1100, 0110\}\\
\{\tau_2\tau_4\bar{J}_1\} &= \{1100, 0001, 0110; 1000, 0010\}\\
\{\tau_1\tau_2\tau_4\bar{J}_1\} &= \{0100, 0001, 1110; -1000, 0010\}\\
\{\tau_3\tau_2\tau_4\bar{J}_1\} &= \{1110, 0001, 0100; 1000, -0010\}\\
\{\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} &= \{0110, 0001, 1100; -1000, -0010\}\\
\{\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} &= \{0010, 0101, 1000; -1100, -0110\}\\
\{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} &= \{0010, 0100, 1000; -1101, -0111\}
\end{aligned}$$

For $d = e, \tau_1\tau_2\tau_4, \tau_3\tau_2\tau_4, \tau_4\tau_2\tau_1\tau_3\tau_2\tau_4$ we have $d\Psi_1 \cap \Psi_1^{'} = \emptyset$. Since

$$e_{J_1, J_1'} = \{\bar{J}\} - \{\tau_1\tau_2\tau_4\bar{J}\} - \{\tau_3\tau_2\tau_4\bar{J}\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}\}$$

then $\{J_1, J_1^{'}\}$ is a good system in $\Phi$.

Now let $K$ be a field with Char$K$=0. Let $M^{\Delta_1}$ be $K$-space whose basis elements are the $\Delta_1$-tabloids. Let $S^{\Delta_1, \Delta_1^{'}}$ be the corresponding $K\mathcal{W}$-submodule of $M^{\Delta_1}$, then by definition of the Specht module we have

$$S^{\Delta_1, \Delta_1^{'}} = Sp\{e_{J_1, J_1'}, e_{\tau_4 J_1, \tau_4 J_1'}, e_{\tau_2\tau_4 J_1, \tau_2\tau_4 J_1'}\}$$

where

$$\begin{aligned}
e_{J_1, J_1'} &= \{\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\}\\
e_{\tau_4 J_1, \tau_4 J_1'} &= \{\tau_4\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\}\\
e_{\tau_2\tau_4 J_1, \tau_2\tau_4 J_1'} &= \{\tau_2\tau_4\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_1\tau_3\tau_2\tau_4\bar{J}_1\}.
\end{aligned}$$

Let $T_1$ be the matrix representation of $\mathcal{W}$ afforded by $S^{\Delta_1, \Delta_1^{'}}$ with character $\psi_1$ and let $\tau_2$ be the representative of the conjugate class $C_2$. Then

$$\begin{aligned}
\tau_2(e_{J_1, J_1'}) &= \{\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} = e_{J_1, J_1'}\\
\tau_2(e_{\tau_4 J_1, \tau_4 J_1'}) &= \{\tau_2\tau_4\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} = e_{\tau_2\tau_4 J_1, \tau_2\tau_4 J_1'}\\
\tau_2(e_{\tau_2\tau_4 J_1, \tau_2\tau_4 J_1'}) &= \{\tau_4\bar{J}_1\} - \{\tau_1\tau_2\tau_4\bar{J}_1\} - \{\tau_3\tau_2\tau_4\bar{J}_1\} + \{\tau_2\tau_1\tau_3\tau_2\tau_4\bar{J}_1\} = e_{\tau_4 J_1, \tau_4 J_1'}
\end{aligned}$$

Thus we have

$$T_1(\tau_2) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi_1(\tau_2) = 1.$$

Table 2

By a similar calculation to the above it can be shown that $\psi_1 = \chi_4$. By the same method to the above, we get Table 2.
We note that the irreducible modules corresponding to the characters $\chi_{10}$, $\chi_{11}$, $\chi_{12}$ have not been obtained. We now show how an additional irreducible character is obtained. Let $\Psi_2^{'} = \mathbf{A_1}$ be the subsystem of $\Phi$ with simple system $J_2^{'} = \{1101\}$. Then $\{J_1, J_2^{'}\}$ is a useful system in $\Phi$. Since $\Psi_2^{'} \subset \Psi_1^{'}$, by Lemma 3.15 $S^{\Delta_1,\Delta_1^{'}}$ is a $K\mathcal{W}$-submodule of $S^{\Delta_1,\Delta_2^{'}}$. By a similar calculation to the above it can be shown that the corresponding character of $\mathcal{W}$ afforded by $S^{\Delta_1,\Delta_2^{'}}/S^{\Delta_1,\Delta_1^{'}}$ is $\chi_{11}$.

In fact, we have a *Specht series* corresponding to series

$$\emptyset \subset \{1101\} \subset \{1101, 0111\}.$$

Unfortunately, no further irreducible character of $\mathcal{W}(\mathbf{D_4})$ are obtained by similar calculations for the other subsystems.

We see that not all the irreducible modules for $\mathcal{W}(\mathbf{D_4})$ are obtained in this way. That is, there are not a sufficient supply of subsystems or Weyl subgroups to give a complete set of irreducible modules. In the next section, it will be shown that the irreducible character $\chi_{10}, \chi_{12}$ of degree 4 and 6 respectively can be obtained by considering $\mathcal{W}(\mathbf{B_3})$ and $\mathcal{W}(\mathbf{G_2})$ as Steinberg subgroups of $\mathcal{W}(\mathbf{D_4})$.

## 4. Additional Specht Modules for Weyl Groups

In this section we consider our groups as reflection groups and thus, include reflection subgroups these groups. We modify the algorithm for determining subsystems of $\Phi$ in (2.4) so as to include Steinberg subgroups of the type described in (2.8). Detailed proofs are not always included as they are either modifications of those in Section 3 or in the earlier paper [11]. In this case, it is necessary to express our results in terms of the subgroups rather than the root subsystems as in Section 3.

**Algorithm 4.1:**

*(i) Form the extended Dynkin diagram of $\Phi$. This is obtained by adding one further node to the graph of $\Phi$ corresponding to the negative of the highest root,*

*(ii) Delete one or more nodes in all possible ways from the extended Dynkin diagram,*

*(iii) Let $\rho$ be a symmetry of the remaining Dynkin diagram. Then follow (2.8) to form a Steinberg subgroup for this remaining Dynkin diagram.*

*(iv) Take also the duals of the diagrams obtained in the same way from the dual system $\tilde{\Phi}$ which is obtained from by interchanging long and short roots,*

*(v) Repeat the process with the Dynkin diagram obtained and continue any number of times.*

**Definition 4.2** An *extended subsystem* $\Psi$ is the root system corresponding to the Dynkin diagram obtained by Algorithm 4.1.

Let $\Psi$ be an extended subsystem of $\Phi$ with simple system $J$ and Dynkin diagram $\Delta$. By Algorithm 4.1 we have

(i) If the symmetry $\rho$ is the identity, then $\mathcal{W}(J)$ is a Weyl subgroup of $\mathcal{W}(\Phi)$.

(ii) If $\rho$ is a non-trivial symmetry, then $\mathcal{W}(J)$ is a Steinberg subgroup of $\mathcal{W}(\Phi)$.

**Definition 4.3** If $\Psi$ is an extended subsystem of $\Phi$, then the reflection subgroup $\mathcal{W}(J)$ of $\mathcal{W}$ which is generated by the $\tau_\alpha, \alpha \in \Psi$ is called a *Steinberg-Weyl ($S\mathcal{W}$)-subgroup* of $\mathcal{W}$.

We now define equivalence relation on the elements of $\mathcal{W}$.

$$w^{'} \sim w \quad \text{if and only if} \quad w^{'}w^{-1} \in \mathcal{W}(J) \text{ for } w, w^{'} \in \mathcal{W}(\Phi).$$

**Definition 4.4** Let $w \in \mathcal{W}$ . The equivalence class

$$\{w\mathcal{W}(J)\} = \{w^{'} \mid w^{'} \sim w\}$$

is called a $\Delta$-*tabloid.*

Let $\tau_\Delta$ be the set of all $\Delta$-tabloids. In this new setting it is clear that the elements of $\tau_\Delta$ are the left coset representatives of $\mathcal{W}(J)$ in $\mathcal{W}$ and the number of distinct elements in $\tau_\Delta$ is $[\mathcal{W} : \mathcal{W}(J)]$.

The group $\mathcal{W}$ acts on $\tau_\Delta$ according to

$$w^{'}\{w\mathcal{W}(J)\} = \{w^{'}w\mathcal{W}(J)\} \quad \text{for all } w^{'} \in \mathcal{W}.$$

This action is again easily seen to be well defined.

Now if K is arbitrary field , let $M^\Delta$ be the K-space whose basis elements are $\Delta$ - tabloids. Extend the action of $\mathcal{W}$ on $\tau_\Delta$ linearly on $M^\Delta$, then $M^\Delta$ becomes $K\mathcal{W}$-module. Then we have the following theorem.

**Theorem 4.5** *The $K\mathcal{W}$-module $M^{\Delta}$ is the permutation module on the $S\mathcal{W}$-subgroup $\mathcal{W}(J)$ $M^{\Delta}$ is a cyclic $K\mathcal{W}$-module generated by any one tabloid and $dim_K M^{\Delta} = [\mathcal{W} : \mathcal{W}(J)]$.*

Now we can consider the possibility of constructing a $K\mathcal{W}$-module $S^{J,J'}$ which generalises the Specht modules described in Section 3. In this direction we first define a useful dual of a $S\mathcal{W}$-subgroup $\mathcal{W}(J)$.

**Definition 4.6** A *useful dual* of $\mathcal{W}(J)$ is a $S\mathcal{W}$-subgroup $\mathcal{W}(J')$ of $\mathcal{W}(\Phi)$ which satisfies $\mathcal{W}(J) \cap \mathcal{W}(J') = < e >$.

Then we have the following lemma.

**Lemma 4.7** *If $\mathcal{W}(J')$ is a useful dual of $\mathcal{W}(J)$ then $\mathcal{W}(w\ J')$ is also a useful dual of $\mathcal{W}(J)$ for all $w \in \mathcal{W}(J)$.*

**Definition 4.8** Let $\mathcal{W}(J)$ be a $S\mathcal{W}$-subgroup of $\mathcal{W}(\Phi)$ and $\mathcal{W}(J')$ a useful dual of $\mathcal{W}(J)$. Let

$$\kappa_{J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma \quad \text{and} \quad e_{J,J'} = \kappa_{J'}\{\mathcal{W}(J)\},$$

where $s$ is the sign function defined in Section 2. Then $e_{J,J'}$ is called a *generalized $\Delta$-polytabloid.*

Let $S^{J,J'}$ be the subspace of $M^{\Delta}$ spanned by all the generalized $\Delta$-polytabloids $e_{wJ,wJ'}$, $w \in \mathcal{W}$. By the same method as in Section 3, $S^{J,J'}$ is a $K\mathcal{W}$-submodule of $M^{\Delta}$, which is called a *generalized Specht module.*

If $\mathcal{W}(J') = < e >$ then $S^{J,J'} \cong M^{\Delta}$.

**Theorem 4.9** *$S^{J,J'}$ is a cyclic submodule generated by any $\Delta$-polytabloid.*

**Lemma 4.10** *Let $\mathcal{W}(J)$ be a $S\mathcal{W}$-subgroup of $\mathcal{W}$ and let $\mathcal{W}(J')$ be a useful dual of $\mathcal{W}(J)$. Let $w'$ be a left coset representative of $\mathcal{W}(J')$ in $W$. Then $S^{J,J'}$ is spanned by $w' e_{J,J'}$.*

**Lemma 4.11** *Let $\mathcal{W}(J)$ be a $S\mathcal{W}$-subgroup of $\mathcal{W}$. If $\{w'\mathcal{W}(J)\}$ appears in $e_{J,J'}$ then it appears only once.*

*Proof.* See Lemma 3.11. □

**Corollary 4.12** *If $\mathcal{W}(J) \cap \mathcal{W}(J') = < e >$, then $e_{J,J'} \neq 0$.*

**Lemma 4.13** *Let $\mathcal{W}(J_1')$ and $\mathcal{W}(J_2')$ be useful duals of $\mathcal{W}(J)$. If $\mathcal{W}(J_1')$ is a $S\mathcal{W}$-subgroup of $\mathcal{W}(J_2')$, then $S^{J,J_2'}$ is a $K\mathcal{W}$-submodule of $S^{J,J_1'}$.*

*Proof.* This follows from Lemma 3.15 □

**Lemma 4.14** *If $w \in \mathcal{W}$ and $\mathcal{W}(J')$ is a useful dual of $\mathcal{W}(J)$, then the following conditions are equivalent:*
*(i) $\{w\mathcal{W}(J)\}$ appears with non-zero coefficient in $e_{J,J'}$.*
*(ii) There exists $\sigma \in \mathcal{W}(J')$ such that $\sigma\{\mathcal{W}(J)\} = \{w\mathcal{W}(J)\}$.*
*(iii) There exist $\rho \in \mathcal{W}(J)$, $\sigma \in \mathcal{W}(J')$ such that $w = \sigma\rho$.*

*Proof.* The equivalence of (i) and (ii) follows directly from the formula

$$e_{J,J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma\{\mathcal{W}(J)\}.$$

(ii) $\Rightarrow$ (iii): Suppose that there exists $\sigma \in \mathcal{W}(J')$ such that $\sigma\{\mathcal{W}(J)\} = \{w\mathcal{W}(J)\}$. Then we have $\sigma^{-1}w\{\mathcal{W}(J)\} = \{\mathcal{W}(J)\}$. By the definition of equivalence, $\sigma^{-1}w \in \mathcal{W}(J)$ and there exists $\rho \in \mathcal{W}(J)$ such that $\sigma^{-1}w = \rho$. Hence $w = \sigma\rho$, where $\sigma \in \mathcal{W}(J')$ and $\rho \in \mathcal{W}(J)$.
(iii) $\Rightarrow$ (ii): If $w = \sigma\rho$, then since $\rho \in \mathcal{W}(J)$, $\rho\{\mathcal{W}(J)\} = \{\mathcal{W}(J)\}$ and $\{w\mathcal{W}(J)\} = \{\sigma\mathcal{W}(J)\}$. □

**Definition 4.15** A useful dual $\mathcal{W}(J')$ of $\mathcal{W}(J)$ is called a *good dual* of $\mathcal{W}(J)$ if $\kappa_{J'}\{w\mathcal{W}(J)\} \neq 0$, then $\{w\mathcal{W}(J)\}$ appears in $e_{J,J'}$.

**Lemma 4.16** *Let $\mathcal{W}(J')$ be a good dual of $\mathcal{W}(J)$.*
*(i) If $\{w\mathcal{W}(J)\}$ does not appear in $e_{J,J'}$ then $\kappa_{J'}\{w\mathcal{W}(J)\} = 0$.*
*(ii) If $\{w\mathcal{W}(J)\}$ appears in $e_{J,J'}$, then there exists $\sigma \in \mathcal{W}(J')$ such that*

$$\kappa_{J'}\{w\mathcal{W}(J)\} = s(\sigma)e_{J,J'}.$$

*Proof.* See Lemma 3.20. □

**Corollary 4.17** *If $m \in M^{\Delta}$, then $\kappa_{J'}m$ is a multiple of $e_{J,J'}$.*

We now define a bilinear form $<\ ,\ >$ on $M^{\Delta}$ by setting

$$<\{w_1\mathcal{W}(J)\}, \{w_2\mathcal{W}(J)\}> = \begin{cases} 1 & \text{if } w_1 = w_2 \\ 0 & \text{otherwise.} \end{cases}$$

This is a symmetric, non-singular, $\mathcal{W}$-invariant, bilinear form on $M^{\Delta}$.

Now the analogue of James' submodule theorem can be proved in this more general setting.

**Theorem 4.18** *Let $U$ be submodule of $M^{\Delta}$. Then either $S^{J,J'} \subseteq U$ or $U \subseteq S^{J,J'^{\perp}}$ where $S^{J,J'^{\perp}}$ is complement of $S^{J,J'}$ in $M^{\Delta}$.*

We can now prove our principal result.

**Theorem 4.19** *The $K\mathcal{W}$-module $D^J = S^{J,J'}/S^{J,J'} \cap S^{J,J'^{\perp}}$ is zero or irreducible.*

*Proof.* See Theorem 3.23. □

Now to illustrate the above we show how a complete system of irreducible modules is determined in the case $\Phi = \mathbf{D_4}$.

**Example 4.20** Let $\Phi = \mathbf{D_4}$ with $\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \epsilon_3 + \epsilon_4\}$ and let $\Psi_1 = \mathbf{A_3}$ be the extended subsystem of $\mathbf{D_4}$ with simple system $J_1 = \{1000, 0100, 0010\}$. Let $\Psi_1^{'} = \mathbf{2A_1}$ be the extended subsystem of $\mathbf{D_4}$ with simple system $J_1^{'} = \{1101, 0111\}$. Since $\mathcal{W}(J_1) \cap \mathcal{W}(J_1^{'}) =< e >$, then $\mathcal{W}(J_1^{'})$ is a useful dual of $\mathcal{W}(J_1)$. Then $\tau_{\Delta_1}$ contains the $\Delta_1$-tabloids $\{\mathcal{W}(J_1)\}$, $\{\tau_4\mathcal{W}(J_1)\}$, $\{\tau_2\tau_4\mathcal{W}(J_1)\}$, $\{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\}$, $\{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}$, $\{\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}$, $\{\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}$, $\{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}$.

For $w = e, \tau_1\tau_2\tau_4, \tau_3\tau_2\tau_4, \tau_4\tau_2\tau_1\tau_3\tau_2\tau_4$, we have $\kappa_{J_1^{'}}\{w\mathcal{W}(J_1)\} \neq 0$. Since

$$e_{J_1,J_1'} = \{\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\},$$

then $\mathcal{W}(J_1^{'})$ is a good dual of $\mathcal{W}(J_1)$.

Now let $K$ be a field with Char$K$=0. Let $M^{\Delta_1}$ be $K$-space whose basis elements are the $\Delta_1$-tabloids. Let $S^{J_1,J_1^{'}}$ be the corresponding $K\mathcal{W}$-submodule of $M^{\Delta_1}$, then by definition of the Specht module we have

$$S^{J_1,J_1^{'}} = Sp\{e_{J_1,J_1'}, e_{\tau_4 J_1,\tau_4 J_1'}, e_{\tau_2\tau_4 J_1,\tau_2\tau_4 J_1'}\}$$

where

$$\begin{aligned}
e_{J_1,J_1'} &= \{\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}\\
e_{\tau_4 J_1,\tau_4 J_1'} &= \{\tau_4\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}\\
e_{\tau_2\tau_4 J_1,\tau_2\tau_4 J_1'} &= \{\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}.
\end{aligned}$$

Let $T_1$ be the matrix representation of $\mathcal{W}$ afforded by $S^{J_1,J_1^{'}}$ with character $\psi_1$ and let $\tau_2$ be the representative of the conjugate class $C_2$. Then

$$\begin{aligned}
\tau_2(e_{J_1,J_1'}) &= \{\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_4\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}\\
&= e_{J_1,J_1'}\\
\tau_2(e_{\tau_4 J_1,\tau_4 J_1'}) &= \{\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}\\
&= e_{\tau_2\tau_4 J_1,\tau_2\tau_4 J_1'}\\
\tau_2(e_{\tau_2\tau_4 J_1,\tau_2\tau_4 J_1'}) &= \{\tau_4\mathcal{W}(J_1)\} - \{\tau_1\tau_2\tau_4\mathcal{W}(J_1)\} - \{\tau_3\tau_2\tau_4\mathcal{W}(J_1)\} + \{\tau_2\tau_1\tau_3\tau_2\tau_4\mathcal{W}(J_1)\}\\
&= e_{\tau_4 J_1,\tau_4 J_1'}
\end{aligned}$$

Thus we have

$$T_1(\tau_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi_1(\tau_2) = 1.$$

By a similar calculation to the above it can be shown that $\psi_1 = \chi_4$. By the same method to the above, in addition to the irreducible modules in Example 3.24 we have,

We note that the irreducible modules corresponding to the characters $\chi_{10}, \chi_{11}$ have not been obtained. We now show how additional irreducible characters are obtained.

Let $\Psi_2' = \mathbf{A_1}$ be the extended subsystem of $\Phi$ with simple system $J_2' = \{1101\}$. Then $\mathcal{W}(J_2')$ is a useful dual of $\mathcal{W}(J_1)$. Since $\mathcal{W}(J_2')$ is a $S\mathcal{W}$-subgroup of $\mathcal{W}(J_1')$, by Lemma 4.13 $S^{J_1,J_1'}$ is a $K\mathcal{W}$-submodule of $S^{J_1,J_2'}$. By a similar calculation to the above it can be shown that the corresponding character of $\mathcal{W}$ afforded by $S^{J_1,J_2'}/S^{J_1,J_1'}$ is $\chi_{11}$.

Let $\Psi_2 = \mathbf{A_1}$ be the extended subsystem of $\Phi$ with simple system $J_2 = \{1000\}$. Let $\Psi_1' = \mathbf{B_3}$ be the extended subsystem of $\Phi$ with simple system $J_1' = \{0100, 0001, \frac{1}{2}(1000 + 1211)\}$ and $\Psi_2' = \mathbf{A_3}$ be another extended subsystem of $\Phi$ with simple system $J_2' = \{0100, 0001, 0010\}$. Then $\mathcal{W}(J_2')$ is a useful dual of $\mathcal{W}(J_2)$ and $\mathcal{W}(J_1')$ is a good dual of $\mathcal{W}(J_1)$. Since $\mathcal{W}(J_2')$ is a $S\mathcal{W}$-subgroup of $\mathcal{W}(J_1')$, by Lemma 4.13 $S^{J_2,J_1'}$ is a $K\mathcal{W}$-submodule of $S^{J_2,J_2'}$. By a similar calculation to the above it can be shown that the corresponding character of $\mathcal{W}$ afforded by $S^{J_2,J_2'}/S^{J_2,J_1'}$ is $\chi_{10}$. Thus we have obtained a complete set of irreducible modules for $\mathbf{D_4}$.

## References


[1] E. Al-Aamily, A. O. Morris and M. H. Peel: *The representations of the Weyl groups of type $B_n$*, Journal of Algebra 68 (1981), 298–305.

[2] A. Borel and J. de Siebenthal: *Les sous-groupes fermès connexes de rang maximium des groupes de Lie clos*, Comment. Math. Helv., 23 (1949), 200–221.

[3] N. Bourbaki: *Groupes et algèbres de Lie, Chapters 4,5,6*, Actualites Sci. Indust. 1337, (Hermann, Paris, 1968).

[4] R. W. Carter: *Conjugacy classes in the Weyl group*, Comp. Math., 25 (1972), 1–59.

[5] R. W. Carter: *Simple Groups of Lie Type*, (Wiley, London, Newyork, Sydney, Toronto, 1989).

[6] E. B. Dynkin: *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Trans. (2), 6 (1957), 111–244.

[7] J. E. Humphreys: *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Volume 9 (Springer-Verlag, Berlin, 1972).

[8] G. D. James: *The irreducible representations of the symmetric group*, Bull. Lond. Math. Soc., 8 (1976), 229–232.

[9] G. D. James: *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, Vol. 682 (Springer-Verlag, Berlin, 1978).

[10] G. D. James, A. Kerber: *The Representation Theory of the Symmetric Group*, Addison-Wesley Publishing Company (London, 1981).

[11] A. O. Morris: *Representations of Weyl groups over an arbitrary field*, Astèrisque 87-88 (1981), 267–287.

[12] A. O. Morris and A. J. Idowu: *Some combinatorial results for Weyl groups*, Proc. Camb. Phil. Soc. 101 (1987), 405–420.