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# The Semiring of Immersions of Manifolds

Dedicated to the 50th birthday of Prof. Bang-Yen Chen

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**Abstract.** In [2], [3], [4], [5] B.-Y. Chen introduced the tensor product immersion of a given Riemannian manifold. In this paper we study the tensor product of two immersions of differentiable manifolds. Although inspired by Chen's definition, our tensor product is a somewhat different concept, since it realizes an immersion of the product manifold. Basically, we will obtain a commutative semiring structure on the set of transversal immersions of manifolds in Euclidean spaces with operations direct sum $\oplus$ and tensor product $\otimes$. As a geometrical application of our notion of tensor products, we will make a remark on a conjecture of H. Hopf concerning Riemannian metrics on $S^2 \times S^2$.

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## 1. Introduction

We review some elementary facts about the tensor product of Euclidean spaces. Let $V$ and $W$ be two finite–dimensional real vector spaces of dimensions $m$ resp. $n$, equipped with metrics $g_V$ resp. $g_W$. Then the direct sum $V \oplus W$ is a $(m+n)$-dimensional vector space with metric $g_{V\oplus W}$ defined by :

$$g_{V\oplus W}((v,w),(v',w')) = g_V(v,v') + g_W(w,w'),$$

for all $v, v' \in V$ and $w, w' \in W$. The tensor product $V \otimes W$ is a $mn$-dimensional vector space with metric $g_{V\otimes W}$ defined by :

$$g_{V\otimes W}(v\otimes w, v'\otimes w') = g_V(v,v').g_W(w,w'),$$

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for all $v, v' \in V$ and $w, w' \in W$, and extended linearly.

Denote by $\mathbf{E}^n$ the $n$-dimensional Euclidean space with the Euclidean metric $g_n$. Define

$$i_{p,q} : \mathbf{E}^p \to \mathbf{E}^q : (x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_p, 0, \ldots, 0),$$

for each $p \leq q$.

**Lemma 1.** $\mathbf{E}^p \oplus \mathbf{E}^q \cong \mathbf{E}^{p+q}$; (ii) $\mathbf{E}^p \otimes \mathbf{E}^q \cong \mathbf{E}^{pq}$.

*Proof.* The first part is obvious. For (ii), the dimensions do agree. We still check the metrics. If $X = (x_1, \ldots, x_p), X' = (x'_1, \ldots, x'_p) \in \mathbf{E}^p$ and if $Y = (y_1, \ldots, y_q), Y' = (y'_1, \ldots, y'_q) \in \mathbf{E}^q$, then

$$\begin{aligned} g_{\mathbf{E}^p \otimes \mathbf{E}^q}(X \otimes Y, X' \otimes Y') &= g_p(X, X') g_q(Y, Y') = (\sum_{i=1}^{p} x_i x'_i)(\sum_{j=1}^{q} y_j y'_j) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{q} x_i x'_i y_j y'_j = \sum_{i=1}^{p} \sum_{j=1}^{q} (x_i y_j)(x'_i y'_j) \\ &= g_{pq}(X \otimes Y, X' \otimes Y'). \end{aligned}$$

□

## 2. The set of transversal immersions in Euclidean spaces.

**Definition.** Consider two differentiable manifolds $M$ and $N$ of dimensions $r$ resp. $s$ and assume that

$$f : M \to \mathbf{E}^m : p \mapsto (f_1(p), \ldots, f_m(p))$$

and

$$h : N \to \mathbf{E}^n : q \mapsto (h_1(q), \ldots, h_n(q))$$

are two immersions. Then we define the direct sum map

$$\begin{aligned} &f \oplus h : M \times N \to \mathbf{E}^{m+n} : \\ &(p, q) \mapsto (f(p), h(q)) = (f_1(p), \ldots, f_m(p), h_1(q), \ldots, h_n(q)) \end{aligned}$$

and the tensor product map

$$\begin{aligned} &f \otimes h : M \times N \to \mathbf{E}^{mn} : \\ &(p, q) \mapsto f(p) \otimes h(q) = (f_1(p)h_1(q), \ldots, f_1(p)h_n(q), \ldots, f_m(p)h_n(q)). \end{aligned}$$

**Lemma 2.** *Assume $M$ and $N$ are differentiable manifolds and $f : M \to \mathbf{E}^m$ and $h : N \to \mathbf{E}^n$ are two immersions. Then*

(1) $(f \oplus h)_* T_{(p,q)}(M \times N) \cong f_* T_p M \oplus h_* T_q N$;

(2) $(f \otimes h)_* T_{(p,q)}(M \times N) \cong f_*(T_p M) \otimes h(q) + f(p) \otimes h_*(T_q N)$.

*Proof.* Assume $X_p \in T_pM$ and $Y_q \in T_qN$. It is well known that $(f \oplus h)_*(X_p, Y_q) = (f_*(X_p), h_*(Y_q))$ and thus (1) follows. Assume now $\alpha : I \subseteq \mathbf{R} \to M$ and $\beta : J \subseteq \mathbf{R} \to N$ are curves such that $p = \alpha(0), X_p = \alpha'(0)$ and $q = \beta(0), Y_q = \beta'(0)$. Then, the $(ij)$-component of $(f \otimes h)_*(X_p, Y_q)$ equals

$$[(f_i \circ \alpha).(h_j \circ \beta)]'(0,0) = (f_i \circ \alpha)'(0).h_j(q) + f_i(p).(h_j \circ \beta)'(0)$$

and therefore

$$(f \otimes h)_*(X_p, Y_q) = f_*(X_p) \otimes h(q) + f(p) \otimes h_*(Y_q). \qquad \square$$

**Definition.** Let $f : M \to \mathbf{E}^m$ be an immersion of a differentiable manifold in a Euclidean space. Then $f$ is said to be *transversal in a point* $p \in M$ if and only if the position vector $f(p)$ is not tangent to $M$ at $p$, i.e. $f(p) \notin f_*(T_pM)$. If $f$ is transversal in every point of $M$, then $f$ shortly is called *transversal* .

**Proposition 3.** *Consider two differentiable manifolds $M$ and $N$ of dimensions $r$ resp. $s$ and assume that $f : M \to \mathbf{E}^m$ and $h : N \to \mathbf{E}^n$ are two transversal immersions. Then $f \oplus h$ and $f \otimes h$ are transversal immersions of $M \times N$ into $\mathbf{E}^{m+n}$, resp. $\mathbf{E}^{mn}$.*

*Proof.* Assume $X_p \in T_pM, Y_q \in T_qN$. Then, as $f$ and $h$ are immersions, it follows that $(f \oplus h)_*(X_p, Y_q) = (f_*(X_p), h_*(Y_q)) = 0$ if and only if $X_p = Y_q = 0$. Thus $f \oplus h$ is an immersion. Moreover, as $f$ and $h$ are transversal, so is $f \oplus h$. Indeed, assume $(f(p), h(q)) \in (f \oplus h)_*T_{(p,q)}(M \times N)$, then it immediately follows from Lemma 2 that $f(p) \in f_*T_p(M)$ and $h(q) \in h_*T_qN$, a contradiction.

We now prove that $f \otimes h$ is an immersion. Suppose $(f \otimes h)_*(X_p, Y_q) = 0$. Then, by Lemma 2, $f(p) \otimes h_*(Y_q) = -f_*(X_p) \otimes h(q)$ and so there exists $\lambda \in \mathbf{R}_0$ such that $f(p) = -\lambda f_*(X_p)$ and $h_*(Y_q) = (1/\lambda)h(q)$; a contradiction. Moreover, $f \otimes h$ is transversal. For assume that $f \otimes h$ is not transversal. Then there exist $p \in M, q \in N$ such that $(f \otimes h)(p,q) \in T_{(p,q)}(f \otimes h)(M \times N)$. Thus $f(p) \otimes h(q) = f(p) \otimes h_*(w_q) + f_*(v_p) \otimes h(q)$ for $v_p \in T_pM$ and $w_q \in T_qN$. But then $f(p) \otimes (h(q) - h_*(w_q)) = f_*(v_p) \otimes h(q)$ and there exists $\mu \in \mathbf{R}_0$ such that

$$\begin{cases} f(p) = \mu f_*(v_p) \\ h(q) - h_*(w_q) = 1/\mu h(q), \end{cases}$$

a contradiction. $\square$

**Remark.** Note that the transversality condition implies that the images of the immersions do not contain the origins of the Euclidean spaces. But to assure that the tensor product of two immersions is an immersion again, the latter condition is not sufficient, as the following example shows.

Put $f : \mathbf{E}^1 \to \mathbf{E}^2 : s \mapsto (s, (s-1)^2)$ and $h : \mathbf{R}_{>0} \to \mathbf{E}^2 : t \mapsto (t,t)$. Then these are two immersions, yielding a parabola resp. a halfline, but the tensor product $f \otimes h : \mathbf{E}^1 \times \mathbf{R}_{>0} \to \mathbf{E}^4 : (s,t) \mapsto (st, st, (s-1)^2t, (s-1)^2t)$ is not an immersion, as its Jacobian

$$\begin{pmatrix} t & s \\ 2(s-1)t & (s-1)^2 \\ t & s \\ 2(s-1)t & (s-1)^2 \end{pmatrix}$$

has rank one in all points $(1,t), t>0$. Indeed, $f$ is not transversal in 1 and $h$ is nowhere transversal.

It directly follows from its proof that Proposition 3 can also be stated in the following way (in the above terminology) :

**Proposition 3'.** *The tensor product $f \otimes h$ is an immersion if and only if $f$ and $h$ are immersions that satisfy the three following conditions :*
(1) *either $f$ or $h$ is transversal;*
(2) $0 \notin \mathrm{Im}(f)$;
(3) $0 \notin \mathrm{Im}(h)$.
If both $f$ and $h$ are transversal immersions (and then conditions (2) and (3) are automatically satisfied), then the tensor product $f \otimes h$ is a transversal immersion too.

Consequently, for example, the tensor product of the above parabola immersion $f$ with $h' : \mathbf{E}^1 \to \mathbf{E}^2 : t \mapsto (t, t-1)$ is indeed an immersion again (as $h$ is transversal).

We first look how these operations behave under orthogonal transformations :

**Proposition 4.** *Assume $f : M \to \mathbf{E}^m$ and $h : N \to \mathbf{E}^n$ are two transversal immersions. Let $A : \mathbf{E}^m \to \mathbf{E}^m$ and $B : \mathbf{E}^n \to \mathbf{E}^n$ be two orthogonal transformations of $\mathbf{E}^m$, resp. $\mathbf{E}^n$. Let $d_1 : M' \to M$ and $d_2 : N' \to N$ be two diffeomorphisms. Then*
(i) *$Af \oplus Bh$ and $f \oplus h$ are orthogonally equivalent transversal immersions in $\mathbf{E}^{m+n}$;*
(ii) *$Af \otimes Bh$ and $f \otimes h$ are orthogonally equivalent transversal immersions in $\mathbf{E}^{mn}$;*
(iii) *$(i_{m,N} \circ f) \oplus (i_{n,N} \circ h)$ and $i_{m+n,2N} \circ (f \oplus h)$ are orthogonally equivalent transversal immersions for all $N \geq m+n$;*
(iv) *$(i_{m,N} \circ f) \otimes (i_{n,N} \circ h)$ and $i_{mn,N^2} \circ (f \otimes h)$ are orthogonally equivalent transversal immersions for all $N \geq mn$;*
(v) *$(f \circ d_1) \oplus (h \circ d_2) = (f \oplus h) \circ (d_1, d_2)$, where $(d_1, d_2) : M' \times N' \to M \times N : (p', q') \mapsto (d_1(p'), d_2(q'))$ is also a diffeomorphism;*
(vi) *$(f \circ d_1) \otimes (h \circ d_2) = (f \otimes h) \circ (d_1, d_2)$.*

*Proof.* First remark that $Af$ and $Bh$ are still transversal immersions and therefore it follows from Theorem 4 that $Af \oplus Bh$ and $Af \otimes Bh$ are still transversal immersions in $\mathbf{E}^{m+n}$ resp. $\mathbf{E}^{mn}$. Analogously, $(i_{m,N} \circ f) \oplus (i_{n,N} \circ h)$ and $(i_{m,N} \circ f) \otimes (i_{n,N} \circ h)$ are also transversal immersions.
(i) It is clear that $Af \oplus Bh = (A \oplus B)(f \oplus h)$ and obviously, $A \oplus B$ is an orthogonal transformation of $\mathbf{E}^{m+n}$.
(ii) Put $A = (a_{i,j})_{1 \leq i,j \leq m}$ and $B = (b_{i,j})_{1 \leq i,j \leq n}$. Then, the $(i,j)$-component in $\mathbf{E}^{mn}$ $(1 \leq i \leq m, 1 \leq j \leq n)$ of $(Af \otimes Bh)(\alpha, \beta)$ is

$$\sum_{k=1}^{m} \sum_{l=1}^{n} (a_{ik})(b_{jl}) f_k h_l.$$

So $Af \otimes Bh = C(f \otimes h)$, where $C$ is a $(mn \times mn)$-matrix with entries $C_{i,j;k,l} = a_{i,k} b_{j,l}$ for $1 \leq i,k \leq m$ and $1 \leq j,l \leq n$. Therefore $C = A \otimes B$. Since the tensor product of two orthogonal matrices is orthogonal again, this proves the second point of the Theorem.

(iii) until (vi) are easy. □

**Remark.** It is easily seen from the Remark after Proposition 3 that the tensor product is not invariant under translations. Consider also the following example. Put

$$f : \mathbf{E}^1 \to \mathbf{E}^2 : s \mapsto (s, s+1).$$

Then the tensor product $f \otimes f$ is the intersection of the hyperquadric $xu = yz$ with the hyperplane $x - y - z + u = 1$ in $\mathbf{E}^4$ (with coordinates $(x, y, z, u)$). Let $A$ be the translation by $(0, 1)$. Then $Af \otimes Af$ is the intersection of the hyperquadric $xu = yz$ with the hyperplane $x - y - z + u = 4$ in $\mathbf{E}^4$. Clearly these two surfaces are not isometrically equivalent.

Define $Z$ to be the set of all transversal immersions of all differentiable manifolds in Euclidean spaces. We define a symmetric relation $\sim$ on $Z$ as follows :
(1) if $f : M \to \mathbf{E}^m \in Z$, then $f \sim i_{m,n} \circ f$ for all $n \geq m$;
(2) if $f : M \to \mathbf{E}^m$, $h : N \to \mathbf{E}^n \in Z$ and there exists an orthogonal transformation of $\mathbf{E}^N$, $N \geq \max(m, n)$, that carries $i_{m,N} \circ f$ into $i_{n,N} \circ h$, then $f \sim h$;
(3) if $f : M \to \mathbf{E}^m \in Z$ and $d : N \to M$ is a diffeomorphism, then $f \sim f \circ d$.
Obviously, $\sim$ is an equivalence relation on $Z$ and we put $\mathcal{I} = Z/\sim$. Then we obtain

**Theorem 5.** *$\mathcal{I}$, $\oplus$, $\otimes$ is a commutative semiring, with $e : \{0\} \to \mathbf{E}^1 : 0 \mapsto 1$ as identity element.*

*Proof.* It directly follows from Propositions 3 and 4 and the definition of $\mathcal{I}$ that $\oplus$ and $\otimes$ are well defined operations on $\mathcal{I}$. Associativity and distributivity are readily checked. Moreover it is clear that, if $f, h \in \mathcal{I}$, then $f \otimes h$ and $h \otimes f$ are equivalent in $Z$, since $M \times N$ and $N \times M$ are diffeomorphic and the respective images are orthogonally equivalent (by a change of coordinates). Thus they are equal in $\mathcal{I}$. Now take $f : M \to \mathbf{E}^m \in Z$, then

$$f \otimes e : M \times \{0\} \to \mathbf{E}^m : (p, 0) \mapsto f(p) \otimes 1 = f(p).$$

Thus $f \otimes e$ is diffeomorphic to $f$ and thus $f \otimes e \sim f$ in $Z$. □

## 3. An application.

As one example of a geometrical application, we consider the following well known

**Conjecture of H. Hopf** ([1], [6]). *On $S^2 \times S^2$ there exists no Riemannian metric with strictly positive sectional curvature.*

We consider the tensor product of the immersions of two spheres in $\mathbf{E}^3$ each being immersed as an ellipsoid with mass center the origin. Then it follows from Theorem 3 that, eventually after applying an orthogonal transformation of $\mathbf{E}^3$, we may assume the immersions are of the form

$$f_1 : S^2 \times \mathbf{E}^3 : \quad p = (x_1, y_1, z_1) \mapsto (a_1 x_1, b_1 y_1, c_1 z_1),$$
$$f_2 : S^2 \times \mathbf{E}^3 : \quad p' = (x_2, y_2, z_2) \mapsto (a_2 x_2, b_2 y_2, c_2 z_2),$$

where $a_1, b_1, c_1, a_2, b_2, c_2 > 0$. Then the tensor product $f : S^2 \times S^2 \to \mathbf{E}^9$ is given by

$$f(p,p') =$$
$$(a_1a_2x_1x_2, a_1b_2x_1y_2, a_1c_2x_1z_2, b_1a_2y_1x_2, b_1b_2y_1y_2, b_1c_2y_1z_2, c_1a_2z_1x_2, c_1b_2z_1y_2, c_1c_2z_1z_2).$$

We show that, according to the conjecture of H. Hopf, the sectional curvature of this immersed surface can not be strictly positive everywhere. A parametrization around $((1,0,0),(1,0,0))$ is given by

$$\begin{cases} x_1 = \cos\theta_1 \cos\phi_1 \\ y_1 = \cos\theta_1 \sin\phi_1 \\ z_1 = \sin\theta_1, \end{cases}$$

$$\begin{cases} x_2 = \cos\theta_2 \cos\phi_2 \\ y_2 = \cos\theta_2 \sin\phi_2 \\ z_2 = \sin\theta_2. \end{cases}$$

Then a basis of the tangent space at this point is given by

$$\begin{aligned} f_{\theta_1}(0,0,0,0) &= (0,0,0,0,0,0,c_1a_2,0,0), \\ f_{\theta_2}(0,0,0,0) &= (0,0,a_1c_2,0,0,0,0,0,0), \\ f_{\phi_1}(0,0,0,0) &= (0,0,0,b_1a_2,0,0,0,0,0), \\ f_{\phi_2}(0,0,0,0) &= (0,a_1b_2,0,0,0,0,0,0,0). \end{aligned}$$

The normal space is spanned by

$$\begin{aligned} f_{\theta_1\theta_1}(0,0,0,0) &= (-a_1a_2,0,0,0,0,0,0,0,0) = h(f_{\theta_1}, f_{\theta_1}), \\ f_{\theta_1\theta_2}(0,0,0,0) &= (0,0,0,0,0,0,0,0,c_1c_2) = h(f_{\theta_1}, f_{\theta_2}), \\ f_{\theta_1\phi_2}(0,0,0,0) &= (0,0,0,0,0,0,0,c_1b_2,0) = h(f_{\theta_1}, f_{\phi_2}), \\ f_{\theta_2\phi_1}(0,0,0,0) &= (0,0,0,0,0,b_1c_2,0,0,0) = h(f_{\theta_2}, f_{\phi_1}), \\ f_{\phi_1\phi_2}(0,0,0,0) &= (0,0,0,0,b_1b_2,0,0,0,0) = h(f_{\phi_1}, f_{\phi_2}), \end{aligned}$$

where $h$ denotes the second fundamental form of $f(S^2 \times S^2)$ in $\mathbf{E}^9$.

If all sectional curvatures were to be positive, then

$$\begin{aligned} g(R(f_{\theta_1}, f_{\theta_2})f_{\theta_2}, f_{\theta_1}) &= g(h(f_{\theta_1}, f_{\theta_1}), h(f_{\theta_2}, f_{\theta_2})) - g(h(f_{\theta_1}, f_{\theta_2}), h(f_{\theta_1}, f_{\theta_2})) \\ &= a_1^2a_2^2 - c_1^2c_2^2 > 0. \end{aligned}$$

But similarly, by working with a parametrization around ((0,0,1),(0,0,1)), we obtain $c_1^2c_2^2 - a_1^2a_2^2 > 0$, a contradiction. Therefore we obtain

**Proposition 6.** *If two spheres are immersed as ellipsoids in* $\mathbf{E}^3$*, with mass center the origin, then the tensor product immersion of their product manifold confirms the Conjecture of H. Hopf.*

## References

[1] J.-P. Bourguignon: *La conjecture de Hopf sur* $S^2 \times S^2$. Géométrie Riemannienne en dimension 4, Séminaire A. Besse 3 (1981), 347-355.

[2] B.-Y. Chen: *Differential Geometry of semiring of immersions. I: General Theory.* Bull. Inst. Math. Acad. Sinica 21 (1993), 1-34.

[3] B.-Y. Chen: *Classification of tensor product immersions which are of* 1*-type.* Glasgow J. Math. (to appear).

[4] B.-Y. Chen: *Two theorems on tensor product immersions.* Rend. Sem. Math. Messina, Série II, Tomo XIV, Vol I, 69–83 (1991).

[5] B.-Y. Chen: *Differential geometry of tensor product immersions.* Ann. Global Anal. Geom. (to appear).

[6] S.S. Chern: *Differential Geometry : its past and its future.* Actes du Congrès International des Math. 1970 Nice 1 (1970), 41-53.