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The Finite Moufang Hexagons Coordinatized

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Abstract. We introduce coordinates for the finite Moufang hexagons and show with a few applications that this approach makes these objects less mysterious.

1. Introduction

Moufang polygons are the natural geometries of the Chevalley groups with a BN-pair of rank 2, see e.g. Tits [14], [17]. In fact, every finite group with an irreducible rank 2 BN-pair is associated with a Moufang polygon, see Buekenhout & Van Maldeghem [2]. Therefore it seems natural to study these geometries. This can be done algebraically via the corresponding Chevalley group, or geometrically via a construction in a certain projective space. Both approaches have their advantages and disadvantages. The introduction of coordinates provides a third way to look at hexagons and it is in a sense complementary to the other methods. For example, it yields a very quick way to define the Moufang hexagons without the introduction of Chevalley groups, or without the use of cosets in certain groups, or without the help of underlying polar spaces and trialities (of course, this is only an *a posteriori* definition!). Also, the points and lines are very "concrete " and "direct" objects in this approach. On the other hand, the description is *inhomogeneous* : one flag plays a special role. By the Moufang property however, this flag can be chosen arbitrarily. But "homogeneous problems" are best handled without coordinates.

We will introduce coordinates for the Moufang hexagons in section 3 below, after we have defined and constructed the Moufang hexagons in section 2. In the remaining sections we will give some applications. In particular, we will construct explicitly the Ree unital on the quadric $Q(6,q)$, $q=3^{2h+1}$, $h \in \mathbf{N}$ (in **ATLAS** [3] group notation : $O_7(q)$), we deduce from this an explicit form of the point stabilizer (or parabolic subgroup) in the Ree group $R(q) = {}^2G_2(q)$, show that $R(3)$ has a unique oval (up to isomorphism) and compute its isomorphism group. We also present a very simple geometric construction of the 2-designs

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related to the Ree unitals introduced by Assmus & Key [1]. Other applications can be found in various other papers, e.g. Van Maldeghem [18], [19], De Smet & Van Maldeghem [4], Schroth & Van Maldeghem [13] and Van Maldeghem & Bloemen [20].

A similar coordinatization has been introduced for generalized quadrangles in two papers by Hanssens & Van Maldeghem [7], [8]. Applied to projective planes (which are essentially generalized 3-gons), this yields the usual coordinatization method of Hall [6].

The coordinatization method we propose here (already alluded to in Van Maldeghem [18]) can be used to describe *any* generalized hexagon, finite or infinite, Moufang or not. When applied to the Moufang hexagons, it has strong connections with the coordinatization carried out by Faulkner [5].

2. Definitions and construction of the finite Moufang hexagons

A (finite thick) *generalized hexagon of order* (s,t), $s,t>1$ is a point-line incidence geometry $\mathcal{S}$ satisfying (GH 1) up to (GH 4) below. A *flag* is an incident point-line pair and a non-trivial circuit consisting of six points and equaly many lines will be called an *apartment* (language of Tits' buildings, see e.g. Tits [16]).

(GH 1) There are $s+1$ points incident with every line.
(GH 2) There are $t+1$ lines incident with every point.
(GH 3) Every two flags lie in a common apartment.
(GH 4) There are no non-trivial circuits with less than six points.

At present and up to duality, only two classes of finite thick generalized hexagons are known. They are related to the (adjoint and adjoint twisted) Chevalley groups $G_2(q)$ and ${}^3D_4(q)$. Both are constructed originally on the quadric $Q^+(7,q)$ (or $O_8^+(q)$) by Tits [14]. The $G_2(q)$ hexagon lies entirely in a hyperplane and therefore (one of the two mutually dual geometries of) it can be embedded in the quadric $Q(6,q)$. To fix the notation, we will call this hexagon $G_2(q)$ (following Kantor [10]) in order to distiguish it from its dual. This construction, due to Tits [14], runs explicitly as follows. The equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$$

represents the quadric $Q(6,q)$ in the projective space $PG(6,q)$ over the Galois field $GF(q)$ of q elements. The points of $G_2(q)$ are the points of that quadric and the lines of $G_2(q)$ are those lines on $Q(6,q)$ whose *Grassmann coordinates* satisfy

$$p_{12}=p_{34},\ p_{20}=p_{35},\ p_{01}=p_{36},\ p_{03}=p_{56},\ p_{13}=p_{64},\ p_{23}=p_{45}.$$

As a consequence, one has

$$p_{04}+p_{15}+p_{26}=0.$$

The order of $G_2(q)$ is (q,q).

Kantor [10] gives a common construction of the ${}^3D_4(q)$ hexagon and the dual $G_2(q)$ hexagon. This goes as follows. Let

$$Q=\{(a,\beta,c,\delta,e)\|a,c,e\in GF(q);\beta,\delta\in GF(q^3)\}$$

and define the multiplication $(a, \beta, c, \delta, e)(a', \beta', c', \delta', e)$ as

$$(a + a', \beta + \beta', c + c' + a'e - \mathrm{Tr}(\beta'\delta), \delta + \delta', e + e'),$$

where $\mathrm{Tr}(\gamma) \stackrel{\text{def}}{=} \gamma + \gamma^q + \gamma^{q^2}$. This makes Q a group of order q^9. For $0 \leq i \leq 5$, let x_i be the element of Q whose ith coordinate is x and all others 0, and let X_i be the set of all such x_i. Note that X_3 is both the derived group of Q and the center of Q. Define for all $x \in GF(q^3)$ an automorphism x_6 of Q by

$$\begin{aligned}(a, \beta, c, \delta, e)^{x_6} = (a, \beta + ax, c - a^2\mathrm{N}(x) - \mathrm{Tr}(\beta^{q+q^2}x) - \mathrm{Tr}(a\beta x^{q+q^2}),\\ \delta + ax^{q+q^2} + \beta^q x^{q^2} + \beta^{q^2}x^q, e + a\mathrm{N}(x) + \mathrm{Tr}(\beta x^{q+q^2}) + \mathrm{Tr}(\delta x)),\end{aligned}$$

where $\mathrm{N}(x) = x^{1+q+q^2}$.

Now define

$$\begin{aligned}
A_1(\infty) &= X_5 & A_1(t) &= X_1^{t_6},\\
A_2(\infty) &= X_4X_5 & A_2(t) &= (X_1X_2)^{t_6},\\
A_3(\infty) &= X_3X_4X_5 & A_3(t) &= (X_1X_2X_3)^{t_6},\\
A_4(\infty) &= X_2X_3X_4X_5, & A_4(t) &= (X_1X_2X_3X_4)^{t_6}.
\end{aligned}$$

Let t run over $GF(q^3) \cup \{\infty\}$ and g over Q. Then the points of ${}^3D_4(q)$ are the symbol (∞), all cosets $A_4(t)g$ and $A_2(t)g$, and all elements g. The lines of ${}^3D_4(q)$ are the elements t and the cosets $A_3(t)g$ and $A_1(t)g$. Incidence is obtained via ("suitable") inclusion and also t is incident with $A_4(t)g$ and (∞). Restricting β, δ and t above to $GF(q)$ produces the dual of $G_2(q)$. This will be clear after we have coordinatized these geometries.

Note that ${}^3D_4(q)$ has order (q, q^3).

3. Coordinatization of $G_2(q)$ and ${}^3D_4(q)$.

3.1. Generalities about coordinatization

Let us briefly set the general rules for coordinatizing an arbitrary generalized hexagon $\mathcal{S}$. We give ourselves two sets R_1 and R_2 of coordinates, with $|R_1| = s$, $|R_2| = t$ and we assume that both sets contain two distinct elements which we denote by 0 and 1. It is clear that we shall use R_1 to label points on a line (except for one point which will already have been labeled otherwise) and R_2 to label lines through a point (similar remark). We choose an apartment $\mathcal{A}$, fix a point p in $\mathcal{A}$ labelling it (∞); fix a line L in $\mathcal{A}$ through p and label it $[\infty]$. The coordinates of the other elements of $\mathcal{A}$ are determined by

$$[\infty]\, I\, (0)\, I\, [0,0]\, I\, (0,0,0)\, I\, [0,0,0,0]\, I\, (0,0,0,0,0)\, I$$

$$[0,0,0,0,0]\, I\, (0,0,0,0)\, I\, [0,0,0]\, I\, (0,0)\, I\, [0]\, I\, (\infty),$$

where I denotes the incidence relation.

The coordinates of lines will always be denoted with square brackets and those of points with round ones. If we define the *distance* of an element (a point or a line) x to the flag $\mathcal{F} \stackrel{\text{def}}{=} \{(\infty), [\infty]\}$ as

$$\frac{d(x,(\infty)) + d(x,[\infty]) - 1}{2},$$

where d denotes the distance function in the incidence graph, then we want to give an element at distance i, $1 \le i \le 5$ exactly i coordinates. We also want to do this in a consistent and logical way, i.e., we want to be able to see quickly whether two elements are incident or not. This is achieved by the following procedure.

Label the points on $[\infty]$ which have distance 1 from the flag $\mathcal{F}$ (a), $a \in R_1$ (in a bijective manner and consistently with (0)), and dually, label the lines through (∞) (except $[\infty]$) $[k]$, $k \in R_2$. There is a unique point on $[0,0,0,0,0]$ nearest to (a), $a \in R_1$, we label it $(a,0,0,0,0)$. Dually we define $[k,0,0,0,0]$, $k \in R_2$.

From now on, we always assume that a's and b's are elements of R_1 and k's and l's are in R_2. We consider the point p_a on $[1,0,0,0,0]$ nearest to (a). The point on $[0]$ nearest to p_a is labelled $(0,a)$ and the point on $[0,0,0,0]$ nearest to $(0,a)$ is labelled $(0,0,0,0,a)$. Dually we get $[0,k]$ through (0) and $[0,0,0,0,k]$ through $(0,0,0,0)$. This sub-procedure is called *normalization* in Van Maldeghem [18].

Next, we assign to each point on $[0,0]$, except for (0), a coordinate $(0,0,a')$ in a bijective manner (being consistent with $(0,0,0)$); we consider the point on $[0,0,0]$ nearest to $(0,0,a')$ and label it $(0,0,0,a')$. Dually, we define $[0,0,k']$ and $[0,0,0,k']$. The freedom we have here to choose the bijection has as a consequence that the eventual coordinatization is not uniquely determined. But there seems to be no natural standard way to define this bijection.

So far, we have coordinatized all elements incident with one of the elements of $\mathcal{A}$. Now we do the rest. The point on $[k]$ nearest to $(0,0,0,0,b)$ is labelled (k,b) (dually we obtain $[a,l]$); the point on $[a,l]$ nearest to $(0,0,0,a')$ is labelled (a,l,a') (dually we obtain $[k,b,k']$); the point on $[k,b,k']$ nearest to $(0,0,b')$ is labelled (k,b,k',b') (dually we obtain $[a,l,a',l']$) and the point on $[a,l,a',l']$ nearest to $(0,a'')$ is labelled (a,l,a',l',a'') (dually we obtain $[k,b,k',b',k'']$).

Two elements x and y not both at distance 5 from $\mathcal{F}$ are incident if and only if they have an unequal number of coordinates (say x has more coordinates than y) and deleting the last coordinate of x gives us exactly the coordinates of y. There is no such simple rule for elements both at distance 5 from $\mathcal{F}$. In that case, we have to introduce algebraic operations. Therefore we define

$$\begin{aligned}
S_1(k,a,l,a',l',a'') = b &\iff d((k,b),(a,l,a',l',a'')) = 4, \\
S_2'(k,a,l,a',l',a'') = k' &\iff d([k,S_1(k,\ldots,a''),k'],(a,\ldots,a'')) = 3, \\
S_2(a,k,b,k',b',k'') = l &\iff d([a,l],[k,b,k',b',k'']) = 4, \\
S_1'(a,k,b,k',b',k'') = a' &\iff d((a,S_2(a,\ldots,k''),a'),[k,\ldots,k'']) = 3.
\end{aligned}$$

It is now easy to verify that

$$(a,l,a',l',a'')\ I\ [k,b,k',b',k''] \iff \begin{cases} S_1(k,a,l,a',l',a'')=b, \\ S_1'(a,k,b,k',b',k'')=a', \\ S_2(a,k,b,k',b',k'')=l, \\ S_2'(k,a,l,a',l',a'')=k'. \end{cases}$$

We can also define an operation $S_1^*(a,k,b,k',b',k'')$ by the rule : the unique point on $[k,b,k',b',k'']$ nearest to (a) has last coordinate $S_1^*(a,k,b,k',b',k'')$. And dually, the unique line through (a,l,a',l',a'') nearest to $[k]$ has last coordinate $S_2^*(k,a,l,a',l',a'')$. The normalization procedure ensures that

$$S_1^*(a,1,0,0,0,0)=a \qquad \text{and} \qquad S_2^*(k,1,0,0,0,0)=k.$$

As for generalized quadrangles (see Hanssens & Van Maldeghem [9]), one can normalize in a different way to obtain

$$S_1(1,a,0,0,0,0)=a \qquad \text{and} \qquad S_2(1,k,0,0,0,0)=k.$$

We have chosen the former normalization because it is more often used in the applications (see the papers already mentioned in the introduction).

The operations S_1, S_1', S_2 and S_2' determine the generalized hexagon $\mathcal{S}$ completely. Conditions on these operations could be given in order that, given the sets R_1 and R_2 and the operations S_1, S_1', S_2 and S_2' (without the pre-knowledge of $\mathcal{S}$), the geometry of coordinate-tuples defined as above with incidence also defined as above is actually a generalized hexagon. But these conditions seem to be too akward in general to work with.

Part of the previous method is explained and used in general for generalized polygons by Van Maldeghem [18]. In the next paragraph, we will apply this to the known finite generalized hexagons.

3.2. Coordinatization of $G_2(q)$

We use Tits' description [14] of $G_2(q)$ embedded in $PG(6,q)$ (see above). There is no need to introduce new symbols for the coordinates of the points in $PG(6,q)$ because all points have 7 coordinates there. A line through the points p_1 and p_2 is denoted by $\langle p_1, p_2\rangle$.

We start by choosing the apartment $\mathcal{A}$ and the flag $\mathcal{F}$ (we only mention the points of $\mathcal{A}$; the symbol $\perp$ means "*collinear to*") :

$$\mathcal{F}=\{(1,0,0,0,0,0,0),\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle\},$$

$$\begin{aligned}\mathcal{A}:&(1,0,0,0,0,0,0)\perp(0,0,0,0,0,0,1)\perp(0,1,0,0,0,0,0)\\ &\perp(0,0,0,0,1,0,0)\perp(0,0,1,0,0,0,0)\perp(0,0,0,0,0,1,0).\end{aligned}$$

Following the general rules above we first define the 4 bijections we must have (in the following, an arrow ("$\rightarrow$") means *is labelled*).

$$\begin{aligned}
(a,0,0,0,0,0,1) &\rightarrow (a),\\
(0,1,0,0,0,0,-a') &\rightarrow (0,0,a'),\\
\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle &\rightarrow [k],\\
\langle(0,0,0,0,0,1,0),(k',0,1,0,0,0,0)\rangle &\rightarrow [0,0,k'].
\end{aligned}$$

This determines the coordinates of every point and line in $G_2(q)$ as above. After a few calculations, we obtain the coordinatization given in Table 1.

From the coordinates in $PG(6,q)$, we can calculate that (a,l,a',l',a'') is incident with $[k,b,k',b',k'']$ if and only if

$$\begin{cases}
b = a'' - ak,\\
a' = a^2k + b' + 2ab,\\
l = k'' - ka^3 - 3ba^2 - 3ab',\\
k' = k^2a^3 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2,
\end{cases}$$

which is equivalent with

$$\begin{cases} a'' = ak + b, \\ a' = a^2k + b' + 2ab, \\ k'' = ka^3 + l - 3a''a^2 + 3aa', \\ k' = k^2a^3 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2. \end{cases}$$

Hence we deduce from this

$$(*) \qquad \begin{cases} S_1(k,a,l,a',l',a'') = a'' - ak, \\ S_1'(a,k,b,k',b',k'') = a^2k + b' + 2ab, \\ S_2(a,k,b,k',b',k'') = k'' - ka^3 - 3ba^2 - 3ab', \\ S_2'(k,a,l,a',l',a'') = k^2a^3 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2, \end{cases}$$

and

$$\begin{cases} S_1^*(a,k,b,k',b',k'') = ak + b, \\ S_2^*(k,a,l,a',l',a'') = ka^3 + l - 3a''a^2 + 3aa'. \end{cases}$$

From this, we can already see that characteristic 3 will play a special role here. It is obvious that, if the characteristic is not 3, then $G_2(q)$ is not self-dual. We will show the converse in the next section. First, we coordinatize ${}^3D_4(q)$ in the next paragraph.

3.3. Coordinatization of ${}^3D_4(q)$

We use Kantor's description [10] (see above). For the special apartment $\mathcal{A}$, we make the natural choice, namely the apartment containing all $A_i(\infty)$, all $A_i(0)$, $i = 1,2,3,4$, the elements $g = (0,0,0,0,0) \in Q$, $t = 0, \infty$ and the special element (∞). We immediately write down the coordinates (see Table 2). From this information, one can easily reconstruct the whole procedure and all calculations.

An elementary calculation shows us that (a,l,a',l',a'') is incident with $[k,b,k',b',k'']$ if and only

$$(**) \qquad \begin{cases} b = a'' - a\mathrm{N}(k) - \mathrm{Tr}(lk^{q+q^2}) - \mathrm{Tr}(l'k), \\ a' = a^2\mathrm{N}(k) + b' + \mathrm{Tr}(k''^{q+q^2}k) - a\mathrm{Tr}(k''k^{q+q^2}) - \mathrm{Tr}(k'k'') - ab, \\ l = k'' - ka, \\ k' = k^{q+q^2}a + l' + l^qk^{q^2} + l^{q^2}k^q, \end{cases}$$

which is equivalent to

$$\begin{cases} a'' = b + a\mathrm{N}(k) - \mathrm{Tr}(k''k^{q+q^2}) + \mathrm{Tr}(k'k), \\ a' = a^2\mathrm{N}(k) + b' + \mathrm{Tr}(k"^{q+q^2}k) - a\mathrm{Tr}(k"k^{q+q^2}) - \mathrm{Tr}(k'k'') - ab, \\ k'' = l + ka, \\ k' = k^{q+q^2}a + l' + l^qk^{q^2} + l^{q^2}k^q. \end{cases}$$

Hence we deduce from this

$$\begin{cases} S_1(k,a,l,a',l',a'') = a'' - a\mathrm{N}(k) - \mathrm{Tr}(lk^{q+q^2}) - \mathrm{Tr}(l'k), \\ S_1'(a,k,b,k',b',k'') = a^2\mathrm{N}(k) + b' + \mathrm{Tr}(k''^{q+q^2}k) - a\mathrm{Tr}(k''k^{q+q^2}) - \mathrm{Tr}(k'k'') - ab, \\ S_2(a,k,b,k',b',k'') = k'' - ka, \\ S_2'(k,a,l,a',l',a'') = k^{q+q^2}a + l' + l^q k^{q^2} + l^{q^2}k^q, \end{cases}$$

and

$$\begin{cases} S_1^*(a,k,b,k',b',k'') = b + a\mathrm{N}(k) - \mathrm{Tr}(k''k^{q+q^2}) + \mathrm{Tr}(k'k), \\ S_2^*(k,a,l,a',l',a'') = ka + l. \end{cases}$$

Restricting the elements k, k', k'', l, l' to $GF(q)$ (noting $\mathrm{Tr}(x) = 3x$ and $\mathrm{N}(x) = x^3$ for $x \in GF(q)$), we see that we get exactly the dual of the coordinatization of $G_2(q)$ above.

With some changes in notation, this description remains valid for the infinite Moufang hexagons of type G_2 and ${}^3D_4$.

Let's look at some applications.

4. Applications

4.1. Generalized homologies

Van Maldeghem [19] showed that the finite classical generalized hexagons contain a lot of *generalized homologies*, i.e. automorphisms preserving an apartment $\mathcal{A}$ and the set of elements incident with some element of $\mathcal{A}$. This was proved by considering Kantor's description above. With coordinates, it is even simpler. Indeed, let $\mathcal{A}$ be our standard apartment of coordinatization, then it is almost trivial to check that the mappings

$$\begin{aligned} h_1: \quad & (a,l,a',l',a'') \mapsto (a, yl, \mathrm{N}(y)a', y^{q+q^2}l', \mathrm{N}(y)a''), \\ & [k,b,k',b',k''] \mapsto [yk, \mathrm{N}(y)b, y^{q+q^2}k', \mathrm{N}(y)b', yk''] \\ h_2: \quad & (a,l,a',l',a'') \mapsto (xa, xl, x^2a', xl', xa''), \\ & [k,b,k',b',k''] \mapsto [k, xb, xk', x^2b', xk''], \end{aligned}$$

preserve the equations $(**)$ for all $x \in GF(q)$ and $y \in GF(q^3)$, $x, y \neq 0$, hence they define an automorphism of ${}^3D_4(q)$, which we also denote by h_i, $i = 1,2$. The "standard apartment" $\mathcal{A}$ is preserved by both h_1 and h_2 and moreover, h_1 (resp. h_2) fixes every point (resp. line) on $[\infty]$ (resp. through (∞)). So we have lots of generalized homologies in ${}^3D_4(q)$. Restriction to $G_2(q)$ gives us similarly lots of generalized homologies in $G_2(q)$.

One other class of generalized homologies is worth mentioning. If we raise every coordinate to the power q or q^2, then we obtain a generalized homology which fixes elementwise the subhexagon $G_2(q)$ of ${}^3D_4(q)$.

4.2. Regular and half regular points

Let us call a *pre-ideal line* in a generalized hexagon $\mathcal{S}$ a set of all points at distance 4 from some point z and 2 from some point p (which is itself at distance 6 from z). Denote by $p^\perp$ the set of all points collinear to p, including p. If the set of pre-ideal lines and customary lines in $p^\perp$ forms a linear space, then we call p *half regular*, see Van Maldeghem & Bloemen [20]. In this case, the pre-ideal lines in $p^\perp$ are called *ideal lines*, a notion introduced by Ronan [12]. If moreover no two pre-ideal lines through p meet in a point distinct from p, then we call p regular (again see [20]).

We have the following proposition.

Proposition 1. *Let $\mathcal{S}$ be a generalized hexagon of order (s,t) coordinatized by the sets R_1 and R_2 with corresponding operations S_i and S_i', $i = 1,2$ as before. Then the point (∞) is half regular if and only if S_1 is independent from l, a' and l' (or equivalently, S_1^* is independent from k', b' and k''). If, in this case, $s = t$ is finite, then the set of all ideal lines and customary lines in $(\infty)^\perp$ forms a projective plane on $(\infty)^\perp$ which can be coordinatized by the ternary operation $a" = T(a,k,b) \iff a" = S_1^*(a,k,b,0,0,0)$ (by identifying $S_1^*(1,k,0,0,0,0) \in R_1$ with $k \in R_2$). The point (∞) is regular if and only if (∞) is half regular and S_1' is independent from k' and k''. If, in this case $s = t$ is finite, then the two quaternary operations*

$$\begin{aligned} Q_1(a,k,b,b') &= S_1^*(a,k,b,0,b',k''), \\ Q_2(a,k,b,b') &= S_1'(a,k,b,0,b',0), \end{aligned}$$

define a coordinatization of a generalized quadrangle of order (s,s).

Proof. Consider the points $(0,a'')$ and (a). A general point p at distance 4 from both these points and at distance 6 from (∞) has coordinates (a,l,a',l',a'') (l,a',l' are arbitrary). The unique point on $[k]$ nearest to p has coordinates $(k,S_1(k,a,l,a',l',a''))$. This should be independent from the choice of p if (∞) is half regular. This is indeed the case only when S_1 is independent from l,a' and l'. Conversely, suppose that S_1 is independent from l,a' and l'. Let (k_0,b_0) and (k_1,b_1) be two points of $(\infty)^\perp$, with $k_0 \neq k_1$. Let (a) (resp. $(0,a'')$) be a point on $[\infty]$ (resp. $[0]$) on the same pre-ideal line X through (k_0,b_0) and (k_1,b_1). Every point at distance 4 from both (a) and (o,a'') lies also at distance 4 from both (k_0,b_0) and (k_1,b_1) since b_0 (resp. b_1) only depends on k_0,a,a'' (resp. k_1,a,a''). Suppose now p is a point at distance 4 from both (k_0,b_0) and (k_1,b_1) and let L_0 (resp. L_1) be the line through p nearest to (k_0,b_0) (resp. (k_1,b_1)). Furthermore, let x be the point on L_0 nearest to (o,a''), L the line through x nearest to (o,a'') and y the point on L nearest to (a). Then y is a point at distance 4 from both $(0,a'')$ and (a), hence it is at distance 4 from (k_0,b_0), but clearly, this can only happen when $x=y$, in which case $d(x,(k_1,b_1))=4$ implies $x=y=p$. Hence p is at distance 4 from both $(0,a'')$ and (a) and X is an ideal line uniquely determined by (k_0,b_0) and (k_1,b_1). Hence (∞) is half regular.

Similarly, one shows that (∞) half regular is equivalent to S_1^* being independent from k',b' and k''.

Suppose again that (∞) is half regular. The mapping $k \mapsto S_1^*(1,k,0,0,0,0)$ is an injection from R_2 in R_1 by a similar argument as above. So if $s=t$ is finite, then this defines a bijection. It is now easy to see that in this case the ideal lines and customary lines in $(\infty)^\perp$ form a projective plane coordinatized as stated in the proposition. This shows the first part.

We now show that every pre-ideal line through (∞) and a point (a,l,a') is ideal if and only if S_1' is independent from its last argument. A general point at distance 4 from (∞) and (a,l,a') has coordinates (k,b,k',b'), where k,b and k' are arbitrary and b' is a solution of

$$\begin{cases} S_1'(a,k,b,k',b',k'') = a', \\ S_2(a,k,b,k',b',k'') = l, \end{cases}$$

where k'' is also an unknown. Now note that every pre-ideal line through (∞) and (a,l,a') is ideal if and only if they all contain (a,l^*,a'), for any $l^* \in R_2$. Indeed, the point $(0,0,0,a')$ is at distance 4 from all these points and from (∞). So let $l^* \in R_2$ be arbitrary, then (a,l^*,a') is at distance 4 from (k,b,k',b') if and only if there exists $k_1'' \in R_2$ such that

$$\begin{cases} S_1'(a,k,b,k',b',k_1'') = a', \\ S_2(a,k,b,k',b',k_1'') = l^*. \end{cases}$$

But the mapping $l^* \mapsto k_1''$ obtained by the last equation is a bijection. Hence the claim.

Now, given that S_1' is independent from k'', one shows similarly that S_1' is independent from its fourth argument k' if and only if every point of any pre-ideal line through (∞) and (k,b,k',b') defined by all points collinear to (k,b) and at distance 4 from a point (a,l,a') (which is itself chosen at distance 4 from (k,b,k',b') of course), is independent from that choice of (a,l,a'). Note that these pre-ideal lines consist of (∞) and all points (k,b,k_1',b'), where k,b,b' are fixed and k_1' varies.

We now have to show that this condition implies that (∞) is regular. So let $p = (k, b, k', b')$, $p_1 = (k, b, k_1', b')$ and $p_0 = (k_0, b_0, k_0', b_0')$ with $d(p, p_0) = 4$ and $k \neq k_0$. We have to show that $d(p_0, p_1) = 4$. Let x be the point collinear to both p and p_0; let p_1' be the unique point at distance 4 from p_0 and incident with $[k, b, k_1']$; let x' be collinear to both p_0 and p_1'. Both x and x' are at distance 4 from both (k, b) and (k_0, b_0), hence since (∞) is half regular, they are at distance 4 from a common point z on $[\infty]$. Let y (resp. y') be collinear to both x and z (resp. x and y), then y is at distance 4 from both p and p_1 by the remark above. Now both p_0 and p_1 are at distance 4 from both (∞) and y, hence they determine the same pre-ideal line through (∞) and y in $z^\perp$. Since $y' \in z^\perp$ and $d(p_0, y') = 4$, y' is on that pre-ideal line. Hence $d(p_1, y') = 4$. Unless $p_1 = p_1'$, we have a pentagon $p_1 \perp p_1' \perp x' \perp y' \perp \ldots \perp p_1$. This shows our claim.

By Van Maldeghem & Bloemen [20], with every regular point in a generalized hexagon of finite order (s, s) is associated a generalized quadrangle of order (s, s). The last part of the proposition now follows from direct coordinatization (in a natural way) of that quadrangle by the method of Hanssens & Van Maldeghem [8]. Details of these computations would require new long — and for this paper uninformative — definitions and are left to the reader. $\square$

By this proposition, one can easily see that $G_2(q)$ and the dual of ${}^3D_4(q)$ contains regular points. Moreover, in $G_2(q)$, there is a generalized quadrangle associated with every (regular) point. Of course, this is the residue of a point in the polar space $Q(6, q)$, while the projective plane associated with a (half regular) point is the residue of a plane in $Q(6, q)$.

4.3. The Ree unitals

4.3.1. Description

Looking at the dual of $G_2(q)$, we see that it has regular points if and only if q is a power of 3. So that is the only case in which $G_2(q)$ could be isomorphic to its dual. An explicite isomorphism from $G_2(q)$ to its dual is given by

$$(a, l, a', l', a'') \mapsto [a^3, l, a'^3, l', a''^3],$$
$$[k, b, k', b', k''] \mapsto (k, b^3, k', b'^3, k'').$$

We now investigate when there exists a polarity in $G_2(q)$, $q = 3^h$, $h \in \mathbf{N}^\times$. Suppose θ is a polarity and take any point p in $G_2(q)$. Then p^θ is at distance $1, 3$ or 5 from p, hence there is a unique chain of consecutively incident elements connecting p and p^θ. Let x and y be the middle elements of that chain, then obviously $y = x^\theta$. An element z with the property $z\ I\ z^\theta$ is called an *absolute element.* We just showed that every polarity defines absolute elements. Now we coordinatize $G_2(q)$ in such a way that (∞) and $[\infty]$ are absolute elements. Moreover, we can choose $(0)^\theta = [0]$, $(0,0)^\theta = [0,0]$, $(0,0,0)^\theta = [0,0,0]$, etc. In fact, we can also assume that $(1)^\theta = [1]$. We have a mapping $GF(q) \to GF(q) : a \mapsto a^\phi$ defined by $(a)^\theta = [a^\phi]$. There is a second map $GF(q) \to GF(q) : a \mapsto a^\psi$ defined by $(0, 0, a)^\theta = [0, 0, a^\psi]$. It is straight forward to calculate that

$$(a, l, a', l', a'')^\theta = [a^\phi, l^{\phi^{-1}}, a'^\psi, l'^{\psi^{-1}}, a''^\phi],$$
$$[k, b, k', b', k'']^\theta = (k^{\phi^{-1}}, b^\phi, k'^{\psi^{-1}}, b'^\psi, k''^{\phi-1})$$

If we now express that this mapping preserves the incidence relation, then we find that $\phi = \psi$, both are field automorphisms and $\phi^2 = 3$. Hence h is odd. The polarity θ maps (a, l, a', l', a'') to $[a^\phi, l^{\phi/3}, a'^\phi, l'^{\phi/3}, a''^\phi]$, the point (a, l, a', l', a'') is absolute if and only if

$$\begin{cases} a'' = aa^\phi + l^{\phi/3}, \\ a' = a^2 a^\phi + l'^{\phi/3} + 2al^{\phi/3}. \end{cases}$$

So the set of all absolute points, called the Ree unital $U_R(q)$, is (putting $s = 3^{h+1}$ and $3^{2h+1} = q$)

$$\{(a, a''^s - a^{3+s}, a', a^{3+2.s} + a'^s + a^s a''^s, a'')\|a, a', a'' \in GF(3^{2h+1})\}.$$

By definition, a subset of points of $U_R(q)$ forms a line if it is the set of fixed points of an involution of $G_2(q)$ preserving $U_R(q)$ (see Tits [15]). A geometric construction of these lines is contained in [4]. The coordinates of the points of this unital in $PG(6, q)$ can be explicitly obtained from table 1.

Next, as a further illustration, we calculate the order of the automorphism group in the group $G_2(q)$ of the Ree unital. This group is the twisted Chevalley group ${}^2G_2(q)$ discovered by Ree [11] and also denoted by $R(q)$.

First we note that every automorphism of $G_2(q)$ preserving $U_R(q)$ and (∞) must also fix $[\infty]$, as $[\infty]$ is an absolute line. We first find the stabilizer H in $R(q)$ of (∞). It is clear that H is the semidirect product of a Sylow 3-subgroup (which is by the way generated by root-elations) and a group generated by generalized homologies fixing the apartment $\mathcal{A}$. From subsection 4.1 (dualizing), we derive that a general element of the group generated by the generalized homologies fixing $\mathcal{A}$ maps (a, l, a', l', a'') to $(xa, x^3yl, x^2ya', x^3y^2l', xya'')$ and by the above description of the Ree unital in coordinates, we see that this preserves $U_R(q)$ if and only if $y = x^{3^{h+1}}$. This defines a group of order $q - 1$. Similarly, a general element of the Sylow 3-subgroup P maps (a, l, a', l', a'') to

$$(a+A, l+L-Ka^3-KA^3, a'+A'-A''a+Ka^2-AA'', l'+L'+K^2a^3+Kl+K^2A^3, a''+A''+Ka)$$

and $[k, b, k', b', k'']$ to

$$[k+K, b+A''-Ak-AK, k'+L'-Lk+A^3k^2-KL, b'+A'+A^2k+Ab+A^2K, k''+L+A^3k]$$

(this is readily verified). It needs an elementary calculation to see that such a transformation preserves the Ree unital if and only if

$$K = A^{3^{h+1}}, \qquad L' = A'^{3^{h+1}}, \qquad L = A''^{3^{h+1}},$$

and we obtain a group acting regularly on the points of $U_R(q)$ distinct from (∞). So $R(q)$ is doubly transitive and has order $(q^3 + 1)q^3(q - 1)$. For later reference we denote the above transformation by $E(A, A', A'')$.

Let σ be an involution of $R(q)$ fixing at least one point of $U_R(q)$. Then σ is conjugated to the generalized homology η with $x = y = -1$ (using the notation of the previous paragraph). Indeed, this follows immediately from the fact that $q-1 \equiv 2 \bmod 4$. Clearly η fixes the points (∞) and $(0,l,0,0,a'')$, $l, a'' \in GF(q)$, in $G_2(q)$. Hence η fixes the $q+1$ points (∞) and $(0, a''^s, 0, 0, a'')$, $a'' \in GF(q)$. These points form by definition a line of $U_R(q)$. So a general line of $U_R(q)$ is the set of fixed points of σ. There are q^2 lines through one point and $q^2(q^2-q+1)$ lines in total.

Let's take a closer look at the Sylow 3-subgroup S. An element of $U_R(q)$ is completely determined by the coordinates a, a' and a''. So we can attach to that point in a unique way the triple $((a, a'', a' - aa''))$ (double parentheses to avoid confusion with points in $G_2(q)$ with three coordinates). The reason to define this strange last coordinate this way will become clear below. We now identify the point $((x,y,z)) \in U_R(q)$ with the element of P defined by $A = x$, $y = A''$ and $A' = z - xy$. It is clear that this element maps $(0,0,0,0,0)$ to $(x, \ldots, z + xy, \ldots, y)$ and this is denoted above by $((x,y,z))$. So every point p is identified in this way with the unique group element in P mapping $(0,0,0,0,0)$ to p. This identification defines a natural multiplication in $U_R(q) \setminus \{(\infty)\}$ and after a short calculation one finds

$$((x,y,z)).((x',y',z')) = ((x + x', y + y' + x^s x', z + z' - xy' + yx' - x^{s+1}x')),$$

which is exactly the operation used by Tits [15] to define the Ree unitals.

4.3.2. Hyperovals in $U_R(3)$

An *arc* C in $U_R(q)$ is a non-empty set of points no three of which are collinear. Clearly, the maximal number of points of an arc is q^2+1. In this case, every line of $U_R(q)$ meets C in 0 or 2 points (indeed, let B be a line of $U_R(q)$ meeting C in at least one point p, then the q^2 lines of $U_R(q)$ through p must contain at most one point of C (because C is an arc) and consequently at least one point (because C contains q^2 points other than p), hence B meets C in two points). We call such an arc a *hyperoval* (see Assmus & Key [1] who called this an *oval*).

Suppose now $h = 0$, i.e. we consider the smallest Ree Unital $U_R(3)$. Here both ϕ and $\phi/3$ are the identity and this is responsible for some remarkable properties of $U_R(3)$. One of them is the fact that $U_R(3)$ contains hyperovals. Let us take a closer look at this situation.

One calculates that the following sets of $G_2(3)$ are lines of the Ree unital $U_R(3)$:

$$\begin{aligned}
L_1 &:= \{(0,0,0,0,0),(0,1,2,2,1),(2,1,0,0,2),(2,2,1,0,0)\},\\
L_2 &:= \{(0,0,0,0,0),(0,0,2,2,0),(2,2,2,1,0),(1,2,0,1,0)\},\\
L_3 &:= \{(\infty),(0,0,0,0,0),(0,1,0,0,1),(0,2,0,0,2)\}.
\end{aligned}$$

Now we let the Sylow 3-subgroup described above (of order 27 here) act on these sets and we obtain 2 orbits of size 27 (of resp. L_1 and L_2) and an orbit of size 9 (of L_3). Hence we obtain in this way all of the 63 lines of $U_R(3)$.

Now let O be an hyperoval of $U_R(3)$. By the 2-transitivity of $R(3)$, we can assume that $p_1 := (\infty)$ and $p_2 := (0,0,0,0,0) \in O$. We can choose a third point of O on $L_3^{E(0,2,0)}$ having 3 possibilities. Let us take $p_3 := (0,0,2,2,0)$. The line p_2p_3 is L_2 and this meets

$L_3^{E(1,0,0)}$ in $(1,2,0,1,0)$ and $L_3^{E(2,2,0)}$ in $(2,2,2,1,0)$. On $L_3^{E(1,0,0)}$, there are two points left, namely $(1,0,0,2,1)$ and $p_4 := (1,1,0,0,2)$. Suppose $p_4 \in O$. Similarly on $L_3^{E(2,2,0)}$, there are two points left, namely $(2,0,2,0,1)$ and $p_5 := (2,1,2,2,2)$. Suppose $p_5 \in O$ (so the choice of (p_3, p_4, p_5) is one out of 12). The line $L_3^{E(1,2,0)}$ meets the line p_2p_4 (resp. p_2p_5) in $(1,2,2,0,0)$ (resp. $(1,0,2,1,1)$), so there is only one point left on $L_3^{E(1,2,0)}$ and that is $p_6 := (1,1,2,2,2)$. Similarly p_3p_6 and p_2p_5 rule out the points $(1,0,1,0,1)$ and $(1,2,1,2,0)$ on $L_3^{E(1,2,1)}$ and leave $p_7 := (1,1,1,1,2) \in O$. Going on like that (also using the fact that O has no tangents), we find 3 more points of O, namely $p_8 := (2,1,0,0,2)$, $p_9 := (0,0,1,1,0)$ and $p_{10} := (2,1,1,1,2)$. This constitutes indeed an hyperoval. Checking the other 11 possibilities, we end up with 9 more hyperovals through (∞) and $(0,0,0,0,0)$. We will show :

Proposition 2. *All hyperovals in $U_R(3)$ are isomphic to O above. The group stabilizing O fixes (∞) and acts on the other points of O as the pointwise stabilizer of a line in $PG(2,3)$ acts on the points off that line, hence $R(3)_O \cong 3^2 : 2$.*

Proof. One can easily check that the group elements $E(0,1,0)$ and $E(1,0,2)$ preserve O and they generate a group of order 9, acting regularly on the points of O except (∞). Hence, no other transformation of the form $E(A,A',A'')$ stabilizes O. The generalized homology with $x = y = 2$ also fixes O. So we have the group $3^2 : 2$ stabilizing O. If $R(3)_O$ did not fix (∞), then $R(3)_O$ would be a 2-transitive group on 10 points and hence $R(3)_O$ is isomorphic to either S_{10}, A_{10} or $L_2(9)$. But none of these groups have a point stabilizer of the form $3^2 : 2$. So $R(3)_O \cong 3^2 : 2$. It follows that the number of hyperovals in $U_R(3)$ isomorphic to O is equal to

$$\frac{|R(3)|}{|R(3)_O|} = \frac{27.28.2}{18} = 84.$$

Counting the triples (a, b, O') in two ways, where a, b are points of $R(3)$ and O' is an hyperoval isomorphic to O containing a and b, we obtain

$$28.27.|\{\text{hyperovals through two fixed points}\}| = 84.10.9,$$

hence the number of hyperovals through two fixed points and isomorphic to O is 10, which is exactly the number of hyperovals we found through (∞) and $(0,0,0,0,0)$. $\square$

4.3.3. 2-designs related to $U_R(q)$

Consider the following incidence structures $\mathcal{S}_1$ and $\mathcal{S}_2$. Both have as point set the set of points of the Ree unital $U_R(q)$, q an odd power of 3. Let L be a line of the generalized hexagon $G_2(q)$ not incident with any point of $U_R(q)$. Denote the set of points of $U_R(q)$ collinear to some point on L by B_L. Then the set $\mathcal{B}_1$ of blocks of $\mathcal{S}_1$ consists of all such sets B_L. The set $\mathcal{B}_2$ of blocks of $\mathcal{S}_2$ is the union of $\mathcal{B}_1$ and the set of lines of $U_R(q)$. Evidently every block has $q+1$ points. We now present a geometric proof of the fact that two blocks of $\mathcal{S}_2$ have at most 2 points in common. Consequently the elements of $\mathcal{B}_1$ are arcs of $U_R(q)$ and $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) is a $2-(q^3+1, q+1, q+1)$ design (resp. $2-(q^3+1, q+1, q+2)$ design).

We denote by θ the polarity of $G_2(q)$ defining the Ree unital $U_R(q)$. Let x and y be two different points of $U_R(q)$, then there are exactly $q+1$ lines of $G_2(q)$ at distance 3 from both of them. Let L and M be two such lines and suppose that z is a point of $U_R(q)$ also at distance 3 from both L and M. By the fact that $G_2(q)$ and its dual has ideal lines (see Ronan [12]), we can choose L and M such that L (resp. M) meets x^θ (resp. y^θ). So L^θ is incident with M, collinear with x and it must be at distance 3 from z^θ. Since z is incident with z^θ, this implies that z^θ meets M, a contradiction.

All that is left to show is that the elements of $\mathcal{B}_1$ are arcs of $U_R(q)$. Suppose again x, y, z are points of $U_R(q)$ at distance 3 from a common line L and suppose an involution σ fixes x, y and z. Then it fixes all points on L, all points of the block B_L and also all lines u^θ where $u \in B_L$. Since none of these lines meets L, this implies that σ fixes all lines through x, and hence also all lines through any of the points of B_L. So σ is the identity, a contradiction. This shows our assertion.

Finally, these designs coincide with the ones defined by Assmus & Key [1], Theorem C, since the stabilizer of a line L in $G_2(q)$ not incident with a point of $U_R(q)$ contains a subgroup H of order q of the parabolic subgroup fixing the unique point x of $U_R(q)$ for which x^θ meets L. This is readily verified using the structure of the parabolic subgroup given above taking $x = (\infty)$, $L = [0,0,0]$ (in which case H consists of the elements of the form $E(0,0,A''))$.

More and new properties of the Ree unitals using this coordinatization method are proved in [4].

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