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On the Existence and Constructibility of Inscribed Polygons

Peter Schreiber
Fachrichtungen Mathematik/Infomatik, Ernst-Moritz-Arndt Universität,
Jahnstr. 15a, D - 17489 Greifswald

In a little paper [1] for pupils in honour of the 450th birthday of François Viète in 1990 I revisited his construction of the inscribed quadrangle from its given four sides (first published in 1596 as an appendix on Vieta's paper "Pseudomesolabum" and reprinted 1646 in [2]). This has been the starting point for my further investigations on the existence and constructibility of inscribed polygons which I therefore want to dedicate to the memory of Vieta. My interest in this somewhat oldfashioned problem may be apologized by the variety of used proof techniques that include algebraical methods as well as geometrical variants of Bolzano's intermediate value theorem and — last not least — by the fact that there are hitherto not too many really interesting construction problems from which we definitively know the unsolvability by compass and ruler.

Theorem 1. *Let be $n > 2$ a natural number and $a_1, \ldots, a_n$ positive reals. An inscribed polygon (i.e. a convex polygon with vertices on a circle) with sides $a_1, \ldots, a_n$ in this cyclic order exists if and only if the n inequalities*

$$\begin{aligned} a_1 &< a_2 + a_3 + \cdots + a_n, \\ a_2 &< a_3 + \cdots + a_n + a_1, \\ &\vdots \\ a_n &< a_1 + a_2 + \cdots + a_{n-1} \end{aligned} \tag{1}$$

hold true.

The necessity of these inequalities (even for arbitrary closed polygons) is obvious. We shall prove the sufficiency by induction. For this proof we need

Lemma. *If $n > 4$ and $a_1, \ldots, a_n$ satisfy (1) then there is at least one i such that $a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_n$ (where the indices are to understand modulo n) satisfy the respective inequalities (1) for $n-1$ cyclic ordered reals.*

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Figure 1

Remark. For $n = 4$ and $a_1 = a_3$ and $a_2 = a_4$ this is obviously not true, i.e. it is impossible to deform a rectangle in a triangle by stretching two neighbouring sides in one side. Our lemma says: For all $n > 4$ such a deformation of an n-sided polygon into an $(n-1)$-sided polygon is always possible.

Proof: Let us assume

$$\begin{aligned} a_1 + a_2 &\geq a_3 + a_4 + \cdots + a_n, \\ a_2 + a_3 &\geq a_4 + \cdots + a_n + a_1, \\ &\vdots \\ a_n + a_1 &\geq a_2 + a_3 \cdots + a_{n-1}. \end{aligned}$$

By adding these inequalities we obtain

$$2(a_1 + \cdots + a_n) \geq (n-2)(a_1 + \cdots + a_n),$$

i.e, $n \leq 4$. Therefore there exists at least one i such that

$$a_i + a_{i+1} < a_1 + \cdots + a_{i-1} + a_{i+2} + \cdots + a_n.$$

From (1) follows that also

$$a_1 < a_2 + \cdots + (a_i + a_{i+1}) + \cdots + a_n$$

and so on.

Proof of Theorem 1: For $n = 3$ we obtain from (1) the wellknown existence of a triangle with sides a_1, a_2, a_3 which of course has a circumcircle. For $n = 4$ Vieta has shown the constructibility by compass and ruler but not strongly the existence under the only conditions (1). This existence (as a symmetrical trapezium) is obvious in the special case $a_1 = a_3$ (or $a_2 = a_4$). Let be $a_1 \neq a_3$ and $a_2 \neq a_4$, say $a_1 < a_3$ and $a_2 < a_4$, then $a_1 + a_2 < a_3 + a_4$, i.e. we can stretch the quadrangle (assumed as a linkage) to a triangle with sides $a_1 + a_2$, a_3, a_4 (Fig. 1). Herein we have (with the denotations of Fig. 1) $\alpha + \gamma < 180^o$, $180^o < \beta + \delta < 360^o$.

If we deform this special quadrangle continuously by diminishing β we arrive at a quadrangle with $\alpha = 180^o$ or $\gamma = 180^o$ (or both if $a_1 + a_4 = a_2 + a_3$). In each case we have $\beta + \delta < 180^o$ because of $\alpha + \beta + \gamma + \delta = 360^o$. The continuity of the motion ensures that there is a value of β for which $\beta + \delta = \alpha + \gamma = 180^o$ and this is sufficient for being an inscribed quadrangle.

This step from the existence of (always inscribed) triangles to that of inscribed quadrangles gives the hint how to conclude from the existence of inscribed $(n-1)$-angles that of inscribed n-angles using our lemma proved above. Let be $n > 4$ and $a_1, \ldots, a_n$ positive reals which satisfy (1). According to the lemma let be $a_1 + a_2 < a_3 + \cdots + a_n$. By induction we may assume that there exist inscribed $(n-1)$-angles with sides $x, a_3, \ldots, a_n$ (Fig. 2) for each x such that $x < a_1 + a_2$ and

$$\begin{aligned} a_3 &< x + a_4 + \cdots + a_n, \\ a_4 &< x + a_3 + a_5 + \cdots + a_n, \\ &\vdots \\ a_n &< x + a_3 + \cdots + a_{n-1}, \end{aligned} \tag{2}$$

i.e. $x, a_3, \ldots, a_n$ satisfy the conditions (1). We have to show that by diminishing of β from its initial value 180^o and shortening the chord x from its initial value $a_1 + a_2$ we may get an inscribed polygon with sides $a_1, a_2, \ldots, a_n$.

Figure 2

Indeed, by diminishing β the vertex $B(\beta)$ moves in direction of the exterior of the circumcircle $k(\beta)$ (which continuously depends on β) until the polygon stops to be convex (or one of the conditions (2) becomes false). At this moment we have that a_2 (or a_1 or both, but let us assume a_2 as in Fig. 2) is the straight line prolongation of its neighbouring side a_3 and hence $B(\beta)$ obviously is located outside of the circles $k(\beta)$. Because of the continuity of this motion there exists a value of β for which $B(\beta)$ lies exactly on $k(\beta)$. It remains to show that it is impossible that one of the conditions (2), say the last one, becomes false before $B(\beta)$ reaches $k(\beta)$. If a_n is near to the sum $x + a_3 + \cdots + a_{n-1}$ then we have a situation as shown in Fig. 3 and obviously the radius of $k(\beta)$ goes to infinity, but the distance of the vertex $B(\beta)$ from the chord $x(\beta)$ is growing continuously.

Figure 3

Theorem 2. *The inscribed quintangle is in general not constructible from its five sides by compass and ruler.*

Proof: We show the nonconstructibility even in special cases if $a_1 = a_3 = a,\ a_2 = a_4 = b,\ a_5 = c$ (Fig. 4). Then three diagonals x_1, x_2, x_3 have the same length x. According to tho wellknown Ptolemaean theorem on inscribed quadrangles we have (quadrangle $ABCD$) $xy = ac + xb$ and hence

$$bxy = abc + b^2x, \tag{3}$$

(quadrangle $ACDE$) $x^2 = a^2 + by$ and hence

$$bxy = x^3 + a^2x. \tag{4}$$

From (3) and (4) we get

$$x^3 - (a^2 + b^2)x - abc = 0. \tag{5}$$

If the inscribed quintangle $ABCDE$ would be constructible from given a, b, c (which satisfy (1)) by compass and ruler then the diagonal x would be constructible and hence it would be a quadratic irrationality in a, b, c. Therefore it remains to show that in some cases (5) is not solvable by quadratic irrationals. We assume the reader to be familiar with the fact that we only must show that for special integer values of a, b, c (5) is not factorizable in integer coefficients. Indeed, for $a = 3,\ b = 4,\ c = 5$ we get from (5)

$$x^3 - 25x - 60 = 0.$$

Setting $x^3 - 25x - 60 = (x^2 + Ax + B)(x + D)$ leads to $A + D = O,\ AD + B = -25,\ BD = -6O$, i.e.

$$AB = 60 \quad \text{and} \quad A^2 = B + 25.$$

This system is not solvable in integers: For $A < 6$ we have $B > 1O$, i.e $A^2 < 36$ and $B + 25 > 35$. For $A > 5$ we have $B < 11$ and hence $A^2 > 35$ and $B + 25 < 36$.

Figure 4

Figure 5

Theorem 3. *For $n > 5$ the n-sided inscribed polygon is in general not constructible from its sides by compass and ruler.*

Proof: Because of Theorem 2 it is sufficient to show that from a general solution for any n we could get a general solution for $n-1$. Let us assume for the n-sided inscribed polygon a special co-ordinate system as given in Fig. 5.

If we assume the constructibility by compass and ruler in general then $x_1, y_1, \ldots, x_{n-2}$, y_{n-2} (and of course also x_n) are quadratic irrationalities R_i $(i = 1, \ldots, 2n-4)$ depending on the variables $a_1, \ldots, a_n$ and as such they are continuous functions of their n variables. On the other hand, the geometrical dependence of $x_i,\ y_i$ $(i = 1, \ldots, n-2)$ on the variables $a_1, \ldots, a_n$ is described by continuous functions f_i $(i = 1, \ldots, 2n-4)$ of the variables $a_1, \ldots, a_n$. Because for $a_n \to 0$ the n-sided inscribed polygon is converging to the $(n-1)$-sided inscribed polygon with sides $a_1, \ldots, a_{n-1}$ and for each i both functions R_i and f_i are identical for $a_n \neq 0$ and both are continuous, the R_i must have the same limit values for $a_n \to 0$ as the f_i, i.e. for $a_n = 0$ the quadratic irrationalities R_i describe the constructibility of the $(n-1)$-sided inscribed polygon as depending on its sides $a_1, \ldots, a_{n-1}$.

Finally let us remark that $a_1 = a_2 = \cdots = a_n$ forces that the inscribed polygon with these sides is regular. Hence from the wellknown nonconstructibility of the regular heptagon by compass and ruler Theorem 3 would follow for $n > 6$. But this does not provide an answer to the problem for $n = 5,\ 6$.

References

[1] Schreiber, P.: *Der gallische Apollonios. Zum 45O. Todestag von François Viète.* Math. Schülerzeitschrift alpha 199O, Heft 6, 122-124.

[2] Francisci Vietae Opera omnia (ed. F. van Schooten). Lugduni Batavorum 1646.