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On the Length of Space Curves of Constant Width

Eberhard Teufel
Mathematisches Institut B, Universität Stuttgart
Pfaffenwaldring 57, D-70550 Stuttgart 80

Abstract. The length of a curve of constant width d in the euclidean plane, i.e. the perimeter of a Gleichdick, is always equal to πd (E. Barbier, 1860). The length of a space curve of constant width d is always greater or equal to πd, with equality holding only for plane curves of constant width d (H. Bückner, 1937). In this note we obtain an upper bound for the length of space curves of constant width (Proposition 1). Furthermore, we get upper bounds for the length of 1-transnormal curve segments in non-euclidean planes (see Proposition 2 and Proposition 3).

1. Space curves of constant width

A space curve c of constant width d means a closed smooth (i.e. C^1-)curve in the 3-dimensional euclidean space E^3 with the following property: Each normal plane of c intersects c in exactly two points, and that orthogonally and at distance d (cf. [3]).

Proposition 1. *Let L denote the length of a space curve c of constant width d, then*

$$L < 3\sqrt{2} \cdot \pi d \quad . \tag{1}$$

Proof. Following B. Wegner [3] [4], we isotop c to a space curve of the same constant width d, lying on a sphere $S^2_{d/2}$ of radius $d/2$ and with the same length L. Moreover there exists a pair of antipodal points which pencil of great circles does not touch the curve, s. [4]. Hence w.l.o.g. we assume c to lie antipodally symmetrically on $S^2_{d/2}$ as a spherical graph w.r.t. the equator e. In the following we consider c as an antipodal symmetric spherical curve with the following property: Each normal of c (i.e. normal great circle = intersection of $S^2_{d/2}$ and the normal plane of c as space curve) intersects c in exactly two antipodal points.

Let A and B be two points of intersection of c and the equator e, and let D denote the part of $S^2_{d/2}$ between the great circles through the poles of e and A, resp. B, such that the length $\overline{AB}$ of the geodesic segment $D \cap e$ is less or equal to $\frac{1}{2} \cdot \pi d$. In the following we consider the curve $D \cap c$.

Let c_1 be the fragments of $D \cap c$ making absolute angles between 0 and $\pi/4$ with the parallels of latitude of $S^2_{d/2}$. Because c is a graph w.r.t. e, and representing the curve w.r.t. geodesic parallel coordinates based on e, the length L_1 of c_1 can estimated analytically, namely

$$L_1 \leq \sqrt{2} \cdot \overline{AB} \quad . \tag{2}$$

Let c_2 be the fragments of $D \cap c$ making absolute angles between $\pi/4$ and $\pi/2$ with the parallels of latitude of $S^2_{d/2}$. To estimate the length L_2 of c_2 we use some geometrical observations in order to apply the Crofton formula of the spherical integral geometry.

Lemma 1. *Each normal of $D \cap c$ intersects $D \cap e$ in exactly one point.*

Proof: Supposed a normal of $D \cap c$ avoids $D \cap e$. The normal separates D into two connected components. The end points A, B of $D \cap c$ lie in the one component, the connected curve $D \cap c$ runs into both components. Therefore the normal intersects $D \cap c$ in at least two points. Hence we have a contradiction to the special property of c. Because $\overline{AB} \leq \frac{1}{2}\pi d$ we get exactly one point of intersection. □

Definition 1. Let $P \in D \cap c$ ($P \notin e$). Let F_P, resp. N_P denote the point of intersection of $D \cap e$ and the meridian of $S^2_{d/2}$ through P, resp. the normal of c at P. Then the members of the pencil of great circles through $\pm P$ which intersect $D \cap e$ between F_P and N_P are called subnormals of $D \cap c$ at P.

Lemma 2. (i) *Each subnormal of $D \cap c$ intersects $D \cap e$ in exactly one point.*
(ii) *Each subnormal of $D \cap c$ intersects $D \cap c$ in exactly one point.*

Proof. (i) follows through Definition 1 and Lemma 1. (ii): Assume that a subnormal of $D \cap c$ at some point P intersects $D \cap c$ in more than the point P. Because $D \cap c$ is a graph w.r.t. $D \cap e$ this causes that the normal of c at P intersects $D \cap c$ in at least two points. Hence a contradiction to the special property of c. □

Now we want to apply the Crofton formula. The kinematic density of geodesics (great circles), parameterized along c, writes (cf. [2] (3.16), (17.42))

$$dG_g = \cos\rho \cdot ds\, d\rho \tag{3}$$

where s denotes the arc length parameter on c, and ρ denotes the absolute angle that is included by the geodesic g and the normal of c at the point of intersection $c \cap g$.

By definition of c_2 the subnormals along c_2 build up angle segments greater or equal to $\pi/4$ w.r.t. the normals at the respective points of c_2. Therefore taking into account only subnormals satisfying $0 \leq \rho \leq \pi/4$ and noting Lemma 2 (ii), integrating (3) yields

$$L_2 \leq \sqrt{2} \cdot m(G_2) \tag{4}$$

where G_2 denotes the set of subnormals of c_2 and $m(G_2)$ denotes its total kinematic measure.

Because of Lemma 2 (i) the set G_2 of subnormals of c_2 is contained in the set G_3 of geodesics intersecting $D \cap e$. And integrating (3) we get

$$m(G_2) \leq m(G_3) = 2 \cdot \overline{AB} \quad . \tag{5}$$

Combining (4) and (5) we arrive at

$$L_2 \leq 2\sqrt{2} \cdot \overline{AB} \quad . \tag{6}$$

Finally (2), (6) and going around e yield (1). □

Remark. The upper bound (1) is not smallest possible.

2. 1-transnormal curve segments in non-euclidean planes

Definition 2. (cf. [1]) Let c be a compact connected smooth curve segment in either the euclidean plane E^2 (curvature 0) or the unit sphere S^2 (curvature + 1) or the hyperbolic plane H^2 (curvature - 1) with end points A and B. c is called a 1-transnormal curve segment between A and B, if it meets the following property: Each normal of c intersects c in exactly one point.

Lemma 3. *Let c be a 1-transnormal curve segment between A and B in E^2, S^2 or H^2; suppose $\overline{AB} \leq \pi/2$ in the spherical case. Then c lies inside the intersection K of the geodesic disks of radius $\overline{AB}$ round A and B.*

Proof. Otherwise the normal of c at a suitable of its intersection points with the boundary of K avoids the geodesic segment between A and B, hence a contradiction as in Lemma 1. □

Proposition 2. *Let L denote the length of a 1-transnormal curve segment c with end points A and B. Then*

(i) $L < 3\sqrt{2} \cdot \overline{AB}$, *if* $c \subset E^2$; (7)

(ii) $L < (\cosh \overline{AB} + 2)\sqrt{2} \cdot \overline{AB}$, *if* $c \subset H^2$; (8)

(iii) $L < 3\sqrt{2} \cdot \overline{AB}$, *if* $c \subset S^2$ *and* $\overline{AB} \leq \pi/2$. (9)

Proof. The 1-transnormality of c causes that c is a graph w.r.t. the geodesic e through A and B (i.e. w.r.t. geodesic parallel coordinates based on e) and that c lies inside the strip D built up by the geodesics orthogonal to e and between A and B. This is the case, because otherwise there exists a tangent to c at some point $P \in c$ running orthogonally through e; hence the normal of c at P avoids $D \cap e$, hence a contradiction as in Lemma 1.

Lemma 1, Definition 1 and Lemma 2 are true without change, and the remaining proof runs as in Proposition 1. To the analytical estimation (2) in the hyperbolic case, note Lemma 3: c lies inside K, hence the biggest possible distance of a point of c to e is smaller than $\overline{AB}$. □

Remark. The upper bounds (7), (8), (9) are not smallest possible.

Proposition 3. *Let L denote the length of a 1-transnormal convex smooth curve segment c with end points A and B (convex means that c together with the geodesic segment between A and B build up the boundary of a convex set). Then,*

(i) *if $c \subset E^2$,*

$$L < 2 \cdot \frac{\pi}{3} \cdot \overline{AB}, \tag{10}$$

(ii) *if $c \subset H^2$,*

$$L < 2 \cdot \arccos \frac{\cosh \overline{AB}}{1 + \cosh \overline{AB}} \cdot \sinh \overline{AB}, \tag{11}$$

(iii) *if $c \subset S^2$ and $\overline{AB} \leq \pi/2$,*

$$L < 2 \cdot \arccos \frac{\cos \overline{AB}}{1 + \cos \overline{AB}} \cdot \sin \overline{AB}. \tag{12}$$

Proof. c lies inside K (Lemma 3). Now the monotonicity of the perimeter of convex sets containing one another (proved e.g. through the Crofton formula), and the computation of the length of the circle segments of the boundary of K yield the proposition. □

Remark. The upper bounds (10), (11), (12) are smallest possible.

References

[1] S. A. Robertson: *Generalized constant width for manifolds.* Mich. Math. J. 11 (1964) 97-105.

[2] L. A. Santaló: *Integral Geometry and Geometric Probability.* Addison-Wesley, London etc., 1976.

[3] B. Wegner: *Globale Sätze über Raumkurven konstanter Breite.* Math. Nachr. 53 (1972) 337-344.

[4] B. Wegner: *Globale Sätze über Raumkurven konstanter Breite II.* Math. Nachr. 67 (1975) 213-223.