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On some Packings of the Hyperbolic Plane by Incongruent Hypercyclic Domains

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In the earlier paper [8] packings of the hyperbolic plane with congruent hypercyclic domains have been studied in general. However, we must omit generality in case of incongruent hypercyclic domains since the basic line of a given hypercycle divides the plane into two parts and so the hypercyclic domains can be unluckily separated.

In this paper we consider only packings with those incongruent hypercyclic domains where their distances obtain three different values ($\ell_3 > \ell_2 > \ell_1 > 0$). Obviously, the basic lines of three fixed hypercyclic domains and their common perpendicular lines determine a hexagon with six right angles which is called a rectangular-hexagon. The copies of such a rectangular-hexagon obtained by reflections can form a tiling of the hyperbolic plane which determines a special packing of incongruent hypercyclic domains with respect to the Dirichlet cellular decomposition.

The densities of the packings will be calculated by using the tiling hexagons where the partial densities of the hypercyclic domains will be determined by their Dirichlet cells.

The isometry groups of these tilings belong to the same isomorphism class (see [2],[3]) and each group is generated by six reflections (axial symmetries) on the six edges. Remark that these isometry groups have infinitely many non-isometric realisations and any two different realisations can be carried into each other by a continuous deformation.

We determine the exact upper bounds of the densities of the hypercyclic packings having three different distances. First we prove the following simple lemma.

Lemma 1. *Let be given three arbitrary edges e_1, e_2, e_3. Then there exists uniquely a rectangular-hexagon such that its non-adjacent edge coincide with e_1, e_2, e_3.*

Proof. Let us take the given edges e_1, e_2, e_3 and the straight line a passing through the points X, Y (Fig.1). Draw the perpendicular lines $XX_1 = e_1, YY_1 = e_2$ starting from the points X, Y. Suppose that the perpendiculars x_1 and y_1 to XX_1 resp. YY_1 have a point Z of intersection. If the points X and Y move away from each other, then it is possible that x_1 and y_1 have parallel position. If X and Y further move away, then the straight lines x_1

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Figure 1.

Figure 2.

and y_1 have a common perpendicular Z_1Z_2. The distance Z_1Z_2 is equal to e_3 in a suitable position (Fig. 2).

For a rectangular-hexagon we have the following trigonometric formulas

$$\frac{\sinh \frac{a_1}{\kappa}}{\sinh \frac{e_1}{\kappa}} = \frac{\sinh \frac{a_2}{\kappa}}{\sinh \frac{e_2}{\kappa}} = \frac{\sinh \frac{a_3}{\kappa}}{\sinh \frac{e_3}{\kappa}} \tag{1}$$

$$\cosh \frac{a_1}{\kappa} = \frac{\cosh \frac{e_1}{\kappa}}{\sinh \frac{e_2}{\kappa} \sinh \frac{e_3}{\kappa}} + \coth \frac{e_2}{\kappa} \coth \frac{e_3}{\kappa} \tag{2}$$

(see [5]), therefore the distance Z_1Z_2 is a strictly increasing function of XY. Using the equalities (1) and (2), we obtain the other three edges $XY = a_3,\ X_1Z_1 = a_2,\ Y_1Z_2 = a_1$. Hence, they are determined by $e_1,\ e_2,\ e_3$ which completes the proof. □

Lemma 2. *By reflections on edges it is possible to construct a tiling of congruent rectangular-hexagons without gaps in the hyperbolic plane. If we colour the edges $a_1,\ a_2,\ a_3$ with blue, green resp. red, then we obtain complete blue, green resp. red straight lines in the tiling. (A similar statement is true for the egdes $e_1,\ e_2,\ e_3$.)*

Proof. By the Alexandrov-Poincaré theorem (see [2] and [3]) it suffices to show the existence of a ring consisting of congruent rectangular-hexagons, which encircles the original hexagon (Fig.3). Let us consider the hexagon $A_1 ... A_6$ and the symmetric hexagons $A_1' ... A_6',\ A_1'' ... A_6''$ obtained by the reflections on the axes A_1A_6, A_5A_6, respectively. In the symmetries with respect to the axes $A_5'A_6', A_1''A_6''$ the reflected hexagons coincide, and so the congruent rectangular-hexagon encircle the vertex A_6 without gaps. Similar constructions were applicable to other vertices, and hereby we obtain an encircling ring. The blue, green, red edges constitute complete blue, green, red straight lines, respectively.

Fig. 4 represents a tiling with congruent regular rectangular-hexagons. □

Corollary 1. *If we consider three disjoint hypercyclic domains where the distances measured from one another are $\ell_3 > \ell_2 > \ell_1$ then the basic lines and their common perpendiculars determine a rectangular-hexagon (Fig. 5), which may be an element of the above tiling.*

Figure 3.

Figure 4.

Figure 5.

We must suppose that the hypercyclic domains have only two points of intersection with the boundary of the rectangular-hexagon. In the tilings with such rectangular-hexagons we get incongruent hypercyclic domains belonging to the blue, green, red straight lines, respectively. Therefore we obtain a packing of incongruent hypercyclic domains. We calculate the density of the packing with respect to the elements of the tiling.

Remark 1. The curvature of a hypercycle with the distance ℓ is given by formula (see [9])

$$\mathcal{K}(\ell) = \frac{1}{\kappa} \tanh \frac{\ell}{\kappa}. \tag{3}$$

Corollary 2. *The curvature of the hypercycles with the distances $\ell_3 > \ell_2 > \ell_1$ have the following inequalities*

$$\mathcal{K}(\ell_3) > \mathcal{K}(\ell_2) > \mathcal{K}(\ell_1).$$

The boundary of a hypercyclic domain includes two hypercycles in the two sides of the basic line. One of these hypercycles divides the plane into two parts. One of the two parts contains the basic line, which is called a hypercyclic-half-plane with curvature $\mathcal{K}(\ell)$. If two hypercyclic-half-planes touch and they are included in the same half-plane with respect to the common tangent line, moreover their boundaries have the curvatures $\mathcal{K}_1 < \mathcal{K}_2$, then the hypercyclic-half-plane with curvature $\mathcal{K}_1$ contains the other one with curvature $\mathcal{K}_2$. This statement is true in view of curvatures of their boundaries; resp. when we consider a pencil of touched cycles with common axis.

Remark 2. The hypercycles are uniform curves, therefore three incongruent hypercyclic domains pushed to one another determine a well-defined arc-triangle (Fig. 6). This configuration determines a rectangular-hexagon with area $\kappa^2\pi$. It seems to us that this arc-triangle has the minimal area which is not covered by the hypercyclic domains. Obviously this arc-triangle is included in the rectangular-hexagon. If $\mathcal{T}_{123}$ is the area of this arc-triangle, then the density has the form:

$$d(\ell_1, \ell_2, \ell_3) = \frac{\kappa^2\pi - \mathcal{T}_{123}}{\kappa^2\pi}. \tag{4}$$

In the following we investigate three hypercycles with distances $\ell_3 > \ell_2 > \ell_1$ such that they touch one another and the common tangents are the radical axes of two hypercycles (see [4], [6] and [7]). The points of tangency T_1, T_2, T_3 and the radical centre O divide the rectangular-hexagon with the edges OT_1, OT_2, OT_3 into three symmetric pentagons (Fig. 6). The perpendiculars from O to the basic lines determine six Lambert quadrilaterals with three right angles. Let $\gamma_1, \gamma_2, \gamma_3$ be the acute angles in them. Let $\Pi(c)$ denote the angle of parallelism with respect to a given distance c and let s denote the common length of the segments OT_1, OT_2, OT_3. Take the distances c_i $(i = 1, 2, 3)$ defined by $\Pi(c_i) = \gamma_i$. Two distances ℓ_i and ℓ_i' are called complementary if $\Pi(\ell_i) + \Pi(\ell_i') = \frac{\pi}{2}$ is valid (see [1] §§10, 14, 15). We obtain from the trigonometry of Lambert quadrilaterals

$$\sinh \frac{c_i}{\kappa} \cosh \frac{\ell_i'}{\kappa} = \sinh \frac{s}{\kappa}$$

resp. (5)

$$\cot \gamma_i = \sinh \frac{s}{\kappa} \tanh \frac{\ell_i}{\kappa} \quad (i = 1, 2, 3).$$

Figure 6.

The inequalities $\gamma_3 < \gamma_2 < \gamma_1$ are true, since $\ell_3 > \ell_2 > \ell_1$; and $\gamma_1 + \gamma_2 + \gamma_3 = \pi$. Moreover $a_3 < a_2 < a_1$, since $\ell_1 + \ell_2 < \ell_1 + \ell_3 < \ell_2 + \ell_3$. The area of the arc-triangle is given by

$$\mathcal{T}_{123} = \kappa^2\pi - \kappa\{a_1 \sinh\frac{\ell_1}{\kappa} + a_2 \sinh\frac{\ell_2}{\kappa} + a_3 \sinh\frac{\ell_3}{\kappa}\}. \tag{6}$$

Lemma 3. *Let τ' resp. τ'' be the areas of the arc-triangles bounded with the hypercyclic arcs $\mathcal{L}, \mathcal{L}^*, \mathcal{L}'$ and $\mathcal{L}, \mathcal{L}^*, \mathcal{L}''$ of distances ℓ, ℓ^*, ℓ' and ℓ, ℓ^*, ℓ'', respectively. If $\ell'' > \ell'$, then $\tau'' < \tau'$ (Fig. 7).*

Proof. The arcs $\mathcal{L}, \mathcal{L}'$ and $\mathcal{L}''$ touch one another in the point T where the hypercyclic-half-plane of $\mathcal{L}'$ contains the one of $\mathcal{L}''$. The moving curve $\mathcal{L}''$ rolls along $\mathcal{L}$, and $\mathcal{L}''$ has two intersecting points with $\mathcal{L}'$ - in view of their curvature - inside the rectangular-hexagon. If $\mathcal{L}''$ knocks against $\mathcal{L}^*$, then the moving curve has the position $\hat{\mathcal{L}}''$, where $\hat{\mathcal{L}}''$ has two points of tangency U and V with the curves $\mathcal{L}^*$ and $\mathcal{L}$, respectively which are inside τ'. This implies that $\tau'' < \tau'$ holds. $\square$

Corollary 3. *Let $\mathcal{T}_{ijk}$ be the area of an arc-triangle which is bounded with hypercyclic arcs of distances ℓ_i, ℓ_j, ℓ_k $i, j, k \in \{1, 2, 3\}$. If $\ell_3 > \ell_2 > \ell_1$ is valid, then the following inequalities*

$$\mathcal{T}_{111} > \mathcal{T}_{112} > \mathcal{T}_{122} > \mathcal{T}_{123} > \mathcal{T}_{133} > \mathcal{T}_{233}$$

Figure 7.

resp.

$$\mathcal{T}_{123} > \mathcal{T}_{223} > \mathcal{T}_{233} > \mathcal{T}_{333}$$

hold.

The formula (4) expressing the density can be generalized in the following form:

$$d(\ell_i, \ell_j, \ell_k) = \frac{\kappa^2\pi - \mathcal{T}_{ijk}}{\kappa^2\pi} \quad i, j, k \in \{1, 2, 3\}. \qquad [7]$$

Lemma 4. *Let $\mathcal{H}(\ell)$ be a hypercyclic domain in a given packing and let us take its Dirichlet cell, where a und b are two intersecting half-lines in the vertex P which contain two neighbouring edges of the Dirichlet cell. The basic line A_1B_1, the common perpendiculars of A_1B_1 and a, resp. A_1B_1 and b determine a pentagon, in which the partial-density of $\mathcal{H}(\ell)$ with respect to Dirichlet cell is calculated. The partial-density of $\mathcal{H}(\ell)$ gives the maximum if $\mathcal{H}(\ell)$ and the half-lines a, b touch. - Such a calculation of the density corresponds to the other cases (f.e. for circles), too.*

Proof. If $\mathcal{H}(\ell)$ touches the half-line a in the point of tangency A, and the convex angle of a, b is $\alpha = \sphericalangle(a, b)$, then the hypercycle intersects the boundary of the pentagon in the point B. The tangent of $\mathcal{H}(\ell)$ in the point B intersects the edge PA in the point C (Fig. 8). Denoting by β the angle $\sphericalangle ACB$, we have $\beta > \alpha$. The symmetric pentagon $ACBB_1A_1$

Figure 8. Figure 9.

contains two Lambert quadrilaterals, in which we obtain the following trigonometric formula:

$$\sinh \frac{A_1 B_1}{2\kappa} = \frac{1}{\sinh \frac{\ell}{\kappa}} \cos \frac{\beta}{2}.$$

Let us suppose that the angle α touches $\mathcal{H}(\ell)$ in two points and m is the edge corresponding to $A_1 B_1$. Then we obtain the analogous trigonometric formula:

$$\sinh \frac{m}{2\kappa} = \frac{1}{\sinh \frac{\ell}{\kappa}} \cos \frac{\alpha}{2}.$$

Since $\beta > \alpha$, therefore $m > A_1 B_1$ and we get the required inequality for the densities

$$\frac{A_1 B_1 \sinh \frac{\ell}{\kappa}}{\kappa(\pi - \alpha)} < \frac{m \sinh \frac{\ell}{\kappa}}{\kappa(\pi - \alpha)}.$$

If the hypercycle $\mathcal{H}(\ell)$ lies in the interior of the angle α, then $\mathcal{H}(\ell)$ intersects the considered pentagon in two points A, B, in which the tangents of $\mathcal{H}(\ell)$ have the point C of intersection (Fig. 9). Denoting by β the angle $\measuredangle ACB$, we have $\beta > \alpha$. In this case the proof is similar to the previous one. □

Corollary 4. *If a pentagon - with four right angles and with the angle α in the vertex P - touches the hypercycle $\mathcal{H}(\ell)$ as above (Fig. 10), then the density of $\mathcal{H}(\ell)$ with respect to the pentagon increases if the angle α increases.*

Theorem. *Let be given a packing with incongruent hypercyclic domains of distances $\ell_1 < \ell_2 < \ell_3$ such that it satisfies the conditions written in the introduction and in Corollary 1. If the hypercyclic domains have the maximal density corresponding to their Dirichlet cells* *and* $\Delta(\ell_1, \ell_2, \ell_3)$ *denotes the density of* $\mathcal{H}(\ell_1), \mathcal{H}(\ell_2), \mathcal{H}(\ell_3)$ *on the rectangular-hexagon, then the following inequality holds*

$$\Delta(\ell_1, \ell_2, \ell_3) \leq \frac{1}{2}\{d(\ell_1, \ell_3, \ell_3) + d(\ell_2, \ell_3, \ell_3)\}$$

*(*see Corollary 3. (7)).

Proof. Let us take the tiling with rectangular-hexagons which corresponds to these hypercyclic domains. The radical centre (see [4], [6] and [7]) of three hypercycles $\mathcal{H}(\ell_1), \mathcal{H}(\ell_2)$, $\mathcal{H}(\ell_3)$ may be
(i) in the interior of the rectangular-hexagon,
(ii) in the exterior of the rectangular-hexagon (including the ideal or infinity points),
(iii) on the boundary of the rectangular-hexagon.

Figure 10.

Figure 11.

In case (i) (Fig. 11) the maximal density corresponding to the Dirichlet cells of $\mathcal{H}(\ell_1), \mathcal{H}(\ell_2), \mathcal{H}(\ell_3)$ come into existence, when the three hypercycles mutually touch and determine the arc-triangle $\mathcal{T}_{123}$. Therefore evidently $\Delta(\ell_1, \ell_2, \ell_3) = d(\ell_1, \ell_2, \ell_3)$ (Fig.6) and $\Delta(\ell_1, \ell_2, \ell_3) < \frac{1}{2}\{d(\ell_1, \ell_3, \ell_3) + d(\ell_2, \ell_3, \ell_3)\}$ hold.

In case (ii) the vertices of the Dirichlet cells lie on one of the edges of the fundamental-hexagon $A_1 \ldots A_6$ (Fig. 12). In order to see this fact consider the image-hexagon obtained by the reflection on the axis A_3A_4. Then the adequate radical axes have the intersection points P_1, P_2 on A_3A_4, and the line of A_3A_4 coincides with the radical axis of $\mathcal{H}(\ell_i)$ and $\mathcal{H}'(\ell_i)$. The points P_1 and P_2 are the radical centres of $\mathcal{H}(\ell_i), \mathcal{H}(\ell_j), \mathcal{H}'(\ell_i)$ and $\mathcal{H}(\ell_i), \mathcal{H}(\ell_k), \mathcal{H}'(\ell_i)$, respectively.

P_1 and P_2 belong - in the dual dissection with the common perpendicular of A_1A_6 and $A'_1A'_6$ - to the hexagons $TA_1A_2A'_2A'_1T'$ and $TA_6A_5A'_5A'_6T'$, respectively. If the hypercycles have the maximal density corresponding to the Dirichlet cells - according to Lemma 4 - , then the densities $\Delta(\ell_i, \ell_j, \ell_i)$ resp. $\Delta(\ell_i, \ell_k, \ell_i)$ are obtained in the suitable hexagons. The densities $d(\ell_1, \ell_3, \ell_3)$ and $d(\ell_2, \ell_3, \ell_3)$ were upper bounds of $\Delta(\ell_i, \ell_j, \ell_i)$ and $\Delta(\ell_i, \ell_k, \ell_i)$, consequently,

$$\Delta(\ell_1, \ell_2, \ell_3) \leq \frac{1}{2}\{d(\ell_1, \ell_3, \ell_3) + d(\ell_2, \ell_3, \ell_3)\}.$$

Figure 12. Figure 13.

Fig.13 represents a configuration with the maximal density of $\mathcal{H}(\ell_i), \mathcal{H}(\ell_j), \mathcal{H}(\ell_k)$, which has continuation and realisation in the complete hyperbolic plane. The possibility of the equality is obvious.

Figure 14. Figure 15.

In case (iii) the radical centre lies on one of the edges of the hexagon $A_1 \dots A_6$ (Fig. 14), for example on A_3A_4. Let us consider the image-hexagon obtained by the reflection on the axis A_3A_4. Therefore we get the octagon $A_1A_2A_2'A_1'A_6'A_5'A_5A_6$. If the hypercycles $\mathcal{H}(\ell_i), \mathcal{H}(\ell_j), \mathcal{H}(\ell_k)$ have the maximal densities corresponding to the Dirichlet cells, then we find the configuration according to Fig. 15. Comparing Fig. 6. with Fig. 15, we obtain $2\delta_i + \delta_j + \delta_k = \pi$ and $\gamma_i + \gamma_j + \gamma_k = \pi$. Obviously, $PT^* = PT > OT_1 = OT_2 = OT_3$ and consequently $\delta_i < \gamma_i, \delta_j < \gamma_j, \delta_k < \gamma_k$ hold. (see Corollary 4).

The partial densities of $\mathcal{H}(\ell_i), \mathcal{H}(\ell_j), \mathcal{H}(\ell_k)$ in case of Fig. 15 are less than in case of Fig. 6, consequently $\Delta(\ell_1, \ell_2, \ell_3) < d(\ell_1, \ell_2, \ell_3) < \frac{1}{2}\{d(\ell_1, \ell_3, \ell_3) + d(\ell_2, \ell_3, \ell_3)\}$. □

Remark. We can easily take a simple algorithm for computing the upper bounds $d(\ell_i, \ell_j, \ell_k)$, $i, j, k \in \{1, 2, 3\}$, by using the equalities (2), (6) and (7).

For example, we give the upper bounds when $\ell_1 = c\kappa, \ell_2 = 2c\kappa, \ell_3 = 3c\kappa$ with $c = 0.1; 0.5; 1; 2$ (see the Table). Let us compare them with $\frac{3}{\pi}$ which is the maximal density of the horocyclic-packings in the hyperbolic plane. The maximal density of hypercyclic-packings approximates $\frac{3}{\pi}$ with optional precision, when the distance ℓ increases (see [8]). The Table shows the precise estimations executed with Lemma 3 and Corollary 3, respectively.

	c=0.1	c=0.5	c=1	c=2
$d(\ell_1, \ell_2, \ell_3)$	0.624509562	0.918234909	0.946364326	0.953935465
$d(\ell_1, \ell_3, \ell_3)$	0.666585178	0.926282427	0.947386363	0.953951202
$d(\ell_2, \ell_3, \ell_3)$	0.707628573	0.941041769	0.953686292	0.954904698
$\frac{1}{2}\{d(\ell_1,\ell_3,\ell_3)+d(\ell_2,\ell_3,\ell_3)\}$	0.687106875	0.933662098	0.950536327	0.954427950
$d(\ell_3, \ell_3, \ell_3)$	0.739200414	0.946364563	0.954533554	0.954920855
$\frac{3}{\pi} = 0.954929658$				

Table

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