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On Flag-Transitive Anomalous C_3-Geometries

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Abstract. A finite C_3-geometry is called anomalous if it is neither a building nor the A_7-geometry. It is conjectured that no flag-transitive thick anomalous C_3-geometry exists. For a flag-transitive thick anomalous C_3-geometry, we prove that its 2-order y is odd and that its full automorphism group is non-solvable. As a corollary, there are no flag-transitive circular extensions of duals of anomalous C_3-geometries.

1. Introduction

A *flag-transitive C_3-geometry $\mathcal{G}$ of finite order* (x, y) is a residually connected incidence geometry on $I = \{0, 1, 2\}$, in which the residue $\mathcal{G}_v$ at an element v of type i is isomorphic to a generalized quadrangle of finite order (x, y), to a generalized digon, or to a projective plane of order x, respectively for $i = 0, 1, 2$.

If $x, y \geq 2$, $\mathcal{G}$ is called *thick*. (See [2] for the fundamental terminology.)

It is conjectured that a flag-transitive thick C_3-geometry is either a finite building of type C_3 or the sporadic A_7-geometry. A C_3-geometry is called *anomalous* if it is neither a building nor the sporadic A_7-geometry. M.Aschbacher [1] proved this conjecture assuming that the residues of the elements of type 0 and 2 are classical. In the general case, the second author derived several results, which are summarized in [8], but he did not yet succeed to prove the conjecture.

In particular, he showed that the conjecture can be proved if we weaken the hypotheses of Aschbacher [1], assuming only that residues of elements of type 2 are classical projective planes. This is sufficient to prove that anomalous C_3-geometries cannot appear as rank 3 residues in certain geometries of rank 4. For instance, there are no finite flag-transitive geometries for any of the following two diagrams where C_3-residues are thick and anomalous.

Indeed, the projective planes at the middle of the diagram are forced to be classical here (see [3]). This non-existence lemma is one of the first step in the classification of finite flag-transitive geometries belonging to the above diagrams (see [8],[4] and [5] for the first diagram; [12] and [6] for the second one).

In this paper, we will prove the following.

Theorem 1. *Let $\mathcal{G}$ be a flag-transitive thick anomalous C_3-geometry of order (x, y). Then y is odd and the full automorphism group $Aut(\mathcal{G})$ is not solvable.*

Although this does not yet solve the conjecture completely, it suffices to eliminate the existence of some extensions of anomalous C_3-geometries. A residually connected geometry $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3; *)$ of rank 4 is called a *C_3c^*-geometry*, if it belongs to the following diagram

That is, the residue $\mathcal{G}_\pi$ of each element π of $\mathcal{G}_3$ is a C_3-geometry, the residues $\mathcal{G}_F$ of all flags F of type $\{1, 2\}$ and $\{0, 2\}$ are generalized digons, and the residue $\mathcal{G}_F$ of every flag F of type $\{0, 1\}$ is the dual of a circle geometry.

Theorem 2. *There is no flag-transitive C_3c^*-geometry, in which the residue at each element of type 3 is a thick anomalous C_3-geometry.*

The first author has started a classification program for flag-transitive locally finite C_3c^*-geometries in [14] and [15], but assuming that C_3-residues are buildings. The case with C_3-residues isomorphic to the A_7-geometry is settled in [10]. By Theorem 2, these are the only two cases to consider if C_3-residues are thick.

2. Notation and Preliminaries

Throughout §2–3, $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ is a thick anomalous C_3-geometry of order (x, y) with a fixed maximal flag (a, l, u) ($a \in \mathcal{G}_0$, $l \in \mathcal{G}_1$, $u \in \mathcal{G}_2$), admitting a flag-transitive automorphism group A. Elements of $\mathcal{G}_i$ are called *points*, *lines* and *planes* respectively for

$i = 0, 1, 2$. For a flag F and a type i not contained in the type of F, we use $\mathcal{G}_i(F)$ to denote the set of elements of type i incident to all elements of F. The stabilizer of a flag F in A is denoted by A_F and we use K_F to denote the kernel of the action of A_F on the residue $\mathcal{G}_F$.

We summarize here several known facts about an anomalous C_3-geometry [8], [9]:

Lemma 1. *The following hold.*

(1) *There is at most one line incident to two distinct planes u and v. Such a line will be denoted by $u \cap v$, if it exists.*

(2) *For every point-plane flag (a, u), we denote by $\alpha(a, u)$ the number of planes $v (\neq u)$ of $\mathcal{G}_2(a)$ such that $u \cap v$ exists but a is not incident to $u \cap v$. Then $\alpha := \alpha(a, u)$ is a constant, not depending on the particular choice of (a, u), which is called the Ott-Liebler number of $\mathcal{G}$. For any point-plane antiflag (b, u) of $\mathcal{G}$, $\alpha + 1$ is the number of planes v of $\mathcal{G}_2(b)$ such that $u \cap v$ exists.*

(3) *The locally finite geometry $\mathcal{G}$ is a finite geometry with $(x^2 + x + 1)(x^2 y + 1)/(\alpha + 1)$ points, $(x^2 + x + 1)(x^2 y + 1)(xy + 1)/(\alpha + 1)$ lines and $(x^2 y + 1)(xy + 1)(y + 1)/(\alpha + 1)$ planes.*

(4) *The number x is an even number such that $x + 1 \equiv 0 \pmod 3$ and $x^2 + x + 1$ is a prime. We have $x < y \leq x^2 - x$ and $x + y$ divides $xy(x + 1)(y + 1)$.*

(5) *For $d := g.c.d.(x^2, y)$, we have $d^2 < x$ and $(x - 1)d^2 + d \leq y$. The Ott-Liebler number α is a multiple of xd, and $\alpha + 1$ divides $(xy/d) - (\alpha/xd)$. Furthermore, the number $(x + y)(\alpha + 1)$ divides $(1 + xy)(xy - (\alpha/x))$, and the number $(x^2 + y)(\alpha + 1)$ divides $(1 + x^2 y)(x^3 y - (\alpha/x))$.*

(6) *For a plane u, the group A_u/K_u is isomorphic to a Frobenius group $F^{x+1}_{x^2+x+1}$ with kernel of order $x^2 + x + 1$ and a complement a cyclic group of order $x + 1$. It acts regularly on the set of point-line flags of the projective plane $\mathcal{G}_u$, in which the Frobenius kernel acts regularly on both the sets of points and lines, while a complement fixes a point-line antiflag (b, m) and acts regularly both on the set of lines through b and the set of points on m.*

(7) *Assume that $|A|$ is even, and let $\mathcal{C}$ be the subgeometry of $\mathcal{G}$ fixed by an involution i of A. If y is odd, then $\mathcal{C}$ does not contain any plane and consists of some lines and all points on them. If y is even, then $\mathcal{C}$ consists of a unique plane and all lines and points on it.*

(8) *For a prime $p = x^2 + x + 1$, a Sylow p-subgroup of A is of order p and acts semiregularly on $\mathcal{G}_0$.*

3. Proof of Theorem 1

Lemma 2. *Let Ω be a system of non-trivial proper imprimitivity blocks of $\mathcal{G}_0$ under the action of A. Then $|\mathcal{G}_0(u) \cap \Delta| \leq 1$ for each block Δ of Ω.*

Proof. Suppose $|\Delta \cap \mathcal{G}_0(u)| \geq 2$ for a block Δ. Then the sets $\Delta^g \cap \mathcal{G}_0(u)$ for $g \in A_u$ form a system of non-trivial imprimitivity blocks of $\mathcal{G}_0(u)$ under the action of A_u. By Lemma 1(6), A_u acts primitively on $\mathcal{G}_0(u)$, which implies that $\mathcal{G}_0(u) \subseteq \Delta$.

For any plane $v(\neq u)$ for whch $u \cap v$ exists and any $g \in A$ with $v = u^g$, we have $\emptyset \neq \mathcal{G}_0(u \cap v) \subseteq \mathcal{G}_0(u) \cap \mathcal{G}_0(v) \subseteq \Delta \cap \Delta^g$, and so $\mathcal{G}_0(v) \subseteq \Delta = \Delta^g$. Since $\mathcal{G}$ is residually connected, any plane w can be joined to u by a chain $u = u_0, u_1, \ldots, u_m = w$ such that $u_{i-1} \cap u_i$ exists for each $i = 1, 2, \ldots, m$. Then the above argument shows that $\mathcal{G}_0(w) \subseteq \Delta$ for any plane w, that is, $\Delta = \mathcal{G}_0$, which contradicts the assumption. □

Lemma 3. *For two distinct points b, c, we denote by $\mathcal{L}(b, c)$ the set of lines incident to both b and c, and set $n(b, c) = |\mathcal{L}(b, c)|$. Then the following hold.*
(1) *We have $\alpha = \sum_{b \in \mathcal{G}_0(l)-\{a\}}(n(a, b) - 1)$.*
(2) *If $n(a, b)$ is minimum among $n(a, b)$ for $b \in \mathcal{G}_0(l) - \{a\}$, we have $n(a, b) \leq \alpha/x$.*

Proof. (1) Let $\mathcal{A}(a, u)$ be the set of planes v $(\neq u)$ incident to a such that the line $u \cap v$ is not incident a. By Lemma 1 (2), $\alpha = |\mathcal{A}(a, u)|$. We will count the cardinal number of the set

$$\mathcal{X} := \{(v, m, b) \in \mathcal{A}(a, u) \times (\mathcal{G}_1(a) - \{l\}) \times (\mathcal{G}_0(l) - \{a\}) | m * v, b \in \mathcal{G}_0(m)\}$$

in two different ways. For each plane $v \in \mathcal{A}(a, u)$, the line $u \cap v$ on u is not incident to a. Then l and $u \cap v$ are distinct lines on the projective plane u, and they intersect at the unique point b $(\neq a)$. Since a and b are distinct points on the projective plane v, there is a unique line m $(\neq l)$ on v joining a and b. Thus $\alpha = |\mathcal{X}|$. On the other hand, for each point b on l distinct from a, there are $n(a, b) - 1$ lines through a, b distinct from l. For each such line m, (m, u) is an antiflag of the generalized quadrangle $\mathcal{G}_b$. Then there is a unique plane v $(\neq u)$ of $\mathcal{G}_b$ incident to m such that $u \cap v$ exists. Since $v \in \mathcal{A}(a, u)$, we have $|\mathcal{X}| = \sum_{b \in \mathcal{G}_0(l)-\{a\}}(n(a, b) - 1)$.

(2) By the minimality of $n(a, b)$ and the equation in (1) above, we have $x(n(a, b)-1) \leq \alpha$ or $n(a, b) \leq 1 + (\alpha/x)$. Note that x divides α by Lemma 1(5), hence $1 + (\alpha/x)$ is an integer. Suppose $n(a, b) = 1 + (\alpha/x)$. Then the minimality of $n(a, b)$ implies that $n(a, b)$ is constant for b ranging over the set of points on l distinct from a. By the flag-transitivity of A on $\mathcal{G}$, this implies that $n(a, b)$ is constant for (a, b) ranging over the set of pairs of collinear points. Thus by [8] p.9, Proposition 7, $\mathcal{G}$ is either a building or flat. In the latter case, $\mathcal{G}$ is the sporadic A_7-geometry by [8] p.25, Theorem 8. Since $\mathcal{G}$ is anomalous by our assumption, the case $n(a, b) = 1 + (\alpha/x)$ is ruled out. As (α/x) is an integer, we have the result. □

Lemma 4. *Assume that A is solvable. Then the following hold.*
(1) *There is a system Ω of $p = x^2 + x + 1$ imprimitivity blocks of $\mathcal{G}_0$ on which A induces a Frobenius group A/N with kernel of order p, where $N = O_{p'}(A)$.*
(2) *The group $A_a N/N$ is a cyclic Frobenius complement, which is a direct product of $A_{a,u}N/N$ and $A_{a,l}N/N$.*
(3) *We have $A_u \cap N = K_u = A_{a,l,u}$.*

Proof. (1) Set $p = x^2 + x + 1$. By Lemma 1(8), the stabilizer A_u contains a Sylow p-subgroup P of A of order p. Since A is a solvable group with a cyclic Sylow p-subgroup P, it follows from Ito's lemma ([13] Ch.4, Theorem 5.13 and (5.15)) that the largest normal p'-subgroup $N := O_{p'}(A)$ of A and the full inverse image $H := O_{p',p}(A)$ of $O_p(A/N)$ in A

satisfy the following conditions: $A \trianglerighteq H \trianglerighteq N$, A/H and N are p'-groups, $H/N \cong P$ and $C_{A/N}(H/N) = H/N$. In particular, A/N is isomorphic to a subgroup of the Frobenius group $F_p^{x^2+x}$ with kernel of order p and a complement a cyclic group of order $p-1 = x^2+x$.

By Lemma 1(8), the stabilizer A_a of a point a is a p'-group. Since A is solvable, there is a Hall $S_{p'}$-subgroup S of A containing A_a ([13] Ch.4, Theorem 5.6). Then $A = PS$. Let Δ be the S-orbit on $\mathcal{G}_0$ containing the point a. If $\Delta \cap \Delta^g \neq \emptyset$ for some $g \in A$, we have $a^h = a^{hg^{-1}k}$ for some $h, k \in S$. Then $hg^{-1}kh^{-1} \in A_a \subseteq S$, hence $g \in S$ and $\Delta^g = \Delta$. Thus $\Omega := \{\Delta^g | g \in A\}$ forms a system of imprimitivity blocks under the action of A. Since $A = SP$, Ω consists of exactly p blocks.

The group A acts transitively and so primitively on Ω. Since the normal p'-subgroup $N = O_{p'}(A)$ is contained in any $S_{p'}$-subgroups of A, N acts trivially on Ω. Since any non-trivial normal subgroup of the Frobenius group A/N contains the Frobenius kernel of order p, A/N acts faithfully on Ω. Thus A/N is a Frobenius permutation group on Ω, that is, each non-trivial element of A/N fixes at most one block of Ω.

(3) Since Ω consists of p-blocks, $|\mathcal{G}_0(u) \cap B| = 1$ for each block B of Ω by Lemma 2. Then $A_u \cap N$ fixes each point of $\mathcal{G}_0(u)$, hence fixes each line on the projective plane $\mathcal{G}_u$. Thus $A_u \cap N \subseteq K_u$. The group K_uN/N is a p'-subgroup of the Frobenius group A/N normalized by the kernel PN/N. Thus $A_u \cap N = K_u = A_{a,l,u}$ by Lemma 1(6).

(2) The group A is generated by $A_{a,l}$, $A_{a,u}$ and $A_{l,u}$ by the residual connectedness of $\mathcal{G}$. Since $A_uN/N \cong A_u/K_u$ by the claim (3) above, it follows from Lemma 1(6) that A_uN/N is a split extension of $A_{a,u}N/N$ by PN/N. Then $A/N = \langle A_uN/N, A_aN/N\rangle = (PN/N)(A_aN/N)$, as PN/N is the Frobenius kernel of A/N. Since A_aN/N fixes the block Δ containing a, it is a Frobenius complement. Thus AN/N is a cyclic group generated by $A_{a,u}N/N$ and $A_{a,l}N/N$.

Let $g = hn$ be an element of $A_{a,u} \cap A_{a,l}N$ with $h \in A_{a,l}$, $n \in N$. For each point b on l, $\Omega(b)$ means the block of Ω containing b. We have $\Omega(b) = \Omega(b^n)$ as N is the kernel of the action of A on Ω. Note that $b^n \in \mathcal{G}_0(l)^n = \mathcal{G}_0(l)^g \subseteq \mathcal{G}_0(u)^g = \mathcal{G}_0(u)$. Then both b^n and b are points on the plane u contained in the block $\Omega(b)$. Thus $b = b^n$ by Lemma 2. Since n fixes all points on l, the element $g = hn$ acting on the projective plane $\mathcal{G}_u$ fixes the point a and permutes the points on the line l. Hence g fixes a flag (a, l, u), and therefore $A_{a,u}N \cap A_{a,l}N = (A_{a,u} \cap A_{a,l}N)N = A_{a,l,u}N = N$ by the claim (3) above. Then the cyclic subgroup A_aN/N is a direct product of $A_{a,u}N/N$ and $A_{a,l}N/N$. □

Proposition 5. *There is no locally finite thick anomalous C_3-geometry, admitting a flag-transitive solvable automorphism group.*

Proof. We use the same notation as in Lemma 4. Let b be a point on l distinct from a such that $n(a, b)$ is minimum (see Lemma 3). We denote by $O(b)$ the orbit through b under the action of $N_a := N \cap A_a$. By Lemma 4, $O(b)$ is a subset of the block $\Omega(b)$ containing b. In particular, $|O(b)| \leq |\Omega(b)| = |\mathcal{G}_0|/(x^2+x+1) = (x^2y+1)/(\alpha+1)$ by Lemma 1(3).

The line l is incident to two points a and b. Let $\mathcal{N}$ be the N_a-orbit on $\mathcal{G}_1(a)$ containig l. By Lemma 4(2), we have $|A_a/N_a| = |A_{a,u}/N \cap A_{a,u}||A_{a,l}/N_a \cap A_l|$. Rewriting this using Lemma 4(3), we have $|N_a : N_a \cap A_l| = |A_a : A_{a,u}|/|A_{a,l} : A_{a,l,u}| = (xy+1)(y+1)/(y+1)$, since A_a and $A_{a,l}$ act transitively on the set $\mathcal{G}_1(a)$ of lines and on the set $\mathcal{G}_2(a, l)$ of planes through a line l in the generalized quadrangle $\mathcal{G}_a$ of order (x, y), respectively. Thus there are $xy+1$ lines in the orbit $\mathcal{N}$.

Since b lies on l, each line of $\mathcal{N}$ is incident to a and to at least one point of $O(b)$. By Lemma 2, the points in the block $\Omega(b)$ are mutually non-collinear. Thus each line of $\mathcal{N}$ is incident with a unique point of $O(b)$. For each point $c \in O(b)$, we denote by $\mathcal{N}(c)$ the set of lines of $\mathcal{N}$ incident to c. By the remark above, $\mathcal{N}$ is a disjoint union of sets $\mathcal{N}(c)$ $(c \in O(b))$. Since $O(b)$ is an N_a-orbit, $|\mathcal{N}(c)|$ is constant, and therefore we have $|O(b)| = |\mathcal{N}|/|\mathcal{N}(b)| = (xy+1)/|\mathcal{N}(b)|$. Since $|O(b)| \leq (x^2y+1)/(\alpha+1)$, we have $|\mathcal{N}(b)| \geq (xy+1)(\alpha+1)/(x^2y+1)$. However, since $\mathcal{N}(b)$ consists of lines incident to both a and b, we have $(\alpha/x) \geq n(a,b) \geq |\mathcal{N}(b)|$ by Lemma 3(2), and therefore $(x^2y+1)(\alpha/x) \geq (xy+1)(\alpha+1)$. This implies that $((1/x)-1)\alpha \geq xy+1$, a contradiction. □

We will now complete the proof by establishing the following statement.

Proposition 6. *There is no flag-transitive anomalous C_3-geometry of finite order (x, y) with y even.*

Proof. Assume that $\mathcal{G}$ is an anomalous C_3-geometry of finite order (x, y) with y even, admitting a flag-transitive automorphism group A. For any element $g \in A - A_u$, the subgroup $A_u \cap A_u^g$ fixes two distinct planes u and u^g, hence it has odd order by Lemma 1(7). That is, the proper subgroup A_u is a strongly embedded subgroup of A. Then either a Sylow 2-subgroup of A is cyclic or generalized quarternion, or A acts doubly transitively on the cosets A/A_u, which can be identified with the set $\mathcal{G}_2$ of planes (see e.g. [13] Chap.6, Theorem 4.8).

If A acts doubly transitively on $\mathcal{G}_2$, any two distinct planes are incident to a common line. Thus for a point-plane antiflag (b, u) and for any plane v of $\mathcal{G}_2(b)$, the line $u \cap v$ exists. It follows from Lemma 1(2) that $|\mathcal{G}_2(b)| = (y+1)(xy+1) = \alpha+1$. Since $\alpha+1$ divides x^2y+1 by Lemma 1(3), we have $(y+1)(xy+1) \leq x^2y+1$, or $x^2-x-1 \geq xy$. However, $xy > x^2$ by Lemma 1(4), which is a contradiction.

If A has a cyclic Sylow 2-subgroup S, A is a split extension of S by the odd core $O(A)$, which is solvable by the odd order theorem. Such a possibility is eliminated by Proposition 5. Assume that a Sylow 2-subgroup S of A is a generalized quaternion group. Since $|\mathcal{G}_2|$ is odd by Lemma 1(3), we may take $S \subseteq A_u$. Then $S \subseteq K_u$, as $|A_u/K_u| = (x^2+x+1)(x+1)$ is odd. The unique involution i of S is central in A modulo $O(A)$ (e.g. [13] Chap.6, §2, Example 3). In particular, $M := O(A)\langle i \rangle$ is a normal subgroup of A for which A/M has dihedral Sylow 2-subgroups. Since A is not solvable by Proposition 5, A/M is not solvable. Thus A/M is isomorphic either to A_7 or to a subgroup of $Aut(L_2(q))$ containing $L_2(q)$ for some odd prime power q (e.g. [13] Chap.6, Theorem 8.6, (8.9)).

Let Ω be the set of $O(A)$-orbits on $\mathcal{G}_0$. Each $O(A)$-orbit contains a point a^g for some $g \in A$. Since $gi = ign$ for some $n \in O(A)$,we have $(a^g)^i = (a^i)^{gn} = a^{gn}$. Thus the involution i fixes every $O(A)$-orbit. Since the length of an $O(A)$-orbit is odd, i has a fixed point on each $O(A)$-orbit. Since the set of points fixed by i coincides with $\mathcal{G}_0(u)$ by Lemma 1(8), it follows from Lemma 2 that Ω consists of exactly x^2+x+1 orbits. The subgroup S fixes every point of $\mathcal{G}_0(u)$, and therefore it acts trivially on Ω. Hence the normal subgroup N of A generated by all Sylow 2-subgroups together with M acts trivially on Ω.

If $A/M \cong A_7$, a non-abelian simple group, we have $A = N$, which contradicts the transitivity of A on $\mathcal{G}_0$. Assume that A/M is a subgroup of $Aut(L_2(q))$ containing $L_2(q)$. Then A/N corresponds to a subgroup of the field automorphism group of $L_2(q)$. In particular, A/N is a cyclic group. On the other hand, since N acts trivially on Ω, it follows from Lemma 2 that $A_u \cap N \subseteq K_u$. Thus a subgroup A_uN/N of A/N has a factor group isomorphic to A_u/K_u, which is a non-abelian Frobenius group of order $(x^2+x+1)(x+1)$. This contradiction eliminates the final case. □

4. Proof of Theorem 2

In the remainder of this paper, we assume that $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3; *)$ is a C_3c^*-geometry (see §1), admitting a flag-transitive automorphism group G, in which the residue at $\pi \in \mathcal{G}_3$ is an anomalous C_3-geometry. We may apply the results in §2,3 to $\mathcal{G}_\pi$. We call elements of $\mathcal{G}_i$ *points*, *lines*, *planes* and *quads*, respectively for $i = 0, 1, 2$ and 3. We fix a maximal flag (P, l, u, π) of $\mathcal{G}$ with $P \in \mathcal{G}_0$, $l \in \mathcal{G}_1$, $u \in \mathcal{G}_2$ and $\pi \in \mathcal{G}_3$, and follow the other convention in §2,3.

Lemma 7. *The group G acts regularly on the set of maximal flags of $\mathcal{G}$. In particular,*

(1) *For each flag F, the stabilizer G_F acts faithfully and regularly on the set of maximal flags of the residue $\mathcal{G}_F$.*

(2) *G_u is isomorphic to $F^{x+1}_{x^2+x+1} \times 2$ with the kernel $O_2(G_u) = G_{P,l,u}$ on the projective plane $(\mathcal{G}_0(u), \mathcal{G}_1(u); *)$. Furthermore, $G_{P,u,\pi}$ and $G_{l,u,\pi}$ are cyclic subgroups of order $x+1$.*

Proof. We divide the proof into several steps.

Step 1. *The stabilizer $G_{P,l,u}$ coincides with the kernel of G_u on $\Pi(u) := (\mathcal{G}_0(u), \mathcal{G}_1(u); *)$. Furthermore, $G_{P,l,u,\pi} = K_u$ and $G_u/K_u \cong F^{x+1}_{x^2+x+1} \times 2$.*

Proof. Since $G_{P,l,u}$ induces an automorphism group of the projective plane $\Pi(u)$ fixing a point-line flag (P, l) of $\Pi(u)$, $G_{P,l,u}$ acts trivially on $\Pi(u)$ by Lemma 1 (6). Then $G_{P,l,u}$ is contained in the kernel of the action of G_u on $\Pi(u)$, and the converse inclusion clearly holds.

Since $(\mathcal{G}_3(l), \mathcal{G}_2(l); *)$ is a circle geometry, $\mathcal{G}_3(u)$ consists of two quads including π, on which $G_{P,l,u}$ acts transitively by the flag-transitivity of G. Thus $G_u = G_{P,l,u}G_{u,\pi}$ and $G_{P,l,u} \cap G_{u,\pi} = G_{P,l,u,\pi}$ coincides with the kernel K_u of the action of G_u on the residue $\mathcal{G}_u$ by the paragraph above. Moreover, $G_{P,l,u}/K_u \cong 2$ is the kernel of the action of G_u/K_u on $\Pi(u)$. On the other hand, since G_u/K_u induces the unique flag-transitive automorphism group $F := F^{x+1}_{x^2+x+1}$ of $\Pi(u)$, we have $G_u/G_{P,l,u} \cong G_{u,\pi}/K_u \cong F$. Thus $G_u/K_u = (G_{u,\pi}/K_u) \times (G_{P,l,u}/K_u) \cong F \times 2$. □

Step 2. *The kernel of $G_{P,\pi}$ on the generalized quadrangle $\mathcal{G}_{(P,\pi)}$ coincides with K_P.*

Proof. Let $K = K(P, \pi)$ be the kernel of the action of $G_{P,\pi}$ on the generalized quadrangle $\mathcal{G}_{(P,\pi)}$. Clearly $K_P \subseteq K$. Since $(\mathcal{G}_3(P), \mathcal{G}_2(P), \mathcal{G}_1(P); *)$ is a residually connected $c.C_2$-geometry, any quad μ of $\mathcal{G}_3(P)$ can be joined to π by a chain $\pi_0 = \pi, \pi_1, \ldots, \pi_n = \mu$ of quads of $\mathcal{G}_3(P)$ with $\mathcal{G}_2(\pi_{i-1}) \cap \mathcal{G}_2(\pi_i) \neq \emptyset$ for every $i = 1, \ldots, n$. Thus it suffices to show that $K(P, \pi) \subseteq K(P, \pi')$ for any quad π' with $\mathcal{G}_3(v) = \{\pi, \pi'\}$ for some plane $v \in \mathcal{G}_2(P)$.

The group K fixes v and the unique quad π' of $\mathcal{G}_3(v)$ distinct from π. Let m be any line of $\mathcal{G}_1(P,v)$. The group K fixes every plane of $\mathcal{G}_2(m,\pi)(\subseteq \mathcal{G}_2(P,\pi))$. On the other hand, $\mathcal{G}_2(m,\pi)$ can be identified with the set $\mathcal{G}_3(m)-\{\pi\}$, representing a plane $w\in\mathcal{G}_2(m,\pi)$ by the unique quad incident to it and distinct from π'. Hence K fixes all quads and planes incident to m. Thus $K\subseteq G_{P,m,w,\pi'}=K_w$ for all planes $w\in\mathcal{G}_2(m,\pi')$ by Step 1. That is, the group K acts on the generalized quadrangle $\mathcal{G}_{(P,\pi')}$ fixing a plane v and all planes w intersecting v, together with all lines on them. Then K acts trivially on $\mathcal{G}_{(P,\pi')}$. Thus $K\subseteq K(P,\pi')$, and therefore $K_P=K(P,\pi)$. □

Proof of Lemma 7. Both the kernels K_l and K_P act on the projective plane $\Pi(u)$ fixing a flag (P,l). Thus K_l (resp. K_P) acts trivially on $(\mathcal{G}_0\cup\mathcal{G}_1)(u)$ (resp. $\mathcal{G}_0(l)$). As $\mathcal{G}_3(u)\subseteq\mathcal{G}_3(l)$ and $(\mathcal{G}_2\cup\mathcal{G}_3)(l)\subseteq(\mathcal{G}_2\cup\mathcal{G}_3)(P)$, we have $K_P\subseteq K_l\subseteq K_u$. By Step 2, we have $K_\pi\subseteq K_P$, and so $K_\pi\subseteq K_P\subseteq K_l\subseteq K_u$. Since K_u acts trivially on the projective plane $\Pi(u)$, it follows from Lemma 1(6) that $K_u\subseteq K_\pi$. Then the residual connectivity of $\mathcal{G}$ implies that $K_\pi=K_P=K_l=K_u$ is a normal subgroup of $G=\langle G_P,G_l,G_u,G_\pi\rangle$, which, since it fixes (P,l,u,π), fixes all maximal flags of $\mathcal{G}$. Thus $K_\pi=K_P=K_l=K_u=1$. All statements of the lemma now follow from Steps 1,2 and Lemma 1(6). □

We now consider the action of G_l on the set $\mathcal{G}_3(l)$ of $y+2$ quads.

Lemma 8. *Let M_l and N_l be the kernels of the action of G_l on $\mathcal{G}_3(l)$ and $\mathcal{G}_0(l)$, respectively. Then the following hold.*

(1) *The group M_l is a cyclic group of order dividing $x+1$.*

(2) *The group G_l/M_l is a doubly transitive permutation group on $\mathcal{G}_3(l)$, in which the stabilizer of two distinct quads is a cyclic group of odd order.*

(3) *The group $N_lM_l/M_l\cong N_l$ is a normal subgroup of G_l/M_l of odd index, in which the stabilizer of two distinct quads of $\mathcal{G}_3(l)$ is trivial.*

Proof. Since M_l is a subgroup of the cyclic group $G_{l,u,\pi}$ of order $x+1$ by Lemma 7(2), the claim (1) follows. Since $(\mathcal{G}_3(l),\mathcal{G}_2(l);*)$ is a circle geometry, the set $\mathcal{G}_3(l)-\{\pi\}$ can be identified with the set $\mathcal{G}_2(l,\pi)$ of $y+1$ planes. Then it follows from the flag-transitivity of G that G_l acts doubly transitively on $\mathcal{G}_3(l)$. The stabilizer $G_{l,u,\pi}$ in G_l of two quads $\pi,\pi'\in\mathcal{G}_3(l)$ with $\{\pi,\pi'\}=\mathcal{G}_3(u)$ is a cyclic group of order $x+1$ by Lemma 7(2). The claim (2) follows.

The image N_lM_l/M_l is a normal subgroup of G_l/M_l isomorphic to N_l, since $N_l\cap M_l=K_l=1$ by Prop.2(1). The stabilizer $N_l\cap G_{u,\pi}$ in N_l of two quads π,π' above is contained in $G_{P,l,u,\pi}$, and therefore trivial by Lemma 7.

To complete the proof, it only remains to show that $|G_l/N_l|$ is odd. Let g be an involution of G_l, which has order $(x+1)(y+2)(y+1)$ (see Lemma 7). Since $(\mathcal{G}_3(l),\mathcal{G}_2(l);*)$ is a circle geometry, there is a plane v of $\mathcal{G}_2(l)$ incident to two quads fixed or interchanged by the involution g. Since $|\mathcal{G}_0(l)|=x+1$ is odd, g fixes a point Q. Thus g acts on the projective plane $\Pi(v)$ fixing a flag (Q,l). Since $Aut(\Pi(v))$ is of odd order, g fixes all elements of $(\mathcal{G}_0\cup\mathcal{G}_1)(v)$. In particular, $g\in N_l$. Thus the group G_l/N_l is of odd order. □

Lemma 9. *The permutation group G_l/M_l on the set $\mathcal{G}_3(l)$ is equivalent to a subgroup of $A\Gamma L(1,\mathbf{F}_{q^n})$ containing $T(1,\mathbf{F}_{q^n})$ on the set $\mathbf{F}_{q^n}$ for some prime power $q=p^m$ and* *some n with $y+2=q^n$, where $A\Gamma L(1,\mathbf{F}_{q^n})$ (resp.$T(1,\mathbf{F}_{q^n})$) denotes the group of maps $f(a,b;\alpha): x \mapsto ax^{\alpha}+b$ (resp. $f(1,b;id)$) on $\mathbf{F}_{q^n}$ for all $a \in \mathbf{F}_{q^n}-\{0\}$, $b \in \mathbf{F}_{q^n}$ and $\alpha \in Gal(\mathbf{F}_{q^n}/\mathbf{F}_q)$.*

Proof. By Lemma 8(2), there is a unique minimal non-trivial normal subgroup U_l/M_l of G_l/M_l which is either a non-abelian simple group acting doubly transitively on $\mathcal{G}_3(l)$ or an elementary abelian p-group acting regularly on $\mathcal{G}_3(l)$. In the latter case, G_l/M_l is a split extension of $G_{l,\pi}/M_l$ by U_l/M_l, in which U_l/M_l can be identified with $\mathcal{G}_3(l)$ so that the action of $G_{l,\pi}/M_l$ on the non-trivial elements of U_l/M_l is equivalent to the action of $G_{l,\pi}$ on $\mathcal{G}_3(l)-\{\pi\}$.

Assume that U_l/M_l is a non-abelian simple group. The unique minimal normal subgroup U_l/M_l is contained in a non-trivial normal subgroup N_lM_l/M_l of G_l/M_l. By Lemma 3(3), U_l/M_l is a sharply doubly transitive permutation group on $\mathcal{G}_3(l)$. However, this forces the group U_l/M_l to be a Frobenius group on $\mathcal{G}_3(l)$ of maximal possible order, and therefore it follows from [11] p.62 Prop. 8.4 that it has a regular normal subgroup, which contradicts the simplicity of U_l/M_l. Thus U_l/M_l is an elementary abelian group of order $y+2=q=p^m$ for some prime p.

Since $G_{l,u,\pi}$ is a cyclic group of odd order $x+1$, the stabilizer in G_l/M_l of two quads is of odd order. Thus by applying [7] p.458, Theorem 6.7 to G_l/M_l, either we have the conclusion of the lemma or one of the following.

(i) G_l/M_l on $\mathcal{G}_3(l)$ is equivalent to a subgroup of $GL(2,\mathbf{F}_{q^n})$ on a 2-dimensional space over $\mathbf{F}_{q^n}$ containing $SL(2,\mathbf{F}_{q^n})$ for some prime power $q=p^m$ and some n with $y+2=q^{2n}$.

(ii) $y+2=5^2$, 7^2, 9^2, 11^2, 19^2, 23^2, 29^2, 59^2 or 3^6.

By applying the conditions in Lemma 1(4) to the values of y in Case (ii), we obtain $(x,y)=(434,23^2-2)$, $(1562,59^2-2)$ or $(1886,59^2-2)$. In the first two cases, $d=g.c.d.(x^2,y)=31$ or 71, which does not satisfy the inequality $d^2<x$ in Lemma 1(5). In the last case, we have $d=1$ and we can verify that for each value $t=1,\dots,y-1$, the number $\alpha=xt$ does not satisfy the divisibility condition $\alpha+1|xy-(\alpha/x)$ in Lemma 1(5). Thus the case (ii) does not occur.

Assume that Case (i) occurs. By Lemma 1(4)(5), we have $q^n>3$. Then a perfect subgroup $(G_{l,\pi}/M_l)' \cong SL_2(q^n)$ of G_l/M_l is contained in the normal subgroup N_lM_l/M_l of odd index (see Lemma 8(3)). The stabilizer in $(G_{l,\pi}/M_l)' \cong SL_2(q^n)$ of a non-zero vector of $\mathbf{F}_q^2$ is isomorphic to $\mathbf{F}_q$. In particular, there is a non-trivial element of N_lM_l/M_l fixing two distinct quads of $\mathcal{G}_3(l)$, which contradicts Lemma 8(3). Thus the lemma holds. $\square$

Lemma 10. *A Sylow 2-subgroup T of $G_{P,l,\pi}$ is a Sylow 2-subgroup of G_π, which is cyclic. In particular, $G_\pi=O(G_\pi)T$ is a solvable group.*

Proof. By Lemma 7 and Lemma 1(3), $|G_\pi|=((x^2y+1)/(\alpha+1))(x^2+x+1)(xy+1)(y+1)(x+1)$. As x is even by Lemma 1(4), the group $G_{P,l,\pi}$ of order $y+1$ contains a Sylow 2-subgroup T of G_π. Since $G_{P,l}\cap M_l=G_{P,l,u,\pi}=1$ by Lemma 7, $G_{P,l,\pi}$ is isomorphic to a subgroup of the stabilizer $G_{P,l,\pi}M_l/M_l$ of the quad π in G_l/M_l. This stabilizer corresponds to a subgroup of the group $\{f(a,0;\alpha)|0\neq a\in\mathbf{F}_{q^n},\alpha\in Gal(\mathbf{F}_{q^n}/\mathbf{F}_q)\}$ by Lemma 9, in which the stabilizer $G_{l,u,\pi}/M_l$ of two quads corresponds to a subgroup of the group $\{f(1,0;\alpha)\}$ of the field automorphisms. Since $G_{l,u,\pi}$ is of odd order, a Sylow 2-subgroup T of $G_{P,l,\pi}/M_l$

is isomorphic to a subgroup of the normal cyclic group $\{f(a,0;id) | 0 \neq a \in \mathbf{F}_{q^n}\}$. Thus T is cyclic.

Then the lemma follows from a standard result in group theory and the odd order theorem. □

It now follows from Proposition 5 in §3 and Lemma 10 above that there is no flag-transitive C_3c^*-geometry with residues at elements of type 3 isomorphic to a locally finite thick anomalous C_3-geometry. Theorem 2 is proved.

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