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Transnormal Graph Surfaces

A. M. d'Azevedo Breda and **F. J. Craveiro de Carvalho**
Departamento de Mathemática, Universidade de Aveiro
P-3800 Aveiro, Portugal
Departamento de Mathemática, Universidade de Coimbra
P-3000 Coimbra, Portugal

Graphs of smooth maps very often are examples of transnormal submanifolds of euclidian space $\mathbf{R}^n$. This fact gives us some motivation to look for conditions on the map which ensure us that the graph is transnormal or not.

In this note we study maps f: $\mathbf{R}^2 \to \mathbf{R}$. No doubt that our results (with the possible exception of Theorem 3.2 in **3.**) still hold when we replace $\mathbf{R}^2$ by $\mathbf{R}^n$. However what we gain from working in a more general context is very little. Our results depend heavily on results which have been established previously in [1] and [3].

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1.Introduction

Let f: $\mathbf{R}^2 \to \mathbf{R}$ be a smooth ($= C^\infty$) map. We consider F: $\mathbf{R}^2 \to \mathbf{R}^3$ given by $F(x,y) = (x, y, f(x,y))$. Obviously $F(\mathbf{R}^2)$ is the graph G_f of f. The normal line to F at (x,y) will be denoted by $N_{(x,y)}$ and we say that G_f is transnormal if whenever $(x_1, y_1, f(x_1,y_1)) \in N_{(x,y)}$ we have $N_{(x,y)} = N_{(x_1,y_1)}$. The extension of this notion to maps from $\mathbf{R}$to $\mathbf{R}^2$ is obvious. We remark that the property of being transnormal is invariant under affine isometries.

In what follows $(.|.)$ and $\|.\|$ will denote the standard inner product in $\mathbf{R}^2$ and the norm that it induces. Partial derivatives will be denoted by $D_i f$, i=1,2.

2. Transnormal graphs

Let f: $\mathbf{R}^2 \to \mathbf{R}$ be smooth. Assume that $(x_0, y_0, f(x_0,y_0))$ is such that $D_1 f(x_0,y_0) \neq 0$. Let π denote the affine plane which passes through $(x_0, y_0, f(x_0,y_0))$ and whose direction is $\langle (0,0,1), (D_1 f(x_0,y_0), D_2 f(x_0,y_0), -1) \rangle$, where $\langle \cdots \rangle$ means spanned by. Then $\pi \cap G$ is the graph of h: $\mathbf{R} \to \mathbf{R}^2$ given by $h(t) = (\mu(t), f(t, \mu(t)))$, with $\mu(t) = y_0 + (t - x_0)\frac{D_2 f(x_0,y_0)}{D_1 f(x_0,y_0)}$. Mention to h will be made several times below.

Proposition 2.1 *Let (x_0, y_0) be such that $D_1 f(x_0,y_0) \neq 0$. If G_f is transnormal then G_h is also transnormal.*

Proof. Assume that we have t_0, t_1 such that $(t_1, h(t_1)) \in N_{t_0}$. Since $N_{t_0} \cap \pi = N_{(t_0,h(t_0))}$ it follows from $(t_1, h(t_1)) \in \pi$ that $(t_1, h(t_1)) \in N_{(t_0,h(t_0))}$ and consequently $N_{(t_1,h(t_1))} = N_{(t_0,h(t_0))}$. From this we conclude that $N_{t_1} = N_{t_0}$.

It is known that the graph of a smooth curve $\mathbf{R} \to \mathbf{R}^n$ is transnormal iff every normal n-plane meets the graph in just one point ([4] or [2] for an elementary proof). We then have

Theorem 2.1 *G_f is transnormal iff every normal line meets G_f in just one point.*

Proof. The if part is trivial.

Let $(x_0, y_0) \in \mathbf{R}^2$. If $D_1 f, D_2 f$ both vanish at (x_0, y_0) then $G_f \cap N_{(x_0,y_0)} = \{(x_0, y_0, f(x_0, y_0))\}$. Suppose then that one of the partial derivatives does not vanish. Without loss of generality we may assume that $D_1 f(x_0, y_0) \neq 0$. Consider the curve $h\colon \mathbf{R} \to \mathbf{R}^2$ defined above. Then the assumption $(x_1, y_1, f(x_1, y_1)) \in N_{(x_0,y_0)}$ leads to $(x_1, y_1) = (x_0, y_0)$ due to the transnormality of G_h.

Theorem 2.1 also follows from [5].

The next two results deal with sufficient conditions for the transnormality of G_f.

Theorem 2.2 *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be such that $\|grad\, f(x, y)\| \leq 1, (x, y) \in \mathbf{R}^2$. Then G_f is transnormal.*

Proof. It is well-known that for such an f we have

$$|f(x, y) - f(x', y')| \leq \|(x, y) - (x', y')\|, \quad (x, y), (x', y') \in \mathbf{R}^2.$$

Assume that, for $(x_1, y_1) \neq (x_0, y_0)$, $(x_1, y_1, f(x_1, y_1))$ belongs to $N_{(x_0,y_0)}$. We then have

$$\begin{aligned} x_1 - x_0 &= (f(x_0, y_0) - f(x_1, y_1)) D_1 f(x_0, y_0), \\ y_1 - y_0 &= (f(x_0, y_0) - f(x_1, y_1)) D_2 f(x_0, y_0) \end{aligned}$$

and consequently

$$\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{|f(x_0, y_0) - f(x_1, y_1)|^2} = D_1 f(x_0, y_0)^2 + D_2 f(x_0, y_0)^2.$$

The left-hand side is greater than or equal to 1 while the right hand side is less or equal than 1. It the follows that $\|(x_0, y_0) - (x_1, y_1)\|^2 = |f(x_0, y_0) - f(x_1, y_1)|^2$. Therefore (x_0, y_0) is a maximum for g with $g(x, y) = |f(x, y) - f(x_1, y_1)|^2 - \|(x, y) - (x_1, y_1)\|^2$. Accordingly we must have $D_1 g(x_0, y_0) = D_2 g(x_0, y_0) = 0$. On the other hand it is easy to obtain

$$D_1 g(x_0, y_0) = 4(x_1 - x_0), \quad D_2 g(x_0, y_0) = 4(y_1 - y_0),$$

and at least one of them is nonzero. The contradiction shows that G_f is transnormal.

Theorem 2.3 *Let f: $\mathbf{R}^2 \to \mathbf{R}$ be such that, for i=1,2, $D_i f \geq 0$ or $D_i f \leq 0$. Then G_f is transnormal.*

Proof. Without loss of generality we may assume that $D_i f \geq 0$, $i = 1, 2$. Assume that G_f is not transnormal. Then there exist distinct $(x_1, y_1), (x_2, y_2)$ such that

$$x_2 - x_1 = (f(x_1, y_1) - f(x_2, y_2))D_1 f(x_1, y_1)$$

and

$$y_2 - y_1 = (f(x_1, y_1) - f(x_2, y_2))D_2 f(x_1, y_1).$$

Suppose, for instance, that $x_2 - x_1 \neq 0$. Then $\frac{y_2 - y_1}{x_2 - x_1} \geq 0$. Using the mean value theorem we have

$$x_2 - x_1 = (D_1 f(z_1, z_2)(x_1 - x_2) + D_2 f(z_1, z_2)(y_1 - y_2))D_1 f(x_1, y_1),$$

that is, $1 = \left(-D_1 f(z_1, z_2) - D_2 f(z_1, z_2)\frac{y_2 - y_1}{x_2 - x_1}\right) D_1 f(x_1, y_1)$. The right-hand side is ≤ 0 and we obtain a contradiction.

3. Nontransnormal graphs

Again we take a smooth map f: $\mathbf{R}^2 \to \mathbf{R}$ and assume that $D_1 f(x_0, y_0) \neq 0$. Take the map h: $\mathbf{R} \to \mathbf{R}^2$, with $h(t) = (\mu(t), f(t, \mu(t)))$, already considered at the beginning of **2**. We will write $h_2(t) = f(t, \mu(t))$.

Theorem 3.1 *If $D_1 f(x_0, y_0) < 0$ (resp. > 0) and $\lim_{t\to\infty} h_2'(t) = \infty$ (resp. $-\infty$) then G_f is not transnormal.*

Proof. Assume that G_f is transnormal and define $g : (x_0, \infty) \to \mathbf{R}$ by $g(t) = (t - x_0) + (h(t) - h(x_0)|h'(x_0))$. The transnormality of G_f implies that g never vanishes and since $g < 0$ leads to a contradiction we must have $g > 0$. Therefore

$$\frac{h_2(t) - h_2(x_0)}{t - x_0} < \frac{1}{h_2'(x_0)} \left(-1 - \left(\frac{D_2 f(x_0, y_0)}{D_1 f(x_0, y_0)}\right)^2\right).$$

On the other hand $\lim_{t\to\infty} h_2'(t) = \infty$ implies $\lim_{t\to\infty} h_2(t) = \infty$. By L'Hôpital's rule $\lim_{t\to\infty} \frac{h_2(t) - h_2(x_0)}{t - x_0} = \lim_{t\to\infty} h_2'(t) = \infty$. This contradicts the inequality above.

For the next theorem we need the following lemma

Lemma 3.1 *Let f : $\mathbf{R}^2 \to \mathbf{R}$ be polynomial of positive degree m, depending on both variables, and given by $f(x, y) = \sum_{i+j\leq m} a_{ij} x^i y^j$. Then there exists (x_0, y_0) such that $\frac{D_2 f(x_0, y_0)}{D_1 f(x_0, y_0)}$ is nonzero and it is not a zero of $p(y) = \sum_{i+j=m} a_{ij} y^j$.*

Proof. Start by choosing a point (x_1, y_1) such that $\frac{D_2 f(x_1, y_1)}{D_1 f(x_1, y_1)}$ is nonzero. Take then the connected component A of $\mathbf{R}^2 - \{(x, y) \in \mathbf{R}^2 : D_1 f(x, y) = 0\}$ which contains (x_1, y_1).

If $\frac{D_2 f}{D_1 f}$ is not constant in A then it is clear that we can find (x_0, y_0) as required.

Assume then that $\frac{D_2 f}{D_1 f}$ is constant with value c and that $p(c) = 0$. Since in A $D_2 f = cD_1 f$ it follows that $f(x, y) = g(cy + x)$. A simple argument using Taylor's formula then shows that, in an open neighbourhood of (x_1, y_1), $f(x, y) = \sum_{i=0}^{m} \lambda_i (cy+x)^i$, with $\lambda_m \neq 0$. Keeping in mind that $p(c) = 0$ and equating coefficients we arrive at $c^{2m} + \binom{m}{1} c^{2(m-1)} + \cdots + \binom{m}{m-1} c^2 + 1 = 0$ which is obviously impossible.

Theorem 3.2 *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be polynomial of positive even degree. Then G_f is not transnormal.*

Proof. If f depends only on one variable, say $f(x, y) = \sum_{i=0}^{m} a_i x^i$, choose (x_0, y_0) such that $D_1 f(x_0, y_0) \neq 0$. Then if G_f were transnormal the corresponding curve $h\colon \mathbf{R} \to \mathbf{R}^2$ would have positive even degree and transnormal graph. This cannot happen [1].

If f depends on both variables choose (x_0, y_0) as in the lemma. This ensures again that h is a polynomial curve of positive even degree and consequently G_h cannot be transnormal [1]. Consequently G_f is not transnormal.

4. Transnormal isotopy

Next we show how to deform a map with transnormal graph to a constant map through maps with transnormal graph.

Theorem 4.1 *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ have transnormal graph. Then, for $0 \leq t \leq 1$, $f_t\colon \mathbf{R}^2 \to \mathbf{R}$ given by $f_t(x, y) = tf(x, y)$ has also transnormal graph.*

Proof. We assume that there exists $0 < t < 1$ such that the graph of f_t is not transnormal. Then there are $(x_0, y_0), (x_1, y_1)$, $(x_1, y_1) \neq (x_0, y_0)$, such that $(x_1, y_1, f_t(x_1, y_1))$ belongs to the normal line to F_t at (x_0, y_0), with $F_t(x, y) = (x, y, f_t(x, y))$. Without loss of generality we may assume that $f(x_0, y_0) \neq 0$. Then the normal lines to F and F_t at (x_0, y_0) determine a plane parallel to the z-axis. Using a convenient affine isometry of $\mathbf{R}^3$ if necessary we may suppose that such a plane is determined by $y = 0$.

Let now h and h_t denote the maps relative to f and f_t and to the point $(x_0, 0)$ mentioned at the beginning of **2**. It follows that $h_t(x) = th(x)$. Since G_f is transnormal G_h is transnormal and G_{h_t} is also transnormal [3]. We have now a contradiction because obviously our initial assumption implies that G_{h_t} is not transnormal.

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